INTRODUCTION INTO RIGID ANALYTIC GEOMETRY

Course notes (in progress) for a course
given at Tel Aviv University
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by
Dan Haran

1. Valuation Theory

Only rank 1 valuation, that is, valuations with valuation group contained in $\mathbb{R}^+$.

EXERCISE 1.1:

(a) $|-a| = |a|$.

(b) $|a| < |b|$ *implies* $|a+b| = |b|$.

Proof: (a) $|-a|^2 = |(-a)^2| = |a|^2$, hence $|-a| = |a|$.

(b) On one hand $|a+b| \le \max(|a|,|b|) = |b|$. If $|a+b| < |b|$, then $|b| = |(a+b) + (-a)| \le \max(|a+b|,|a|) < |b|$, a contradiction. ∎

Completion, definition of norm eqivalence of norms over complete fields, uniqueness of extension of valuations from complete fields to finite (and hence algebraic) extensions.

Definition 1.2: Let $(k, |\ |)$ be a valued field.

(a) $k^0 = \{a \in k \mid\ |a| \le 1\}$ is the **valuation ring** of $|\ |$.

It is a valuation subring of k, that is, for each $a \in k$ either $a \in k^0$ or $k^{-1} \in k^0$.

(b) $k^{00} = \{a \in k \mid\ |a| < 1\}$ is the (unique) **maximal ideal** of k^0, because

(c) $U = k^0 \smallsetminus k^{00} = \{a \in k \mid\ |a| = 1\} = (k^0)^\times$.

(d) $\bar{k} = k^0/k^{00}$ is the **residue field** of $|\ |$.

(e) $|k^\times| = \{|a| \mid a \in k^\times\}$ is the **value group** of $|\ |$. ∎

EXERCISE 1.3: *Compute the above objects for $k = \mathbb{Q}$ with p-adic valuation and for $k = k_0(t)$. (Notice that $|k^\times| \cong \mathbb{Z}$ -* **discrete valuation***.)*

Let k_v be the completion of k. Then $\overline{k_v} = \bar{k}$. Indeed, k is dense in k_v. Hence for each $b \in k_v$ with $|b| \le 1$ there is $a \in k$ with $|b-a| < 1$. In particular, $|a| \le 1$.

If $|\ |$ is discrete, then $|k_v^\times| = |k^\times|$. Indeed, if $\{a_n\}$ is a Cauchy sequence in k, then $\lim |a_n| = |a_m|$ for some m or $\lim |a_n| = 0$.

How does $k_v = \mathbb{Q}_p$ look like? Let $b \in k_v^0$. Then there is a unique $a_0 \in \{0,1,\ldots,p-1\} \subseteq \mathbb{Z}$ such that $\bar{a}_0 = \bar{b} \in \bar{k}$, that is $|a_0 - b| < 1$. Thus $b = a_0 + pb_1$, where, $b_1 \in k_v^0$. Again, there is a uniqe $a_1 \in \{0,1,\ldots,p-1\}$ such that $|a_1 - b_1| < 1$. Thus $b = a_0 + pa_1 + p^2 b_2$, where, $b_2 \in k_v^0$. By induction, $b = \sum_{n=0}^\infty a_n p^n$.

For a general $b \in \mathbb{Q}_p$ there is $m \geq 0$ such that $p^m b \in k_v^0$, that is, $b = p^{-m} b'$, where $b' \in k_v^0$. So $b = \sum_{n=N}^{\infty} a_n p^n$, where $N \in \mathbb{Z}$. (This is just like the usual p-adic expansion of numbers, only infinite; the addition and multiplication are the same.) Notice that $(k_v)^0 = \mathbb{Z}_p = \{\sum_{n=0}^{\infty} a_n p^n \mid a_n \in \{0, 1, \ldots, p-1\}\}$.

Similarly, the completion of $k_0(t)$ is $k_0((t)) = \{\sum_{n=N}^{\infty} a_n t^n \mid a_n \in k_0, \ N \in \mathbb{Z}\}$.

2. Banach Spaces

(Some theorems that should be here are at the end of this section.)

Recall the following theorem

BAIRE CATEGORY THEOREM: *Let X be a nonempty complete metric space, and let $\{X_i\}_{i=1}^{\infty}$ be a sequence of closed subsets of X such that $X = \bigcup_{i=1}^{\infty} X_i$. Then not each X_i has empty interior.*

Proof: For $x \in X$ and for a positive number ε denote $B(x, \varepsilon) = \{x' \in X \mid d(x, x') < \varepsilon\}$, the open ball around x of radius ε.

Assume that each X_i has empty interior. Then for each $x \in X$, each $\varepsilon > 0$ and each i the point x is not in the interior of X_i and hence there is $x' \in B(x, \varepsilon)$ such that $x' \notin X_i$. As $B(x, \varepsilon)$ is open and X_i is closed, there is $\varepsilon' > 0$ such that $B(x', \varepsilon') \subseteq B(x, \varepsilon)$ and $B(x', \varepsilon') \cap X_i = \emptyset$.

Fix $x_0 \in X$ and $\varepsilon_0 > 0$. Use the preceding paragraph to construct, by induction, a sequence $x_1, x_2, \ldots \in X$ and a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ such that
(a) $B(x_{i+1}, \varepsilon_{i+1}) \subseteq B(x_i, \varepsilon_i/2) \subseteq B(x_i, \varepsilon_i)$,
(b) $B(x_{i+1}, \varepsilon_{i+1}) \cap X_i = \emptyset$,

By (a), $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence in X, and hence converges to some $x \in X$. Let $i \geq 1$. As $x_j \in B(x_i, \varepsilon_i/2)$ for all $j > i$, this x is in the closure of $B(x_i, \varepsilon_i/2)$ and hence in $B(x_i, \varepsilon_i)$. By (b), $x \notin X_i$. This is a contradiction to $X = \bigcup_{i=1}^{\infty} X_i$. ∎

The actions on a normed vector space (addition and multiplication with scalars) are continuous.

A complete vector space (over a complete field) is called a **Banach space**.

BANACH THEOREM 2.1: *Let $T\colon V \to W$ be a surjective continuous linear map of Banach spaces over a complete field k. Then T is open.*

Proof: Fix $\pi \in k$ with $0 < |\pi| < 1$.

Denote $V^0 = \{v \in V \mid ||v|| < 1\}$. This is an open subset of V; moreover, sets of the form $v + \pi^n V^0$ form a basis for the topology on V. Similarly put $W^0 = \{w \in W \mid ||w|| < 1\}$. We have to show that the image of every open basic set in V is open in W. Since $T(v+\pi^n V^0) = T(v) + \pi^n T(V^0)$, it is enough to show that $U := T(V^0)$ is open in W. Equivalently, as U is an additive subgroup of W, show that 0 is an inner point of U.

CLAIM 1: *0 is an inner point of $\bar{U}$.* Indeed, apply T to $V = \bigcup_{n=1}^{\infty} \pi^{-n} V^0$ to get $W = \bigcup_{n=1}^{\infty} \pi^{-n} U$ and hence $W = \bigcup_{n=1}^{\infty} \pi^{-n} \bar{U}$. By Baire's theorem there is n such that $\pi^{-n}\bar{U}$ has an inner point. Since $\pi^{-n}\bar{U}$ is homeomorphic to $\bar{U}$, also $\bar{U}$ has an inner point u. Then $0 = u - u$ is an inner point of $\bar{U} - u = \bar{U}$.

Thus there is $m \in \mathbb{N}$ such that $\pi^m W^0 \subseteq \bar{U}$.

CLAIM 2: *If $\pi^m W^0 \subseteq \bar{U}$, then $\pi^{m+1} W^0 \subseteq U$.* Indeed, let $w \in \pi^{m+1} W^0$. We will construct a sequence $\{v_n\}_{n=1}^{\infty}$ in V^0 such that

$$w - \sum_{i=1}^{n} \pi^i T(v_i) \in \pi^{n+m+1} W^0. \tag{3}$$

Let $n \geq 1$. Suppose that we have already constructed $v_1, v_2, \ldots, v_{n-1} \in V^0$ such that $w - \sum_{i=1}^{n-1} \pi^i T(v_i) \in \pi^{n+m} W^0$. (For $n = 1$ this is the assumption $w \in \pi^{m+1} W^0$.) Thus there is $w' \in \pi^m W^0$ such that

$$w - \sum_{i=1}^{n-1} \pi^i T(v_i) = \pi^n w'. \tag{4}$$

But $w' \in \pi^m W^0 \subseteq \bar{U} = \overline{T(V^0)}$, hence there is $v_n \in V^0$ such that

$$w' - T(v_n) \in \pi^{m+1} W^0. \tag{5}$$

Multiply (5) by π^n and add it to (4) – and get (3).

Clearly, $\{\sum_{i=1}^{n} \pi^i v_i\}_{n=1}^{\infty}$ is a Cauchy sequence in V^0. Let $v \in V^0$ be its limit. Then $\sum_{i=1}^{n} \pi^i T(v_i) = T(\sum_{i=1}^{n} \pi^i v_i)$ converges to $T(v) \in T(V^0) = U$. But by (3), $\sum_{i=1}^{n} \pi^i T(v_i)$ converges to w. Thus $w \in U$. ∎

COROLLARY 2.2: *There is $C > 0$ such that for every $w \in W$ there is $v \in V$ such that $T(v) = w$ and $||v|| \leq C||w||$.*

Proof: By Banach Theorem, there is $0 < \delta < 1$ such that

$$\{w \in W \mid \ ||w|| < \delta\} \subseteq \{T(v) \mid \ v \in V, \ ||v|| < 1\}$$

That is, replacing w by $\frac{1}{a^r}w$, where $a \in k^\times$ and $r \in \mathbb{Z}$, we have:

(1) If $w \in W$ such that $||w|| < \delta|a^r|$, then there is $v \in V$ such that $w = T(v)$ and $||v|| < |a^r|$.

Choose $a \in k$ such that $|a| > 1$. Put $C = \frac{|a|}{\delta}$. Let $w \in W$. Then there is a unique $r \in \mathbb{Z}$ such that

$$C^{-1}|a|^r = \delta|a|^{r-1} < ||w|| \le \delta|a|^r = \delta|a^r|.$$

By (1) there is $v \in V$ such that $T(v) = w$ and

$$||v|| < |a|^r < C\ ||w||.$$

∎

Corollary 2.3: *Let $T\colon V \to W$ be a linear map of Banach spaces over a complete field k. Then T is continuous if and only if its graph $G = \{(v, T(v) \mid v \in V\}$ is closed in $V \times W = V \oplus W$.*

Proof: Every continuous map $T\colon V \to W$ into a Hausdorff space has a closed graph G. Inded, let $(v, w) \in (V \times W) \smallsetminus G$, that is $T(v) \ne w$. There are disjoint open neighbourhoods: W_1 of $T(v)$ and W_2 of w. The neighbourhood $T^{-1}(W_1) \times W_2$ of $(v, w) \in V \times W$ does not meet G.

Conversely, assume that G is closed. Then it is a complete k-subspace of $V \times W$. The projection $V \times W \to V$ induces a bijective continuous linear map $G \to V$. By Banach Theorem it is also open. Hence its inverse $V \to G$ is also continuous, hence so is its composition with the projection $V \times W \to W$. But this is T. ∎

Definition 2.4: Let k be a complete field. A **Banach algebra** over k is a Banach space A which is also a commutative ring containing k and $||1|| = 1$ and $||ab = ||a|| \cdot ||b||$.

A **Banach module** over A is an A-module M with a norm $||\ ||$ such that M is a Banach space over k and $||am|| \le ||a|| \cdot ||m||$ for all $a \in A$ and $m \in M$.

Theorem 2.5: *Let M be a finitely generated Banach module over a Banach algebra A (over a complete field k). Assume that A is noetherian (every submodule of M is finitely generated). Then every submodule N of M is closed.*

Proof: Let $\tilde{N}$ be the closure of N in M; it is closed and hence complete. By the noetherianity, $\tilde{N}$ has a finite set $e_1, \ldots, e_n$ of generators. Define a norm on A^n by $||(a_1, \ldots, a_n)|| = \max(||a_1||, \ldots, ||a_n|)$. Then A^n is Banach A-module (also a Banach algebra - one can produce examples of Banach algebras this way). The map $A^n \to \tilde{N}$ given by $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i e_i$ is an A-homomorphism (in particular k-linear), continuous and surjective. By Banach Theorem there is $C > 0$ such that every $x \in \tilde{N}$ can be written as $x = \sum_{i=1}^n a_i e_i$ with $||a_i|| \leq C||x||$. Wlog $C > 1$.

Choose $f_1, \ldots, f_n \in N$ such that $||f_i - e_i|| \leq \frac{1}{C^2}$.

CLAIM: *$\hat{N} = \sum_{i=1}^n Af_i$ and hence $\hat{N} = N$.*

Let $x \in \hat{N}$. We wil construct, by induction, convergent series in A

$$a_1 = \sum_{k=1}^{\infty} a_{1k}, \quad a_2 = \sum_{k=1}^{\infty} a_{2k}, \quad , \ldots, \quad a_n = \sum_{k=1}^{\infty} a_{nk},$$

such that $x = a_1 f_1 + \cdots + a_n f_n$. Suppose, by induction, that we have found a_{ik} for $k < l$ such that

$$||x - \sum_{i=1}^{n} (\sum_{k=1}^{l-1} a_{ik}) f_i|| \leq C||x||$$

(for $l = 1$ this is obvious). Then there are $a_{il} \in A$ such that

$$x - \sum_{i=1}^{n} (\sum_{k=1}^{l-1} a_{ik}) f_i = \sum_{i=1}^{n} a_{il} e_i$$

and

$$||a_{il}|| \leq C||x - \sum_{i=1}^{n} (\sum_{k=1}^{l-1} a_{ik})|| \leq C \frac{1}{C^{l-1}} ||x||$$

Hence

$$||\sum_{i=1}^{n} a_{il} e_i - \sum_{i=1}^{n} a_{il} f_i|| = ||\sum_{i=1}^{n} a_{il}(e_i - f_i)|| \leq C \frac{1}{C^{l-1}} ||x|| \frac{1}{C^2} = \frac{1}{C^l} ||x||$$

∎

EXERCISE 2.6: *Let M be a finitely generated module over a noetherian Banach algebra A. Then M is a Banach module.*

Proof: If $M = A^m$, put $||(a_1, \ldots, a_n)|| = \max_i ||a_i||$. In the general case there is a surjective A-homomorphism $s \colon A^n \to M$. Put $||s(x)|| = \inf\{||x - y|| \mid y \in \mathrm{Ker}(s)\}$. Now show that this is a norm on M (here we use that $\mathrm{Ker}(s)$ is closed in A^n) and M is complete w.r.t it.

COROLLARY 2.7: *Every A-homomorphism of finitely generated Banach A-modules is continuous.*

Proof: Let M, N be two A-modules and let $u\colon M \to N$ be an A-homomorphism. Suppose first M is a free A-module with basis $e_1, \ldots, e_n$ and $||\sum a_i e_i|| = \max_i ||a_i||$. Then

$$||u(\sum a_i e_i)|| = ||\sum a_i u(e_i)|| \le \max ||a_i u(e_i)|| \le \max ||a_i|| \cdot \max ||u(e_i)||.$$

In the general case there is a surjective map $s\colon A^n \to M$. By the previous case s and $u \circ s$ are continuous. By Banach theorem s is open. It follows that u is continuous. (Take $U \subseteq N$ open; then $u^{-1}(U) = s(s^{-1}(u^{-1}(U))) = s(u \circ s)^{-1}(U)$ is open.) ∎

Definition 2.11: Let V be a vector space over a complete field k. **Norm** on a E is a function $||\ ||\colon E \to \mathbb{R}$ such that for all $v, v' \in V$ and all $a \in k$

(a) $||v|| \ge 0$.

(b) $||v|| = 0$ implies $v = 0$.

(c) $||av|| = |a| \cdot ||v||$.

(d) $||v + v'|| \le \max(||v||.||v'||)$.

Excluding requirement (b) we get a **semi-norm**.

Two norms $||\ ||_1, ||\ ||_2$ on V are **equivalent norms** if there are positive constants C_1, C_2 such that $C_1||v||_1 \le ||v||_2 \le C_2||v||_1$ for all $v \in V$. ∎

Example 2.12: If $\dim V = n < \infty$, and $v_1, \ldots, v_n$ is its basis,

$$||\sum_{i=1}^{n} a_i v_i|| = \max_i |a_i|$$

defines a norm on V. ∎

LEMMA 2.13: *Let V be a vector space over a complete field k, let $v_1, \ldots, v_n \in V$ be linearly independent, and let $v^{(i)} = \sum_{j=1}^{n} a_j^{(i)} v_j$, for $j = 1, 2, \ldots$ be a Cauchy sequence in V. Then $\{a_j^{(i)}\}_{i=1}^{\infty}$ is a Cauchy sequence in k, for every $1 \le j \le n$.*

Proof: By induction on n.

Corollay 2.13: *In the above lemma,*

$$v^{(i)} \to 0 \leftrightarrow a_j^{(i)} \to 0 \text{ for all } 1 \leq j \leq n.$$

Theorem 2.14: *Let V be a finite dimensional vector space over a complete field k. Then any two norms on V are equivalent: There are positive constants C_1, C_2 such that for every $v \in V$*

$$C_1||v||_1 \leq ||v||_2 \leq C_2||v||_1.$$

Corollay 2.15: *Let E be an algebraic extension of a complete field k. Then the valuation $|\ |$ of k uniquely extends to a valuation of E. Moreover, if E/k is finite, then E is complete.*

Proof: We do not prove the existence of the extension. We proved the completeness and the uniqueness. (Missing.) ∎

3. Affinoids in the projective line

Let K be an algebraically closed valued field wrt to a non-archimedian (multiplicative) valuation $|\ |$. Notice that $|K^\times|$ is dense in $[0, \infty)$.

Let $\mathbb{P} = \mathbb{P}^1(K) = (K \times K \smallsetminus \{(0,0)\})/\sim$ where $(x_0, x_1) \sim (y_0, y_1)$ if there is $a \in K^\times$ such that $y_0 = ax_0$ and $y_1 = ax_1$.

Denote the equivalence class of (x_0, x_1) in $\mathbb{P}$ by $(x_0 : x_1)$ and write $z = (z : 1)$ and $\infty = (1 : 0)$. If $x_1 \neq 0$, then $(x_0 : x_1) = (\frac{x_0}{x_1} : 1) = \frac{x_0}{x_1}$. If $x_1 = 0$, then $x_0 \neq 0$, and hence $(x_0 : x_1) = (1 : 0) = \infty$. Thus $\mathbb{P} = \mathbb{P}^1(K) = K \cup \{\infty\}$. We call $\mathbb{P}$ **the projective line**.

Definition 3.1: A map $\varphi\colon \mathbb{P} \to \mathbb{P}$ is called an **automorphism** of $\mathbb{P}$ if there exists a matrix $A \in \mathrm{Gl}_2(K)$ such that $\varphi(\mathbf{x}) = A\mathbf{x}$. ∎

EXERCISE 3.2: *The set of automorphisms of $\mathbb{P}$ is a group, isomorphic to* $\mathrm{GPl}_2(K)$.

Given distinct $z_1, z_2, z_3 \in \mathbb{P}$ and distinct $z_1', z_2', z_3' \in \mathbb{P}$, there is a unique automorphism φ of $\mathbb{P}$ such that $\varphi(z_i) = z_i'$, for $i = 1, 2, 3$.

Definition 3.3: A subset D of $\mathbb{P}$ is a **closed** [**open**] **disk** if there are $a \in K$ and $\rho \in |K^\times|$ such that

$$D = \{z \in K \mid |z - a| \leq \ [<]\ \rho\} \qquad \text{or} \qquad D = \{z \in K \mid |z - a| \geq \ [>]\ \rho\} \cup \{\infty\}.$$

∎

EXERCISE 3.4: (i) *Let $D = \{z \in \mathbb{P} \mid |z - a| < \rho\}$. If $b \in D$, then $D = \{z \in \mathbb{P} \mid |z - b| < \rho\}$.*

(ii) *Let $D = \{z \in \mathbb{P} \mid |z - a| > \rho\}$. If $b \notin D$, then $D = \{z \in \mathbb{P} \mid |z - b| > \rho\}$.*

(iii) *Analogous statements hold for closed disks.*

LEMMA 3.5: *Let D be an open (closed) disk, and let T be an automorphism of $\mathbb{P}$. Then $T(D)$ is an open (closed) disk.*

Proof: Every automorphism of $\mathbb{P}$ is the product of stretchings ($z \mapsto az$ with $a \in K^\times$), translations ($z \mapsto z + b$ with $b \in K$), and the inversion ($z \mapsto z^{-1}$). (These maps are defined by elementary matrices over K, and every elementary matrix over K is of one of these types, except for $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. But $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Thus we may assume that T is one of these three types.

If T is either stretching or translation, the assertion is obvious. Assume therefore that T is $z \mapsto z^{-1}$. We may also assume that $\infty \notin D$. Otherwise $\mathbb{P} \smallsetminus D$ is a closed (open) disk that does not contain ∞. If $T(\mathbb{P} \smallsetminus D) = \mathbb{P} \smallsetminus T(D)$ is a closed (open) disk, then $T(D)$ will be an open (closed) disk.

This leaves us with four cases. Let $\triangleleft$ be one of the symbols $<, \leq$, and let $\triangleleft'$ be the other one. This notation allows us to deal with a pair of cases simultaneously.

(1) $D = \{z \in \mathbb{P} \mid |z-a| \triangleleft \rho\}$ and $|a| \triangleleft \rho$. Then $0 \in D$, and hence by Exercise 3.4, $D = \{z \in \mathbb{P} \mid |z| \triangleleft \rho\}$. In this case $T(D) = \{w \mid \frac{1}{\rho} \triangleleft |w|\}$, a disk.

(2) $D = \{z \in \mathbb{P} \mid |z-a| \triangleleft \rho\}$ and $\rho \triangleleft' |a|$. Then every $z \in D$ satisfies $|z-a| < |a|$, and hence $|z| = |a|$. Put $D' = \{w \mid |w - \frac{1}{a}| \triangleleft \frac{\rho}{|a|^2}\}$. As $\frac{\rho}{|a|^2} \triangleleft' |\frac{1}{a}|$, every $w \in D'$ satisfies $|w - \frac{1}{a}| < \frac{1}{|a|}$, and hence $|w| = \frac{1}{|a|}$. Therefore

$$T(D) = \{w \mid |\frac{1}{w} - a| \triangleleft \rho,\ |\frac{1}{w}| = |a|\} = \{w \mid |w - \frac{1}{a}| \triangleleft \frac{\rho}{|a|}|w|,\ |w| = \frac{1}{|a|}\} =$$
$$\{w \mid |w - \frac{1}{a}| \triangleleft \frac{\rho}{|a|^2},\ |w| = \frac{1}{|a|}\} = D'. \qquad \blacksquare$$

LEMMA 3.6: *Let D_1, D_2 be two disks (open or closed, not necessarily of the same type!) such that $D_1 \cap D_2 \neq \emptyset$ and $D_1 \cup D_2 \neq \mathbb{P}$. Then either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.*

Proof: Using an autormorphism of $\mathbb{P}$ we may assume that $\infty \notin D_1, D_2$. Thus

$$D_i = \{z \in \mathbb{P} \mid |z - a_i| < \rho_i\} \qquad \text{or} \qquad D_i = \{z \in \mathbb{P} \mid |z - a_i| \leq \rho_i\}, \qquad i = 1, 2.$$

Let $a \in D_1 \cap D_2$. By Exercise 3.4, wlog $a_1 = a_2 = a$. The assertion follows. (If $\rho_i < \rho_j$, then $D_i \subseteq D_j$; if $\rho_i = \rho_j$, and D_i is open or D_j closed, then $D_i \subseteq D_j$.) $\blacksquare$

COROLLARY 3.7: *Let $F' \neq \mathbb{P}$ be the union of finitely many disks. Then F' is the union of finitely many disjoint disks.*

Proof: Let $C_1, \ldots, C_m$ be disks such that $F' = \bigcup_{j=1}^m C_j$. Let $D_1, \ldots, D_r$ be the maximal among $C_1, \ldots, C_m$ (with respect to inclusion of sets). Then $F' = \bigcup_{i=1}^r D_i$. For $i \neq j$, neither $D_i \subseteq D_j$ nor $D_j \subseteq D_i$, and $D_i \cup D_j \neq \mathbb{P}$. Therefore, by Lemma 3.6, $D_i \cap D_j = \emptyset$. $\blacksquare$

Definition 3.8:

(a) A non-empty subset of $\mathbb{P}$ is called a **connected affinoid**, if it is the intersection of finitely many closed disks. Equivalently, the set is the complement of the union of finitely many open disks and the union is not $\mathbb{P}$.

(b) An **affinoid** is the union of finitely many connected affinoids. ∎

The value group $|K^\times|$ is not discrete, and hence it has infinitely many values between ρ_1 and ρ_2. Therefore R is infinite. ∎

LEMMA 3.9: *Let $D_0, \ldots, D_n$ be disks. If $D_i \cup D_j \neq \mathbb{P}$ for all i, j, then $\bigcup_{i=0}^n D_i \neq \mathbb{P}$; moreover, $\mathbb{P} \smallsetminus \bigcup_{i=0}^n D_i$ is an infinite set.*

Proof: Replace $D_0, \ldots, D_n$ by the maximal disks among them to assume that there are no inclusion among them. By Lemma 3.6, $D_0, \ldots, D_n$ are disjoint. By Lemma 3.5 we may assume that either $D_0 = \{z \in \mathbb{P} \mid |z| \geq 1\}$ or $D_0 = \{z \in \mathbb{P} \mid |z| > 1\}$. Let $1 \leq i \leq n$. As $D_0 \cap D_i = \emptyset$, we have $D_i = \{z \in K \mid \ |z - a_i| \triangleleft_i \rho_i\}$, where $\triangleleft_i$ is either $<$ or $\leq$.

PART A: $D_0 = \{z \in \mathbb{P} \mid |z| \geq 1\}$. Let $1 \leq i \leq n$. As $D_0 \cap D_i = \emptyset$, we have $|a_i| < 1$. As $D_0 \cup D_i \neq \mathbb{P}$, also $\rho_i < 1$. Thus $\pi := \max_{1 \leq i \leq n}(|a_i|, \rho_i)$ is smaller than 1, and hence $\{z \in K \mid \ \pi < |z| < 1\}$ is contained in $\mathbb{P} \smallsetminus \bigcup_{i=0}^n D_i$.

PART B: $D_0 = \{z \in \mathbb{P} \mid |z| > 1\}$. Let $1 \leq i \leq n$. As $D_0 \cap D_i = \emptyset$, we have $|a_i|, \rho_i \leq 1$. However, if $\rho_i = 1$ and $\triangleleft_i$ is $\leq$, then $D_i = \{z \in K \mid \ |z| \leq 1\}$, which gives the contradiction $D_0 \cup D_i = \mathbb{P}$. Therefore either $\rho_i \leq 1$ or $\triangleleft_i$ is $<$, and hence $D_i \subseteq \{z \in K \mid \ |z - a_i| < 1\}$. Thus $\mathbb{P} \smallsetminus \bigcup_{i=0}^n D_i$ contains the set

$$\begin{aligned} U :=& \{z \in K \mid \ |z| = 1, |z - a_i| = 1, \ 1 \leq i \leq n\} \\ =& \{z \in K^0 \mid \ \bar{z} \neq 0, \bar{a}_1, \ldots, \bar{a}_r\} \end{aligned}$$

which is infinite, since $\bar{K}$ is infinite. ∎

COROLLARY 3.10: *Let $D_1, \ldots, D_n$ and $C_1, \ldots, C_m$ be disks.*

(a) *If $D_i \cap D_j \neq \emptyset$ for all i, j, then $\bigcap_{i=1}^n D_i \neq \emptyset$.*

(b) *If $\emptyset \neq \bigcap_{i=1}^n D_i \subseteq \bigcup_{j=1}^m C_j \neq \mathbb{P}$, then there are i and j such that $D_i \subseteq C_j$.*

(c) *If $D_1, \dots, D_n$ are disjoint, of the same type (closed or open), then $\mathbb{P}$ is not their disjoint union.*

(d) *If $\bigcup_{i=1}^n D_i = \bigcup_{j=1}^m C_j \neq \mathbb{P}$, and there are no inclusions among the D_i and no inclusions among the C_j, then $n = m$ and, up to a permutation, $D_i = C_i$, for $i = 1, \dots, m$.*

Proof: (a) Apply Lemma 3.9 to the disks $\mathbb{P} \smallsetminus D_1, \dots, \mathbb{P} \smallsetminus D_1$.

(b) If $\bigcap_{i=1}^n D_i \subseteq \bigcup_{j=1}^m C_j$, then $\mathbb{P} = \bigcup_{i=1}^n (\mathbb{P} \smallsetminus D_i) \cup \bigcup_{j=1}^m C_j$. By Lemma 3.9 either $(\mathbb{P} \smallsetminus D_i) \cup (\mathbb{P} \smallsetminus D_{i'}) = \mathbb{P}$ for some i, i', or $C_j \cup C_{j'} = \mathbb{P}$ for some j, j', or $(\mathbb{P} \smallsetminus D_i) \cup C_j = \mathbb{P}$ for some i, j. The first option gives $D_i \cap D_{i'} = \emptyset$, a contradiction to $\emptyset \neq \bigcap_{i=1}^n D_i$. The second option contradicts $\bigcup_{j=1}^m C_j \neq \mathbb{P}$. The third option gives $D_i \subseteq C_j$.

(c) We have $D_i \cup D_k \neq \mathbb{P}$ for all i, k (otherwise $D_i = D_k^c$ are of the same type). Apply Lemma 3.9.

(d) Fix $1 \leq i \leq m$. As $\emptyset \neq D_i \subseteq \bigcup_{j=1}^n C_j \neq \mathbb{P}$, by (b) there is $1 \leq j \leq n$ such that $D_i \subseteq C_j$. Similarly, there is $1 \leq i' \leq m$ such that $C_j \subseteq D_{i'}$. Thus $D_i \subseteq D_{i'}$. By assumption, this implies that $i = i'$. Hence $D_i = C_j$. ∎

Proposition 3.11: *Let F be a connected affinoid, and let $F_1, \dots, F_m$ be disjoint connected affinoids, $m \geq 2$. Then $F \neq \dot\bigcup_{i=1}^m F_i$.*

Proof: Write F as $F = \mathbb{P} \smallsetminus \dot\bigcup_{j=1}^p C_j$, where C_j are disjoint open disks, and $p \geq 0$.

Similarly, for each $1 \leq i \leq m$ we have $F_i = \mathbb{P} \smallsetminus \dot\bigcup_{t_i=1}^{n_i} D_{it_i}$, where the D_{it_i} are open disks.

Assume that $F = \dot\bigcup_{i=1}^m F_i$. Let $\mathbf{T} = \{\mathbf{t} = (t_1, \dots, t_m) \mid 1 \leq t_i \leq n_i\}$. Then

$$\mathbb{P} \neq \dot\bigcup_{j=1}^{p} C_j = (\dot\bigcup_{t_1=1}^{n_1} D_{1t_1}) \cap \dots \cap (\dot\bigcup_{t_m=1}^{n_m} D_{mt_m}) = \bigcup_{\mathbf{t} \in \mathbf{T}} D_{\mathbf{t}}, \tag{3}$$

where $D_{\mathbf{t}} = D_{1t_1} \cap \dots \cap D_{mt_m}$, for each $\mathbf{t} \in \mathbf{T}$.

Part A: *If $D_{\mathbf{t}} \neq \emptyset$, then there is $1 \leq k \leq m$ such that $D_{kt_k} \subseteq D_{it_i}$ for all $1 \leq i \leq m$ and hence $D_{\mathbf{t}} = D_{kt_k}$.*

Indeed, $D_{\mathbf{t}} \subseteq \bigcup_j C_j \neq \mathbb{P}$, so by Corollary 3.10(b) there are $1 \leq k \leq m$ and $1 \leq j \leq p$ such that $D_{kt_k} \subseteq C_j$. In particular, $D_{\mathbf{t}} \subseteq C_j$. As $C_1, \dots, C_p$ are disjoint, this

j is uniquely determined by $\mathbf{t}$. Let $1 \le i \le m$. As $F_i \subseteq F$ and hence $C_j \subseteq \dot{\bigcup}_{s_i=1}^{n_i} D_{is_i}$, by Corollary 3.10(b) there is (a unique) s_i such that $C_j \subseteq D_{is_i}$. Thus there is a unique $\mathbf{s} = (s_1, \dots, s_m) \in \mathbf{T}$ such that $C_j \subseteq D_{\mathbf{s}}$. We get $D_{\mathbf{t}} \subseteq D_{kt_k} \subseteq C_j \subseteq D_{\mathbf{s}}$. But $\mathbf{t} = \mathbf{s}$, since $D_{\mathbf{t}} \cap D_{\mathbf{s}} \ne \emptyset$. Therefore $D_{\mathbf{t}} = D_{kt_k}$, which proves the claim.

Part B: *For all $1 \le i < j \le m$ there are t_i and t_j such that $D_{it_i} \cup D_{jt_j} = \mathbb{P}$.* Indeed, $F_i \cap F_j = \emptyset$, that is, $\bigcup_{t_i=1}^{n_i} D_{it_i} \cup \bigcup_{t_j=1}^{n_j} D_{jt_j} = \mathbb{P}$. By Lemma 3.9, $\mathbb{P}$ is the union of two of the disks on the left handed side. As $\bigcup_{t_i=1}^{n_i} D_{it_i}, \bigcup_{t_j=1}^{n_j} D_{jt_j} \ne \mathbb{P}$, one of the two disks is of the form D_{it_i} and the other one of the form D_{jt_j}.

Part C: *Construction of a special $\mathbf{t} \in T$.* By Part B there are t_1 and t_2 such that $D_{1t_1} \cup D_{2t_2} = \mathbb{P}$. Choose such t_1. For $2 \le i \le m$ choose t_i in the following way:

(a) If there exists t_i such that $D_{1t_1} \cup D_{it_i} = \mathbb{P}$, – choose such t_i.

(b) Otherwise, by Part B, there are $t_1' \ne t_1$ and t_i such that $D_{1t_1'} \cup D_{it_i} = \mathbb{P}$. Choose such t_i. As $D_{1t_1} \cap D_{1t_1'} = \emptyset$, we have $D_{1t_1} \subseteq D_{1t_1'}^c \subseteq D_{it_i}$. Thus we have chosen t_i such that $D_{1t_1} \subseteq D_{it_i}$.

Part D: There is no i such that $D_{it_i} \subseteq D_{1t_1}, \dots, D_{mt_m}$. Observe that (a) applies to $i = 2$, that is, $D_{1t_1} \cup D_{2t_2} = \mathbb{P}$. Thus $D_{1t_1} \not\subseteq D_{2t_2}$. It follows that if i has been chosen by (b), then also $D_{it_i} \not\subseteq D_{2t_2}$. If i has been chosen by (a), then $D_{it_i} \not\subseteq D_{1t_1}$.

Part E: $D_{\mathbf{t}} \ne \emptyset$. By Corollary 3.10(a) it suffices to show for $1 \le i, j \le m$ that $D_{it_i} \cap D_{jt_j} \ne \emptyset$. Suppose first $j = 1$. If t_i has been chosen by (a), then $D_{1t_1} \cup D_{it_i} = \mathbb{P}$, and hence $D_{1t_1} \cap D_{it_i} \ne \emptyset$. If t_i has been chosen by (b), then $D_{1t_1} \subseteq D_{it_i}$, and hence $D_{1t_1} \cap D_{it_i} \ne \emptyset$.

Now the general case: If t_i has been chosen by (b), then $D_{1t_1} \subseteq D_{it_i}$, hence by the previous case $D_{it_i} \cap D_{jt_j} \ne \emptyset$. Similarly if t_j has been chosen by (b). If both t_i and t_j have been chosen by (a), then $D_{1t_1} \cup D_{it_i} = \mathbb{P} = D_{1t_1} \cup D_{jt_j}$, and hence $\emptyset \ne \mathbb{P} \smallsetminus D_{1t_1} \subseteq D_{it_i} \cap D_{jt_j}$. ∎

Exercise 3.12: *Let F_1, F_2 be connected affinoids, $F_1 \cap F_2 \ne \emptyset$. Then both $F_1 \cap F_2$ and $F_1 \cup F_2$ are connected affinoids.*

Proof: The first assertion is trivial. As for the second one, write $\mathbb{P} \smallsetminus F_1$ and $\mathbb{P} \smallsetminus F_2$ as unions of open disks, say, $\mathbb{P} \smallsetminus F_1 = \bigcup_i D_i$ and $\mathbb{P} \smallsetminus F_2 = \bigcup_j E_j$. Then

$$\mathbb{P} \smallsetminus (F_1 \cup F_2) = (\mathbb{P} \smallsetminus F_1) \cap (\mathbb{P} \smallsetminus F_2) = \bigcup_{ij} D_i \cap E_j.$$

The assumption $F_1 \cap F_2 \neq \emptyset$ implies that $D_i \cup E_j \neq \mathbb{P}$, for all i, j. By Lemma 3.6, $D_i \cap E_j$ is either empty or an open disk. ∎

THEOREM 3.13: *Let $F \neq \mathbb{P}$ be an affinoid. There are unique connected affinoids $F_1, \ldots, F_m$ such that $F = \dot{\bigcup}_{i=1}^m F_i$.*

PROOF: *Existence.* Write F as the union of connected affinoids $F_1, \ldots, F_m$. If there are $1 \leq i, j \leq m$ such that $F_i \cap F_j \neq \emptyset$, then $F_i \cup F_j$ is a connected affinoid itself, by Exercise 3.12. Proceed by induction on m.

Uniqueness. Suppose that $F = \dot{\bigcup}_{i=1}^m F_i = \dot{\bigcup}_{j=1}^n G_j$, where F_i, G_j are connected affinoids. Then $F_i = \dot{\bigcup}_{j=1}^n F_i \cap G_j$. By Exercise 3.12, each $F_m \cap G_j$ is either empty or a connected affinoid. Therefore, by Proposition 3.11, there is (a unique) j such that $F_m = F_m \cap G_j$, that is, $F_m \subseteq G_j$. Wlog $j = n$. By a similar argument there is a unique i' such that $G_j \subseteq F_{i'}$. As the F_i are disjoint, $i' = m$. Therefore $F_m = G_n$. Thus $\dot{\bigcup}_{i=1}^{m-1} F_i = \dot{\bigcup}_{j=1}^{n-1} G_j$. It follows by induction on $\min(m, n)$ that $m = n$, and $F_i = G_i$, for $i = 1, \ldots, m$, up to a permutation. ∎

EXERCISE 3.14: *Assume that K is algebraically closed. Let $f \in K(z)$ be a rational function, and let $\rho \in |K^\times|$. Then $F = \{z \mid |f(z)| \leq \rho\}$ is an affinoid.*

Proof: Write f as $c \prod_{i=1}^s (z - a_i)^{n_i}$, where $a_i \neq a_j$ for $i \neq j$, and $n_i \in \mathbb{Z} \smallsetminus \{0\}$. Let $n = \deg(f) = \sum_i n_i$. Replacing ρ by $\frac{\rho}{|c|}$ we may assume that $c = 1$.

PART A: $s = 1$. In this case

$$F = \{z \mid |z - a_1|^{n_1} \leq \rho\} = \begin{cases} \{z \mid |z - a_1| \leq \rho^{\frac{1}{n_1}}\} & \text{if } n_1 > 0; \\ \{z \mid |z - a_1| \geq \rho^{\frac{1}{n_1}}\} & \text{if } n_1 < 0. \end{cases}$$

This is a closed disk.

PART B: *Reduction.* Assume $s \geq 2$. Let T be an automorphism of $\mathbb{P}$. As T^{-1} maps affinoids onto affinoids, it suffices to show that $F' = \{z \mid |f(T(z))| \leq \rho\}$ is an affinoid.

For instance, if T is $z \mapsto az$, where $a \in K^\times$, then

$$F' = \{z \mid \prod_{i=1}^{s} |az - a_i|^{n_i} \le \rho\} = \{z \mid \prod_{i=1}^{s} |z - \frac{a_i}{a}|^{n_i} \le \frac{\rho}{|a|^n}\}$$

Replacing a_i by $\frac{a_i}{a}$ we may assume that

(i) $\max_{i \neq j} |a_i - a_j| = 1$.

If T is $z \mapsto z + a$, where $a \in K$, then $F' = \{z \mid \prod_{i=1}^{s} |z - a_i'|^{n_i} \le \rho\}$, where $a_i' = a_i - a$. Hence we may replace a_i by a_i'. (Observe that $a_i' - a_j' = a_i - a_j$, so that (i) is preserved.)

Apply this with $a = a_1 + u$, where $u \in K$ such that $|u| = 1$ but $\overline{a_i - a_1} \neq \bar{u}$. We have $|a_i'| \le \max(|a_i - a_1|, |u|) \le 1$, but $a_i' = a_i - a_1 - u$ together with $\bar{a}_i - \bar{a}_1 \neq \bar{u}$ implies that but $|a_i'| \not< 1$, otherwise $\bar{a}_i - \bar{a}_1 = \bar{u}$, a contradiction. Replacing a_i by a_i' we may assume that

(ii) $|a_i| = 1$ for each $i = 1, \ldots, s$.

Part C: *Assume that* $|a_i - a_j| = 1$ *for all* $i \neq j$. We have $F = F_0 \cup \bigcup_{i=1}^{s} F_i$, where

$$F_0 = \{z \mid \bigwedge_{j=1}^{s} |z - a_j| \ge 1 \wedge |f(z)| \le \rho\}, \quad F_i = \{z \mid |z - a_i| < 1 \wedge |f(z)| \le \rho\},\ 1 \le i \le s.$$

Let $z \in F_0$. Then $|z - a_i| = |z - a_j|$ for all $i \neq j$. Indeed, if $|z - a_i| > 1$ for some i, this follows from the above assumption; otherwise $|z - a_i| = 1 = |z - a_j|$. Therefore $F_0 = \{z \mid \bigwedge_{j=1}^{s} |z - a_j| \ge 1 \wedge |z - a_i|^n \le \rho\}$ is an affinoid (an intersection of $s+1$ closed disks, by Part A).

Let $1 \le i \le s$ and let $z \in F_i$. Then $|z - a_i| < 1$. By the above assumption $|z - a_j| = 1$ for all $j \neq i$. Therefore

$$F_i = \{z \mid |z - a_i| < 1 \wedge |z - a_i|^{n_i} \le \rho\} =$$
$$= \begin{cases} \{z \mid |z - a_i| \le \rho^{\frac{1}{n_i}}\} & \text{if } \rho < 1 \text{ and } n_i > 0; \\ \emptyset & \text{if } \rho \le 1 \text{ and } n_i < 0; \\ \{z \mid \bigwedge_{j \neq i} |z - a_j| = 1 \wedge |z - a_i| < 1\} & \text{if } \rho \ge 1 \text{ and } n_i > 0; \\ \{z \mid \bigwedge_{j \neq i} |z - a_j| = 1 \wedge \rho^{\frac{1}{n_i}} \le |z - a_i| < 1\} & \text{if } \rho > 1 \text{ and } n_i < 0. \end{cases}$$

It suffices to show that $F_0 \cup F_i$ is an affinoid. By Part A, F_0 is an affinoid. In the first two cases also F_i is an affinoid (possibly empty). Let $U = \{z \mid \bigwedge_{j=1}^{s} |z - a_j| = 1\}$. In the last two cases $F_i \cup U$ is an affinoid; but now $\rho \geq 1$, and hence $U \subseteq F_0$. Therefore $F_0 \cup F_i = F_0 \cup (U \cup F_i)$ is an affinoid.

PART D: Assume that $|a_1 - a_2| \neq 1$. There is k such that $|a_1 - a_k| = 1$, otherwise $|a_1 - a_k| < 1$ for all $k = 2, \ldots, s$, whence $|a_i - a_j| < 1$ for all $i \neq j$, a contradiction to (i). Wlog there is $2 < t < s$ and $\alpha \in |K^\times|$ such that $\alpha < 1$ and $|a_1 - a_i| < \alpha < 1$ for $i = 1, \ldots, t$ and $\alpha < |a_1 - a_i| = 1$ for $i = t+1, \ldots, s$.

If $|z - a_1| \leq \alpha$, then $|z - a_i| = 1$ for $i = t+1, \ldots, s$. If $|z - a_1| \geq \alpha$, then $|z - a_i| = |z - a_1|$ for $i = 1, \ldots, t$. Therefore $F = F_1 \cup F_2$, where

$$F_1 = \{z \mid \ |z - a_1| \leq \alpha \wedge |f(z)| \leq \rho\} = \{z \mid \ |z - a_1| \leq \alpha \wedge \prod_{i=1}^{t} |z - a_i|^{n_i} \leq \rho\}$$

and

$$\begin{aligned} F_2 &= \{z \mid \ |z - a_1| \geq \alpha \wedge |f(z)| \leq \rho\} \\ &= \{z \mid \ |z - a_1| \geq \alpha \wedge |z - a_1|^{n_1 + \cdots + n_t} \prod_{i=t+1}^{s} |z - a_i|^{n_i} \leq \rho\} \end{aligned}$$

Both F_0 and F_1 are affinoids, by induction on s. ∎

LEMMA 3.15: *Let $F_1, F_2, \ldots, F_r$ be disjoint connected affinoids.*

(a) *If $r \geq 2$, there are disjoint closed disks E_1, E_2 such that $F_1 \subseteq E_1$, $F_2 \subseteq E_2$, $F_3, \ldots, F_r \subseteq E_1 \cup E_2$.*

(b) *Suppose $F_1 = \bigcap_{j=1}^{s} D_j$, where D_j are closed disks with disjoint complements. Then $D_1 \cup F_2 \cup \cdots \cup F_r \neq \mathbb{P}$.*

Proof: (a) By induction on the number m of non-disks among $F_1, \ldots, F_r$. If $m = 0$, that is, $F_1, \ldots, F_r$ are disjoint closed disks, this is Corollary 3.10(c). Suppose $m \geq 1$. Then there is t such that F_t is not a disk, and hence F_t is the complement of the disjoint union of open disks $\dot\cup_{j=1}^{s} C_j$. For each $i \neq t$ we have $F_i \subseteq \dot\cup_{j=1}^{s} C_j$, and hence, by Corollary 3.10(b), $F_i \subseteq C_j$, for some (unique) j.

If $t = 1$, wlog $F_2 \subseteq C_1$. Apply the induction hypothesis to $(C_1^c, F_2, \{F_i \mid i \geq 3,\ F_i \subseteq C_1\})$ to get the required assertion. (In detail: the elements of this sequence are disjoint connected affinoids and the number of non-disks among them is $< m$ (we have replaced at least F_1 by a disk C_1^c). So there are disjoint closed disks E_1, E_2 such that $C_1^c \subseteq E_1$ (and hence $F_1 \subseteq E_1$ and $F_i \subseteq E_1$ if $F_i \not\subseteq C_1$), $F_2 \subseteq E_2$, and $F_i \subseteq E_1 \cup E_2$, whenever $i \geq 3$ and $F_i \subseteq C_1$.)

Similarly if $t = 2$.

If $t \neq 1, 2$, wlog $t = r > 2$ and $F_1 \subseteq C_1$. Apply the induction hypothesis to

$$\begin{cases} (F_1, F_2, \{F_i \mid\ i \geq 3,\ F_i \subseteq C_1\}, C_1^c) & \text{if } F_2 \subseteq C_1 \\ (F_1, \{F_i \mid\ i \geq 3,\ F_i \subseteq C_1\}, C_1^c) & \text{if } F_2 \not\subseteq C_1 \end{cases}$$

to get the required assertion. (In detail: the elements of this sequence are disjoint connected affinoids and the number of non-disks among them is $< m$ (we have replaced at least F_r by a disk C_1^c). So there are disjoint closed disks E_1, E_2 such that $F_1 \subseteq E_1$, each F_i is contained in $E_1 \cup E_2$ —either by assumption or because $F_i \subseteq C_1^c \subseteq E_1 \cup E_2$— and $F_2 \subseteq E_2$ —either by assumption or because $F_2 \subseteq C_1^c \subseteq E_1 \cup E_2$—.)

(b) First assume $r = 2$ and F_2 is a closed disk. Then $\bigcap_{j=1}^s D_j \cap F_2 = \emptyset$. By Corollary 3.10(a) there is j such that $D_j \cap F_2 = \emptyset$. Hence if $j = 1$, we have $D_1 \cup F_2 \neq \mathbb{P}$ by an exercise (the union of two disjoint closed disks is not $\mathbb{P}$). If $j \neq 1$, then D_j^c, D_1^c are disjoint, and hence $F_2 \subseteq D_j^c \subseteq D_1$, whence $F_2 \cup D_1 = D_1 \neq \mathbb{P}$.

By induction on the number m of non-disks among $F_2, \ldots, F_r$. If $m = 0$, that is, $F_2, \ldots, F_r$ are disjoint closed disks, by an exercise $F_i \cup F_j \neq P$ for $i \neq j$ and $F_i \cup D_1 \neq P$ by the preceding special case. Hence $D_1 \cup F_2 \cup \cdots \cup F_r \neq \mathbb{P}$ by Lemma 3.9.

Suppose $m \geq 1$. Then $r \geq 2$ and wlog F_r is not a disk. Hence F_r is the complement of the disjoint union of open disks $\dot\cup_{j=1}^s C_j$. For each $i \neq r$ we have $F_i \subseteq \dot\cup_{j=1}^s C_j$, and hence, by Corollary 3.10(b), $F_i \subseteq C_j$, for some (unique) j. Wlog $F_1 \subseteq C_1$. Apply the induction hypothesis to $(F_1, \{F_i \mid\ i \geq 2,\ F_i \subseteq C_1\}, C_1^c)$ (which produces a larger union) to get the required assertion. ∎

Remark 3.16: There are disjoint connected affinoids F_1, F_2, F_3 for which do not exist disjoint closed disks E_1, E_2 such that $F_1 \subseteq E_1$ and $F_2, F_3 \subseteq E_2$. Indeed, let $F_1 = (C_1 \cup C_2)^c$, where $C_1 = \{z \mid |z| < 1\}$ and $C_2 = \{z \mid |z - 1| < 1\}$, and let $F_i \subseteq C_i$ be a closed disk, containing $0, 1$ respectively. If such E_1, E_2 existed, then $0, 1 \in E_1$ and $\infty \notin E_1$. Hence $E_1 = \{z \mid |z| \leq \rho\}$ for some $\rho \in |K^\times|$ and $\rho \geq 1$. But there is $0, 1 \neq \bar{z} \in \bar{K}$. Lift it to $z \in K^o$; then $z \in E_1$ and $z \in F_1$, a contradiction/ ∎

Lemma 3.17: *Let F be a connected affinoid such that $\infty \notin F$. Then either F is a closed disk or a finite union of sets of the form*

$$C_{r,r'} = \{z \in K \mid \ r < |z - a_0| < r'\},$$
$$C_r = \{z \in K \mid \ |z - a_0| = \cdots = |z - a_n| = r\},$$

where $r, r' \in |K^\times|$, $a_0, \ldots, a_n \in K$ such that $|a_i - a_j| = r$.

Proof: If F is not a closed disk, then it is the intersection of $n+1 \geq 2$ closed disks $D_0, \ldots, D_{n+1}$, such that their complements are disjoint. As $\infty \in F^c = \dot{\bigcup}_{i=0}^{n+1} D_i^c$, wlog $\infty \in D_{n+1}^c$. Thus

$$D_i = \{z \mid \ |z - a_i| \geq \pi_i\}, \quad i = 0, \ldots, n, \qquad \text{and} \qquad D_{n+1} = \{z \mid \ |z - a_{n+1}| \leq \pi_{n+1}\}.$$

Put

$$F_k = \{z \in F \mid \ |z - a_k| \leq |z - a_i|, \ i = 0, \ldots, n\}, \quad k = 0, \ldots, n.$$

Then $F = \bigcup_{k=0}^{n} F_k$. (By an exercise each F_k is a connected affinoid, but we will not use this.) Thus it suffices to present each F_k as a finite union of sets of the form $C_{r,r'}$ and C_r. **Wlog $k = 0$.**

As translations move $C_{r,r'}$ and C_r into sets of the same form, we may assume that $a_0 = 0$. Then $0 = a_0 \in D_0^c \subseteq D_{n+1}$; by Exercise 3.4, wlog $a_{n+1} = 0$. Thus

$$D_0 = \{z \mid \ |z| \geq \pi_0\}, \ D_i = \{z \mid \ |z - a_i| \geq \pi_i\}, \ i = 1, \ldots, n, \qquad D_{n+1} = \{z \mid \ |z| \leq \pi_{n+1}\}$$

and

$$F_0 = \{z \in F \mid \ |z| \leq |z - a_i|, \ i = 1, \ldots, n\}.$$

The disjointness of $D_0^c, \ldots, D_{n+1}^c$ implies, in particular,

$$\pi_0 \leq |a_i| \leq \pi_{n+1}, \quad i = 1, \ldots, n,$$
$$\pi_i \leq |a_i|, \quad i = 1, \ldots, n.$$

(Indeed, $a_i \in D_i^c \subseteq D_0, D_{n+1}$, hence $|a_i| \geq \pi_0$, $|a_i| \leq \pi_{n+1}$. Further, $0 \in D_0^c \subseteq D_i$, hence $|a_i| \geq \pi_i$.)

Let $\pi_0 = r_0 < r_1 < \cdots < r_s = \pi_{n+1}$ be all the distinct numbers in the set $\{\pi_0, |a_1|, \ldots, |a_n|, \pi_{n+1}\}$. Then

$$F_0 = \cup_{t=1}^{s}\{z \in F_0 \mid \ r_{t-1} < |z| < r_t\} \cup \cup_{t=1}^{s}\{z \in F_0 \mid \ |z| = r_t\}.$$

But if $r_{t-1} < |z| < r_t$, then $\pi_0 \leq |z| \leq \pi_{n+1}$, and for every $1 \leq i \leq n$

$$|z - a_i| = \begin{cases} |z| > r_{t-1} \geq |a_i| & \text{if } |a_i| \leq r_{t-1}; \\ |a_i| \geq r_t > |z| & \text{if } |a_i| > r_{t-1}, \text{ and hence } |a_i| \geq r_t. \end{cases}$$

In both cases, $|z - a_i| \geq |z|$ and $|z - a_i| \geq |a_i| \geq \pi_i$. Hence $z \in F_0$. Thus

$$\{z \in F_0 \mid \ r_{t-1} < |z| < r_t\} = \{z \in K \mid \ r_{t-1} < |z| < r_t\} = Cr_{t-1}, r_t.$$

Similarly if $|z| = r_t$, then $\pi_0 \leq |z| \leq \pi_{n+1}$, and for every $1 \leq i \leq n$

$$|z - a_i| = \begin{cases} |z| = r_t > |a_i| & \text{if } |a_i| < r_t; \\ \leq r_t & \text{if } |a_i| = r_t; \\ |a_i| > r_t = |z| & \text{if } |a_i| > r_t. \end{cases}$$

Thus if $|a_i| \neq r_t$, then $|z - a_i| \geq |z| = r_t$ and $|z - a_i| \geq |a_i| \geq \pi_i$. If $|a_i| = r_t$, then $|z - a_i| \geq |z| = r_t, \pi_i \leftrightarrow |z - a_i| = r_t (= |a_i| \geq \pi_i)$. Hence

$$\begin{aligned} \{z \in F_0 \mid \ |z| = r_t\} &= \{z \in K \mid \ |z| = r_t, \pi_0 \leq |z| \leq \pi_{n+1}, \ \bigwedge_{\substack{i=1 \\ |a_i| = r_t}}^{n} \ |z - a_i| \geq r_t, \pi_i\} \\ &= \{z \in K \mid \ |z| = r_t \ \bigwedge_{\substack{i=1 \\ |a_i| = r_t}}^{n} \ |z - a_i| \geq r_t, \pi_i\} \\ &= \{z \in K \mid \ |z| = r_t, \ \bigwedge_{\substack{i=1 \\ |a_i| = r_t}}^{n} \ |z - a_i| = r_t\}, \end{aligned}$$

The last set is of the form C_r. Indeed, if for $1 \leq i < j \leq n$ we have $|a_i| = |a_j| = r_t$, then $|a_i - a_j| \leq r_t$. If $|a_i - a_j| < r_t$, then from $|z - a_i| = r_t$ follows $|z - a_j| = r_t$. Therefore we may throw away the condition $|z - a_j| = r_t$. Thus wlog $|a_i - a_j| = r_t$ for all $i < j$.

∎

4. Holomorphic functions

Let $(K, |\ |)$ be an algebraically closed **complete** non-archimedian valued field. Recall that K^o is its valuation ring and K^{oo} is its maximal ideal.

Let F be a subset of $\mathbb{P} = \mathbb{P}(K)$. For a function $f\colon F \to K$ define the **norm** $||f|| = ||f||_F := \sup_{z\in F} |f(z)| \in K$. Observe that

(1) $||f + g|| \le \max(||f||, ||g||)$;

(2) $||fg|| \le ||f|| \cdot ||g||$;

(3) $||cf|| = |c| \cdot ||f||$, for every $c \in K^\times$.

Let $F \subset \mathbb{P}$ be an affinoid.

Definition 4.1: A function $f\colon F \to K$ is **holomorphic** if for every $\varepsilon \in |K^\times|$ there is a rational function $g \in K(z)$ without poles in F such that $||f - g||_F < \varepsilon$. ∎

We set:

(i) $\mathcal{O}(F)$ = the set of K-holomorphic functions on F.

(ii) $\mathcal{O}^o(F) = \{f \in \mathcal{O}(F) \mid\ ||f|| \le 1\}$;

(iii) $\mathcal{O}^{oo}(F) = \{f \in \mathcal{O}(F) \mid\ ||f|| < 1\}$;

(iv) $\overline{\mathcal{O}(F)} = \mathcal{O}^o(F)/\mathcal{O}^{oo}(F)$.

EXERCISE 4.2: *Let $g \in K(z)$ be without poles in F. Show that $||g||_F < \infty$. Deduce that $||f||_F < \infty$ for every holomorphic function f on F.*

Proof: As K is algebraically closed, g is the product of a constant function, linear functions $z - c$, with $c \in K$, and the inverses of linear functions, all of them without poles in F. Thus we may assume that g is one of them. In particular, g has only one pole in $\mathbb{P}$. As F is the union of connected affinoids, we may assume that F is connected. But then F is the intersection of closed disks, and the single pole of g is not in all of them. Therefore we may assume that F is a disk. In this case the assertion is easy. ∎

LEMMA 4.3:

(a) *$\mathcal{O}(F)$ is complete.*

(b) $\mathcal{O}(F)$ *is a* K*-algebra,* $\mathcal{O}^o(F)$ *is a* K^o*-algebra,* $\mathcal{O}^{oo}(F)$ *is an ideal of it, and* $\overline{\mathcal{O}(F)}$ *is an algebra over* $\bar{K} = K^o/K^{oo}$.

Proof: (a) Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{O}(F)$. Let $z \in F$. Obviously, $\{f_n(z)\}$ is a Cauchy sequence in K. As K is complete, this sequence has a limit, say, $f(z) \in K$. This yields a function $f\colon F \to K$.

Let $\varepsilon > 0$. There is N such that for all $n, m \geq N$ and each $z \in F$ we have $|f_n(z) - f_m(z)| \leq ||f_n - f_m|| < \varepsilon$. In particular, $|f_n(z) - f(z)| \leq \varepsilon$ for all $n \geq N$ and each $z \in F$. Hence $||f_n - f|| \leq \varepsilon$ for all $n \geq N$. Thus $f_n \to f$.

Finally, for each $\varepsilon > 0$ there is f_n such that $||f_n - f|| < \varepsilon$ and there is $g \in K(z)$ without poles in F such that $||f_n - g|| < \varepsilon$. Then $||f - g|| < \varepsilon$. ∎

PROPOSITION 4.4: *Let* $D = \{z \in K \mid |z| \leq 1\}$.

(a) $\mathcal{O}(D) = \{\sum_{n=0}^{\infty} a_n z^n \mid a_n \in K$ *and* $\lim_{n\to\infty} a_n = 0\} =: \mathcal{O}$.

(b) $\mathcal{O}(D)^o = \{\sum_{n=0}^{\infty} a_n z^n \mid a_n \in K^o$ *and* $\lim_{n\to\infty} a_n = 0\}$.

(c) $\mathcal{O}(D)^{oo} = \{\sum_{n=0}^{\infty} a_n z^n \mid a_n \in K^{oo}$ *and* $\lim_{n\to\infty} a_n = 0\}$.

(d) $\overline{\mathcal{O}} = \bar{K}[\bar{z}]$, *the ring of polynomials in one variable over* $\bar{K}$.

(e) *Let* $f, g \in \mathcal{O}$. *Then* $||fg|| = ||f|| \cdot ||g||$.

(f) *If* $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}$, *then* $||\sum_{n=0}^{\infty} a_n z^n||_D = \max |a_n|$. *Moreover, there is* $c \in D$ *such that* $|\sum_{n=0}^{\infty} a_n c^n| = \max |a_n|$.

Proof:

PART A: *First part of (a).* Let us denote the right handed side by $\mathcal{O}$. Its elements are convergent sequences of powers of z, hence $\mathcal{O} \subseteq \mathcal{O}(D)$.

PART B: *Proof of (f).*

If $\sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}$, then clearly $||\sum_{n=0}^{\infty} a_n z^n||_D \leq \max |a_n|$. To show "=", we may assume, by (3), that $\max ||a_n|| = 1$, and we have to show that there is $z \in D$ such that $|f(z)| = 1$. Let $\bar{f} := \sum_{n=0}^{\infty} \bar{a}_n Z^n$. This is a nonzero polynomial over $\bar{K}$. Thus there is $\bar{z} \in \bar{K}$ such that $\bar{f}(\bar{z}) \neq 0$. It is the residue of some $z \in K^o = D$. Then $\overline{f(z)} = \bar{f}(\bar{z}) \neq 0$. This means that $|f(z)| = 1$.

PART C: *$\mathcal{O}$ is complete.* Let $\{\sum_{n=0}^{\infty} a_n^{(i)} z^n\}_{i=1}^{\infty}$ be a Cauchy sequence. By the above formula for the norm $\{a_n^{(i)}\}_{i=1}^{\infty}$ is a Cauchy sequence for each $n \geq 0$. Hence it converges to some $a_n \in K$. It is easy to see that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty} a_n^{(i)} z^n \to \sum_{n=0}^{\infty} a_n z^n$. (Indeed, let $\varepsilon > 0$. There is i such that if $j \geq i$, then $|a_n^{(i)} - a_n^{(j)}| \leq \varepsilon$ for all n; hence $|a_n^{(i)} - a_n| \leq \varepsilon$ for all n. There is also N such that if $n \geq N$ then $|a_n^{(i)}| \leq \varepsilon$. Thus $|a_n| \leq \varepsilon$ for all $n \geq N$.)

PART D: *Second part of (a).* As $\mathcal{O}$ is complete, to show that $\mathcal{O}(D) \subseteq \mathcal{O}$, it suffices to show that every rational function $f \in K(z)$ with no poles in D is in $\mathcal{O}$. As $\mathcal{O}$ is a K-algebra (check!), we may assume that f is either a polynomial over K (whence $f \in \mathcal{O}$) or $f = \frac{1}{z-b}$, where $b \notin D$, that is, $|b| > 1$, whence $\frac{1}{z-b} = \frac{1}{-b}\frac{1}{1-\frac{1}{b}z} = \frac{1}{-b}\sum_{n=0}^{\infty}\frac{1}{b^n}z^n = \sum_{n=0}^{\infty} -\frac{1}{b^{n+1}}z^n \in \mathcal{O}$.

(b),(c) – clear.

(d) Let $\bar{z}$ be a variable over $\bar{K}$. The map $\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} \overline{a_n}\bar{z}^n$ is a well defined homomorphism $\mathcal{O}^o \to \bar{K}[\bar{z}]$. The sequence $0 \to \mathcal{O}^{oo} \to \mathcal{O}^o \to \bar{K}[\bar{z}] \to 0$ is exact. Hence $\mathcal{O}^o/\mathcal{O}^{oo} \cong \bar{K}[\bar{z}]$.

(e) Clearly $||fg|| \leq ||f|| \cdot ||g||$. Wlog $||f|| = ||g|| = 1$, and we have to show that $||fg|| = 1$. That is, $\bar{f}, \bar{g} \neq 0$, and we have to show that $\overline{fg} = \bar{f}\bar{g} \neq 0$. This follows from (d), since $\bar{K}[\bar{z}]$ is an integral domain. ∎

EXERCISE 4.5: *Let φ be an automorphism of $\mathbb{P}$. Let F be an affinoid. Show that $f \mapsto f \circ \varphi$ is an isomorphism $\mathcal{O}(\varphi(F)) \to \mathcal{O}(F)$ of K-algebras that preserves the norm.*

EXERCISE 4.6: *Let $c \in K$ and $\pi \in K^\times$.*

(a) *Let $F = \{z \mid |z - c| \leq |\pi|\}$. Then*

$$\begin{aligned}\mathcal{O}(F) &= \{\sum_{n=0}^{\infty} a_n (z-c)^n \mid a_n \in K \text{ and } \lim_{n\to\infty} a_n \pi^n = 0\} \\ &= \{\sum_{n=0}^{\infty} b_n \left(\frac{z-c}{\pi}\right)^n \mid b_n \in K \text{ and } \lim_{n\to\infty} b_n = 0\}\end{aligned}$$

and $||\sum_{n=0}^{\infty} a_n (z-c)^n||_F = \max |a_n||\pi|^n = \max |b_n|$.

(b) *Let* $F = \{z \mid \ |z-c| \geq |\pi|\}$. *Then*

$$\begin{aligned}\mathcal{O}(F) =& \{\sum_{n=0}^{\infty} a_n (z-c)^{-n} \mid \ a_n \in K \text{ and } \lim_{n\to\infty} a_n \pi^{-n} = 0\} \\ =& \{\sum_{n=0}^{\infty} b_n (\frac{\pi}{z-c})^n \mid \ b_n \in K \text{ and } \lim_{n\to\infty} b_n = 0\}\end{aligned}$$

and $||\sum_{n=0}^{\infty} a_n (z-c)^n||_F = \max|a_n||\pi|^{-n} = \max|b_n|$.

Proof: An application of Exercise 4.5 to Proposition 4.4:

(a) The automorphism $z \mapsto \frac{z-c}{\pi}$ maps F onto the unit disk.

(b) The automorphism $z \mapsto \frac{\pi}{z-c}$ maps F onto the unit disk. ∎

For an affinoid F adopt the following notation: For $c \in F$ let $\mathcal{O}(F)_c = \{f \in \mathcal{O}(F) \mid \ f(c) = 0\}$. Furthermore, let $\mathcal{C}(F)$ be the algebra of constant K-holomorphic functions on F. Clearly $\mathcal{C}(F) \cong K$.

PROPOSITION 4.7 (Decomposition of Mittag-Leffler): *Let* $D_1, \ldots, D_m$ *be* m **disjoint** *open disks. Let* F_i *be the complement of* D_i *and let* $F = \bigcap_{i=1}^m F_i$. *Let* $c \in F$. *Then*

(a) $\mathcal{O}(F) = \mathcal{C}(F) \oplus \oplus_{i=1}^m \mathcal{O}(F_i)_c$.

(b) *Let* $f_0 \in \mathcal{C}(F)$ *and let* $f_i \in \mathcal{O}(F_i)_c$, *for* $i = 1, \ldots, m$. *Then* $||\sum_{i=0}^m f_i||_F = \max||f_i||_{F_i}$. *Moreover, there is* $z \in F$ *such that* $|\sum_{i=0}^m f_i(z)| = \max||f_i||_{F_i}$.

Proof: (b) We may assume that $||f_0||_F \leq \max_{1\leq i\leq m} ||f_i||_{F_i}$, otherwise for every $z \in F$ we have $|\sum_{i=0}^m f_i(z)| = |f_0(z)|$. Using (3) we may normalize the f_i to assume that $\max_{1\leq i\leq m} ||f_i||_{F_i} = 1 \geq ||f_0||_F$, and we have to show that there is $z \in F$ such that $|\sum_{i=0}^m f_i(z)| = 1$.

By Exercise 4.5 we may assume that $c = \infty$. Hence $F_i = \{z \mid \ |z - a_i| \geq |\pi_i|\}$, for each i.

Reordering $F_1, \ldots, F_m$ we may assume that

(i) there is $1 \leq s \leq m$ such that $||f_i||_{F_i} = 1$ for $i = 1, \ldots, s$ and $||f_i||_{F_i} < 1$ for $i = s+1, \ldots, m$;

(ii) $|\pi_1| \geq |\pi_i|$ for $i = 1, \ldots, s$.

By Exercise 4.5 we may assume that $a_1 = 0$ and $|\pi_1| = 1$.

Let $2 \leq i$. As $D_1 \cap D_i = \emptyset$ and hence $a_i \notin D_1$ and $a_1 = 0 \notin D_i$,

(x) $|a_i| \geq |\pi_1|, |\pi_i|$, for $2 \leq i \leq m$.

Therefore, reordering $F_2, \ldots, F_s$ we may assume that

(iii) there is $1 \leq r \leq s$ such that $|a_i| = |\pi_1|$ for $i = 2, \ldots, r$ and $|a_i| > |\pi_1|$ for $i = r+1, \ldots, s$.

Put $I = \{1\} \cup \{2 \leq i \leq m \mid |a_i| = |\pi_1|\}$ and

$$G = \bigcap_{i \in I} \{z \in K \mid |z - a_i| = |\pi_1|\}.$$

We claim that

(iv) $G \subseteq F$;

(v) every $z \in G$ satisfies $|f_i(z)| < 1$ for $i = r+1, \ldots, m$; and

(vi) there is $z \in G$ such that $|\sum_{i=0}^r f_i(z)| = 1$.

It then follows that there is $z \in F$ such that $|\sum_{i=1}^m f_i(z)| = 1$, whence $||\sum_{i=1}^m f_i|| = 1$.

(iv) Let $z \in G$ and let $1 \leq i \leq m$. If $i = 1$, then $|z| = |\pi_1|$, and hence $z \in F_1$. If $i \geq 2$ and $i \in I$, then $|a_i| = |\pi_1|$, so $|z - a_i| = |\pi_1| = |a_i| \geq |\pi_i|$, by (x), whence $z \in F_i$. If $i \notin I$, then $i \geq 2$ and $|a_i| > |\pi_1| = |z|$, hence $|z - a_i| = |a_i| \geq |\pi_i|$, by (x), whence $z \in F_i$. Thus $z \in \bigcap_{i=1}^m F_i = F$.

(v) For $s < i \leq m$ this follows from (i). If $r < i \leq s$ we have $|z| = |\pi_1|$ and $|a_i| > |\pi_1|$, hence $|z - a_i| = |a_i| > |\pi_1|$.

(vi) Let $1 \leq i \leq r$. Recall that $||f_i||_{F_i} = 1$. Hence by Exercise 4.6(b), $f_i = \sum_{n=1}^\infty b_n^{(i)} (\frac{\pi_i}{z - a_i})^n$, where $b_n^{(i)} \in K^o$, not all in K^{oo}, $|\pi_i| \leq 1$, and $|a_i| = 1$. Therefore $\overline{f_i} = \sum_{n=1} \overline{b_n^{(i)}} (\frac{\overline{\pi_i}}{\bar{z} - \overline{a_i}})^n \in K(\bar{z})$. Moreover, $\overline{f_1} \neq 0$ (as $||f_1|| = 1$), and has a pole in $\bar{z} = \overline{a_1} = 0$, whereas $\overline{f_i}$, for $i = 2, \ldots, r$, has a pole in $\overline{a_i} \neq \overline{a_1} = 0$ (or $\overline{f_i} = 0$), and f_0 has no poles. Therefore $\sum_{i=0}^r \bar{f}_i$ has a pole in 0. In particular, $\sum_{i=0}^r \bar{f}_i \neq 0$. Hence there is $\bar{z} \in \bar{K}$ such that $|\sum_{i=0}^r \bar{f}_i(\bar{z})| \neq 0$ and $\bar{z} \neq \bar{a}_i$, for each $i \in I$. Lift $\bar{z}$ to an element $z \in K$ with $|z| = 1$. Then $z \in G$ and $|\sum_{i=0}^r f_i(z)| = 1$.

(a) Again, we may assume that $c = \infty$. We have to show that for every $f \in \mathcal{O}(F)_\infty$ there are unique $f_i \in \mathcal{O}(F_i)_\infty$, $i = 1, \ldots, m$, such that $f = \sum_{i=1}^m f_i$. The uniqueness follows from (b): If $0 = \sum_{i=1}^m f_i$, where $f_i \in \mathcal{O}(F_i)_\infty$, then $0 = \max(||f_1||_{F_1}, \ldots, ||f_m||_{F_m})$, and hence $f_1 = \cdots = f_m = 0$.

To show the existences, it suffices to assume that f is rational. (Why?) As K is algebraically closed, f can be written as a finite sum of the form

$$f = \sum_b \sum_k \frac{a_{k,b}}{(z-b)^k}, \tag{6}$$

where $k \geq 1$, and $b \in K \smallsetminus F$ and $a_{k,b} \in K$. Put

$$f_i = \sum_{b \in D_i} \sum_k \frac{a_{k,b}}{(z-b)^k}. \tag{7}$$

Then $f = \sum_{i=1}^m f_i$ and $f_i \in \mathcal{O}(F_i)_\infty$.

∎

EXAMPLE 4.8: *Let* $0 < r_1 \leq r_2$ *and let* $F = \{z \mid r_1 \leq |z| \leq r_2\}$. *For each* $n \in \mathbb{Z}$ *put* $\tilde{r}_n = \begin{cases} r_1 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ r_2 & \text{if } n > 0 \end{cases}$. *Then*

(a) $\mathcal{O}(F) = \{\sum_{n=-\infty}^{\infty} a_n z^n \mid a_n \in K$ *and* $\lim_{n \to \pm\infty} |a_n| \tilde{r}_n^n = 0\}$.

(b) $|| \sum_{n=-\infty}^{\infty} a_n z^n ||_F = \max |a_n| \tilde{r}_n^n$.

Proof: We have $F = D_1 \cap D_2$, where $D_1 = \{z \in \mathbb{P} \mid r_1 \leq |z|\}$ and $D_2 = \{z \in \mathbb{P} \mid |z| \leq r_2\}$. Let $f \in \mathcal{O}(F)$. Choose $c \in F$. By Mittag-Leffler there are $f_0 \in K$ (a constant function), $f_1 \in \mathcal{O}(D_1)_c$, $f_2 \in \mathcal{O}(D_2)_c$, such that $f = f_0 + \mathrm{res}_F f_1 + res_F f_2$, and $||f||_F = \max(|f_0|, ||f_1||_{D_1}, ||f_2||_{D_2})$.

Choose ρ_1, ρ_2 such that $|\rho_i| = r_i$. By Exercise 4.6(a), $f_2(z) = \alpha_2 + \sum_{n=1}^{\infty} a_n z^n$, where $\lim_{n\to\infty} |a_n| r_2^n = 0$. As $f_2(c) = 0$, we have $\alpha_2 = -\sum_{n=1}^{\infty} a_n c^n$.

Similarly, by Exercise 4.6(b), changing n to $-n$, we have $f_1(z) = \alpha_1 + \sum_{n=-1}^{-\infty} a_n z^n$, where $\lim_{n\to-\infty} |a_n| r_1^n = 0$. As $f_1(c) = 0$, we have $\alpha_1 = -\sum_{n=-1}^{-\infty} a_n c^n$.

Thus $f(z) = f_0 + f_1(z) + f_2(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, where $a_0 = f_0 - \alpha_1 - \alpha_2$ and $\lim_{n\to-\infty} |a_n| r_1^n = 0$ and $\lim_{n\to\infty} |a_n| r_2^n = 0$.

(The a_n as above are unique; this follows from (b).)

(b) Observe that $|\alpha_1| \leq \max_{n<0}(|a_n| \tilde{r}_n^n)$ and $|\alpha_2| \leq \max_{n>0}(|a_n| \tilde{r}_n^n)$. Therefore

$$||f||_F = \max(|f_0|, ||f_1||_{D_1}, ||f_2||_{D_2}) = \max_{n\neq 0}(|f_0|, |\alpha_1|, |\alpha_2|, |a_n| \tilde{r}_n^n) = \max_{n\neq 0}(|f_0|, |a_n| \tilde{r}_n^n).$$

We have to show that this is M, where $M = \max_{n\neq 0}(|f_0 - \alpha_1 - \alpha_2|, |a_n|\tilde{r}_n^n)$. Clearly $M \le ||f||_F$. Also, if $|f_0| \le \max_{n\neq 0}(|a_n|\tilde{r}_n^n)$, then $||f||_F \le M$. If $|f_0| > \max_{n\neq 0}(|a_n|\tilde{r}_n^n)$, then $|f_0 - \alpha_1 - \alpha_2| = |f_0|$, so $M = ||f||_F$. ∎

LEMMA 4.9: *Let $F_1, \ldots, F_r$ be disjoint connected affinoids in $\mathbb{P}$. Put $F = \cup_{i=1}^r F_i$. Then $\mathcal{O}(F) \cong \prod_{i=1}^r \mathcal{O}(F_i)$, via $f \mapsto (\mathrm{res}_{F_1} f, \ldots, \mathrm{res}_{F_r})$.*

Proof: Wlog $r \ge 2$.

The map res: $\mathcal{O}(F) \to \prod_{i=1}^r \mathcal{O}(F_i)$ is clearly injective. Each $(f_1, \ldots, f_r) \in \prod_{i=1}^r \mathcal{O}(F_i)$ is the sum of elements of the form $(0, \ldots, 0, f_k, 0, \ldots, 0)$, where $1 \le k \le r$ and $f_k \in \mathcal{O}(F_k)$. Therefore it suffices to show that the latter element is in the image of res. Wlog $k = 1$.

PART A: *$f_1(z) = 1$ for all $z \in F_1$.* Let $1 \le l \le r$ such that $l \ne 1$. By Lemma 3.15(a) there are two disjoint closed disks D' and D'' such that $F \subseteq D' \cup D''$ and $F_1 \subseteq D'$ and $F_l \subseteq D''$.

Wlog $D' = \{z \mid |z| \le \rho'\}$ and $D'' = \{z \mid |z| \ge \rho''\}$, where $\rho' < 1 < \rho''$. The sequence $g_n(z) = \frac{1}{z^n+1}$ (of rational functions without poles in $D' \cup D''$) converges (uniformly!) to 1 on D' and to 0 on D''. Its restriction to F is a function $f_{1,l} \in \mathcal{O}(F)$ that is 1 on F_1 and 0 on F_l.

Let $f = \prod_{l\neq} f_{1,l}$. Then $f \in \mathcal{O}(F)$, and $\mathrm{res} f = (1, 0, \ldots, 0)$.

PART B: *Arbitrary $f_1 \in \mathcal{O}(F_1)$.* Write F_1 as $\bigcap_{i=1}^s D_i$, where $D_1, \ldots, D_s$ are closed disks such that $\mathbb{P} \smallsetminus F_1 = \cup_{j=1}^s D_j^c$. By Mittag-Leffler-Decomposition, $f_1 = g_0 + g_1 + \cdots + g_s$, where g_0 is constant and $g_l \in \mathcal{O}(F)$ extends to a function $g_l \in \mathcal{O}(D_l)$, for each $1 \le l \le s$. Wlog $f_1 = g_l$ for some l and wlog $l = 1$.

Apply an automorphism of $\mathbb{P}$ to Lemma 3.15(b) to assume that $0 \in D_1$ and $\infty \notin D_l \cup F_2 \cup \cdots \cup F_r$. Then wlog D_1 is the unit disk.

We can write $f_1 \in \mathcal{O}(D_1)$ as $f_1(z) = \sum_{n=1}^\infty a_n z^n$, where $|a_n| \to 0$. For each N the function $f_1^{(N)} = \sum_{n=1}^\infty a_n z^n$ has a pole only in ∞, and hence $f_1^{(N)} \in \mathcal{O}(F)$. By Part A there is $g \in \mathcal{O}(F)$ such that g is 1 on F_1 and 0 on the rest. Then $\{g f_1^{(N)}\}_{N=1}^\infty \subseteq \mathcal{O}(F)$ is a Cauchy sequence. Its limit $f \in \mathcal{O}(F)$ satisfies the required conditions. ∎

LEMMA 4.10: *Let F be a connected affinoid, and let D be a closed disk contained in F. Let $0 \neq f \in \mathcal{O}(F)$. Then $\mathrm{res}_D f \neq 0$.*

Proof: Write F as the intersection of r closed disks $D_1, \ldots, D_r$ such that their complements $D_1^c, \ldots, D_r^c$ are disjoint. Wlog $\infty \in D$ and $0 \notin D$. Thus

$$D_k = \{z \mid |z - a_k| \geq |\pi_k|\}, \text{ for } k = 1, \ldots, r, \quad \text{and} \quad D = \{z \mid |z| \geq |\rho|\}.$$

Wlog $||f||_F = 1$. If $f(\infty) \neq 0$, the assertion is trivial. So assume that $f(\infty) = 0$. By Mittag-Leffler there are unique $f_1 \in \mathcal{O}(D_1), \ldots, f_r \in \mathcal{O}(D_r)$ vanishing at ∞, such that $f = \mathrm{res}_F f_1 + \cdots + \mathrm{res}_F f_r$. As $1 = ||f||_F = \max_k ||f_k||_{D_k}$, we have $||f_k||_{D_k} \leq 1$ for each k, and there is k with $||f_k||_{D_k} = 1$.

PART A: $r = 1$. We may assume that $a_1 = 0$ and $\pi_1 = 1$. Thus $D_1 = \{z \mid |z| \geq 1\}$, and $D = \{z \mid |z| \geq |\rho|\}$, where $|\rho| \geq 1$. Then $f(z) = \sum_{i=0}^{\infty} b_i (\frac{1}{z})^i$, where $\max(|b_i|) = ||f||_F > 0$. Thus not all b_i are 0. Now, $\mathrm{res}_D f(z) = \sum_{i=0}^{\infty} \frac{b_i}{\rho^i} (\frac{\rho}{z})^i$, and $||f||_D = \max(|\frac{b_i}{\rho^i}|)$. Hence $||f||_D > 0$.

Assume, by induction, that $r \geq 2$ and that the assertion is true for less than r disks.

PART B: *Reductions.* Wlog (apply the automorphism $z \mapsto \frac{z}{\pi}$ of $\mathbb{P}$) $\max(|a_k - a_l|) = 1$. For distinct $1 \leq k, l \leq r$ we have $D_k^c \cap D_l^c = \emptyset$, and hence $|a_k - a_l| \geq |\pi_k|, |\pi_l|$. Thus

$$(1) \qquad |\pi_1|, \ldots, |\pi_r| \leq 1.$$

Furthermore, wlog $|\rho|$ is very large, say

$$(2) \qquad |\rho| > 1, |a_k|, \ |\pi_k|, \quad k = 1, \ldots, r.$$

Indeed, let $|\rho'| \geq |\rho|$ and let $D' = \{z \mid |z| \geq |\rho'|\}$. Then $D' \subseteq D \subseteq F$. If $\mathrm{res}_{D'} f \neq 0$, then also $\mathrm{res}_D f \neq 0$.

PART C: *Reduction to $|a_k - a_l| = 1$ and $\pi_k = 1$ for all $k \neq l$.* By Mittag-Leffler there are unique $f_1 \in \mathcal{O}(D_1), \ldots, f_r \in \mathcal{O}(D_r)$ vanishing at ∞, such that $f = \mathrm{res}_F f_1 + \cdots + \mathrm{res}_F f_r$. As $f \neq 0$, not all f_k are 0.

For each $1 \le k \le r$ let $D'_k = \{z \mid |z - a_k| \ge 1\}$. By Part C, $D \subseteq D'_k$. By (1), $D'_k \subseteq D_k$. Some of the disks in the sequence $D'_1, \dots, D'_r$ may coincide (see below). Let $E_1, \dots, E_s$ be the distinct elements of this sequence, and for each $1 \le j \le s$ let $\mathcal{K}(j) = \{k \mid D'_k = E_j\}$.

More precisely, if $|a_k - a_l| < 1$, then $D'_k = D'_l$. If, on the other hand, $|a_k - a_l| = 1$, then the complements of D'_k and D'_l are disjoint, and hence $D'_k \ne D'_l$. As there are k, l such that $|a_k - a_l| = 1$, not all the disks in the sequence $D'_1, \dots, D'_r$ are equal. Thus $2 \le s$ and $\#\mathcal{K}(j) < r$ for each $1 \le j \le s$. Furthermore, the complements of $E_1, \dots, E_s$ are disjoint.

Put $G = \bigcap_{j=1}^{s} E_j$. This is a connected affinoid. We claim that $\mathrm{res}_G f \ne 0$. Indeed, for each $1 \le j \le s$ let $g_j = \sum_{k \in \mathcal{K}(j)} \mathrm{res}_{E_j} f_k \in \mathcal{O}(E_j)$. Then $\mathrm{res}_G f = \sum_{j=1}^{s} \mathrm{res}_G g_j$. Therefore this is the Mittag-Leffler decomposition of $\mathrm{res}_G f$. Hence it suffices to show that there is j such that $g_j \ne 0$.

There is k_0 such that $f_{k_0} \ne 0$. Let j be such that $k_0 \in \mathcal{K}(j)$. Now, $F_j = \bigcap_{k \in \mathcal{K}(j)} D_k$ is the intersection of $\#\mathcal{K}(j) < r$ closed disks with disjoint complements. Put $g'_j = \sum_{k \in \mathcal{K}(j)} \mathrm{res}_{F_j} f_k \in \mathcal{O}(F_j)$. This is the Mittag-Leffler decomposition of g'_j. Therefore, as $f_{k_0} \ne 0$, also $g'_j \ne 0$. But $g_j = \mathrm{res}_{E_j} g'_j$. As $\#\mathcal{K}(j) < r$, by the induction hypothesis we have $g_j \ne 0$. This shows that $\mathrm{res}_G f \ne 0$.

Now, either $s < r$ or or $s = r$. In the first case, by the induction hypothesis (applied to $D \subseteq G = \bigcap_{j=1}^{s} E_j$) $\mathrm{res}_D f \ne 0$. In the second case we may replace F with G (and D_k with D'_k for each k) and thus assume that $|a_k - a_l| = 1$ and $\pi_k = 1$ for all $k \ne l$.

Part D: *Assume that $|a_k - a_l| = 1$ for all $k \ne l$ and $|\pi_i| = 1 \le \rho$ for all i.*

Write f_k as $\sum_{j=1}^{\infty} b_j^{(k)} (\frac{1}{z - a_k})^j$.

Then

(i) $|a_k|, |b_j^{(k)}| \le 1$ for all j and k; in particular, $\overline{a_k}, \overline{b_j^{(k)}} \in \bar{K}$ are defined.

(ii) $\overline{a_1}, \dots, \overline{a_r}$ are distinct;

(iii) There are j and k such that $|b_j^{(k)}| = 1$; that is, not all $\overline{b_j^{(k)}}$ are 0.

Furthermore, $|b_j^{(k)}| \to 0$, for each $1 \le k \le r$. Therefore

(iv) there is m such that $\overline{b_j^{(k)}} = 0$ for all k and all $j \geq m$.

It follows that $\bar{f}(t) = \sum_{k=1}^{r} \sum_{j=1}^{\infty} \overline{b_j^{(k)}} (\frac{1}{t - \overline{a_k}})^j \neq 0$ is a non-trivial rational function over $\bar{K}$. Therefore there is $\bar{c} \neq 0$ in (the algebraic closure of) $\bar{K}$ such that $\bar{f}(\bar{c}) \neq 0$.

Thus there is c in the algebraic closure of K such that $|c| = 1$ and $f(c) \neq 0$. In particular, the restriction of f to $D' = \{z \mid |z| \geq 1\}$ is not trivial. Since $|\rho| \geq 1$, we have $D \subseteq D'$. Hence by Part A also $\mathrm{res}_D f \neq 0$. ∎

5. Factorization

The aim of this section is to prove the following

THEOREM 5.1: *Let F be a connected affinoid in $\mathbb{P}$ such that $\infty \notin F$. Let $0 \neq f(z)$.*

(a) *f has finitely many zeroes in F. Moreover, there are $c_1, \ldots, c_m \in F$ such that $f(z) = g(z) \prod_{i=1}^{m} (z - c_i)$, where $g \in \mathcal{O}(F)$ has no zeroes in F.*

(b) *The following are equivalent:*

  (i) *$f \in \mathcal{O}(F)^{\times}$;*

  (ii) *f has no zeros in F;*

  (iii) *There is $\theta > 0$ such that $|f(z)| > \theta$ for all $z \in F$.*

(b) *The ring $\mathcal{O}(F)$ is a principal ideal domain; its maximal ideals are $(z - c)\mathcal{O}(F)$, where $c \in F$.*

We prove this in several steps:

LEMMA 5.2 (Factorization): *Let F be an affinoid in $\mathbb{P}$. Let $\infty \neq c \in F$ and let $f \in \mathcal{O}(F)$ such that $f(c) = 0$. Then there is a unique $g \in \mathcal{O}(F)$ such that $f(z) = (z - c)g(z)$ on $F \smallsetminus \{\infty\}$.*

Proof: To show the uniqueness, it suffices to prove that if $0 \neq g \in \mathcal{O}(F)$, then $(z - c) \cdot g(z) \neq 0$. There is $a \in F$ such that $g(a) \neq 0$. As g is continuous (it is the limit of rational functions, which are continuous on F), we may assume that $a \neq c, \infty$. (There is $0 \neq d \in K$ with $|d|$ sufficiently small; then $g(a + d) \neq 0$ and $a + d \neq c, \infty$.) Then $(a - c) \cdot g(a) \neq 0$.

PART A: *Reduction to a connected affinoid.* Write F as the disjoint union of connected affinoids $F_1, \ldots, F_r$. Wlog $c \in F_1$. For $2 \leq i \leq r$ we have $c \notin F_i$ and hence $(z - c)^{-1} \in \mathcal{O}(F_i)$, whence $g_i := (z - c)^{-1} f_i \in \mathcal{O}(F_i)$ satisfies $\mathrm{res}_{F_i} f = (z - c) g_i(z)$. Suppose there is $g_1 \in \mathcal{O}(F_1)$ such that $\mathrm{res}_{F_1} f = (z - c) g_1(z)$. Then by Lemma 4.9 there is a unique $g \in \mathcal{O}(F)$ such that $\mathrm{res}_{F_i} g = g_i$. Clearly $f(z) = (z - c)g(z)$.

PART B: *Reduction to a closed disk.* Write F as the intersection of closed disks $\bigcap_{j=1}^{s} D_j$. By Mittag-Leffler, $f = \sum f_i$, where $f_i \in \mathcal{O}(F)_c$ extends to a holomorphic function on D_i. It suffices to prove the assertion for each f_i. Therefore wlog f extends to a holomorphic function on D_i. So wlog $F = D_i$.

PART C: *f is the restriction of an automorphism of $\mathbb{P}$ to F.* Say, $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, where $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{Gl}_2(K)$. Since $f(c) = 0$, we have $\alpha c + \beta = 0$. Thus $f(z) = \frac{\alpha(z-c)}{\gamma z+\delta} = (z-c)\cdot\frac{\alpha}{\gamma z+\delta}$.

PART D: *F is the unit disk.* Suppose that $F = U := \{z \mid |z| \le 1\}$ and $c = 0$. By Proposition 4.4, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_n \to 0$. As $f(c) = 0$, we have $a_0 = 0$. Moreover, $h(z) := \sum_{n=1}^{\infty} a_n z^{n-1} \in \mathcal{O}(U)$. Therefore $f(z) = zh(z)$.

PART D: *The general case.* There is an automorphism φ of $\mathbb{P}$ such that $\varphi(F) = U$ and $\varphi(c) = 0$. There is $f_1 \in \mathcal{O}(U)_0$ such that $f(z) = f_1(\varphi(z))$. By Part D, $f_1 = z\cdot g_1$, where $g_1 \in \mathcal{O}(U)$. Thus $f = f_1(\varphi(z)) = \varphi(z)\cdot g_1(\varphi(z))$, and $g_1(\varphi(z)) \in \mathcal{O}(F)$. By Part C, $\varphi(z) = (z-c)g_2(z)$ for some $g_2 \in \mathcal{O}(F)$. So $f = (z-c)g_1(z)g_2(z)$. ∎

The main tool is a lemma we already proved:

LEMMA 3.17: *Let F be a connected affinoid such that $\infty \notin F$. Then either F is a closed disk or a finite union of sets of the form*

$$C_{r,r'} = \{z \in K \mid r < |z-a_0| < r'\},$$
$$C_r = \{z \in K \mid |z-a_0| = \cdots = |z-a_n| = r\},$$

where $r, r' \in |K^\times|$, $a_0, \dots, a_n \in K$ such that $|a_i - a_j| = r$.

LEMMA 5.3: *Let $D = \{z \mid |z| \le 1\}$ be a closed disk. Let $0 \ne f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(D)$, and let $m = \max(n \mid |a_n| = ||f||_D)$.*

(a) *If $m \ge 1$, then f has a zero in D; more precisely –*

(b) *There are $c_1, \dots, c_m \in D$ and $g \in \mathcal{O}(D)$ with no zeros in D such that $f(z) = g(z)\prod_{i=1}^{m}(z-c_i)$.*

(c) *The following are equivalent:*

(i) *$f \in \mathcal{O}(D)^\times$;*

(ii) *f has no zeros in D;*

(iii) *$f = c(1+s)$, where $c \in K^\times$ and $s \in \mathcal{O}^{oo}(D)$ (that is, $m = 0$);*

(iv) $|f(z)| = ||f||_D$ *for each* $z \in D$.

Proof: Wlog $a_m = 1$. Hence $f \in \mathcal{O}^o(D)$.

(a) For $k \geq m$ let $f_k(z) = \sum_{n=0}^{k} a_n z^n$. Then

$$\bar{f}(z) = \bar{f}_k(z) = z^m + \bar{a}_{m-1} z^{m-1} + \cdots + \bar{a}_0.$$

Write f_k as

$$f_k(z) = \lambda' \prod_{i=1}^{s} (z - c_{ik}) \prod_{j=1}^{t} (z - d_{jk}),$$

where $|c_{ik}| \leq 1$ and $|d_{jk}| > 1$. Put $\lambda = \lambda'(-d_{1k}) \cdots (-d_{tk})$ (and λ' is the leading coefficient of f_k, which is not necessarily a_k, because the latter could be 0). Then we can write the preceding equation as

$$f_k(z) = \lambda \prod_{i=1}^{s} (z - c_{ik}) \prod_{j=1}^{t} (1 - d_{jk}^{-1} z).$$

Comparing norms on both sides we get $|\lambda| = 1$. Taking bar on both sides we see that

$$z^m + \bar{a}_{m-1} z^{m-1} + \cdots + \bar{a}_0 = \bar{f}_k(z) = \bar{\lambda} \prod_{i=1}^{s} (z - \bar{c}_{ik}).$$

Hence $\bar{\lambda} = 1$ and $m = s$.

For each k put $Z_k = \{c_{1k}, \ldots, c_{mk}\}$. Then $\# Z_k \leq m$.

Fix k and let $c_{k+1} \in Z_{k+1}$. Then

$$\prod_{i=1}^{m} |c_{k+1} - c_{ik}| = |f_k(c_{k+1})| = |f_k(c_{k+1}) - f_{k+1}(c_{k+1})| \leq ||f_k - f_{k+1}||.$$

Hence there is $c_k = c_{ik} \in Z_k$ such that $|c_{k+1} - c_k| \leq ||f_k - f_{k+1}||^{\frac{1}{m}}$. Choose this $c_k \in Z_k$; this defines a map : $Z_{k+1} \to Z_k$ by $c_{k+1} \mapsto c_k$. Now, $\varprojlim Z_n \neq \emptyset$, so there is a sequence $\{c_k\}_k \subseteq D$ such that $f_k(c_k) = 0$ and $|c_{k+1} - c_k| \leq ||f_k - f_{k+1}||^{\frac{1}{m}}$ for every k. Thus $\{c_k\}_k$ is a Cauchy sequence. Hence its limit $c \in D$ is a zero of f.

(c) (i) $\Rightarrow$ (ii) - clear.

(ii) $\Rightarrow$ (iii): If f has no zeros in D, then by (a), $m = 0$. Hence $f = a_0 + s = 1 + s$, where $s = \sum_{n=1}^{\infty} a_n z_n$ satisfies $||s||_D < 1$.

(iii) $\Rightarrow$ (iv): Let $z \in D$. Then $|s(z)| \le ||s||_D < 1$, hence $|1 + s(z)| = 1$.

(iv) $\Rightarrow$ (i): Write f as the limit of a sequence of rational functions f_k without poles in D (for instance, the partial sums $f_k(z) = \sum_{n=0}^{k} a_n z^n$). We may assume that $||f_k - f|| < 1$ for each k, and hence f_k has no zeros in D; in fact, for every $z \in D$ we have $|f_k(z) - f(z)| < 1$, but $|f(z)| = 1$, whence $|f_k(z)| = 1$. Thus $\frac{1}{f_k}$ is a sequence of rational functions with no poles in D. Check that $\frac{1}{f_k} \to \frac{1}{f}$.

(b) By induction on m. Assume first that $m = 0$. Then $||1 - f|| < 1$, hence by (c), $f \in \mathcal{O}(D)^\times$.

Assume that $m \ge 1$. By (a), f has a zero $c \in D$. Then f can be written as $f(z) = \sum_{n=0}^{\infty} b_n (z-c)^n$, where $|b_n| \le 1$. As $f(c) = 0$, we have $b_0 = 0$. Thus $f(z) = (z-c)h(z)$, where $h(z) = \sum_{n=1}^{\infty} b_n (z-c)^n \in \mathcal{O}^o(D)$. Write $h(z)$ as $h(z) = \sum_{n=0}^{\infty} a'_n z^n$, and put $m' = \max(n \mid |a'_n| = 1)$. From $\bar{f}(z) = (z - \bar{c})\bar{h}(z)$ we see that $m' = m - 1$. By the induction hypothesis $h(z) = g(z) \prod_{i=1}^{m-1} (z - c_i)$, where $c_1, \ldots, c_{m-1} \in K$ and $g \in \mathcal{O}(D)$ has no zeros in D. Put $c = c_m$. Then $f(z) = g(z) \prod_{i=1}^{m} (z - c_i) g(z)$. ∎

Remark 5.4: *Let C be a subset of an affinoid F, and let $f, q \in \mathcal{O}(F)$ such that $||f - q||_C < ||f||_C$. Then*

(i) $||f||_C = ||q||_C$.

(ii) *If $z \in C$ and $|f(z)| = ||f||_C$, then $|f(z)| = |q(z)|$.*

Proof: Let $C' = \{z \in C \mid |f(z)| > ||f - q||_C\}$. As $\sup_{z \in C} |f(z)| = ||f||_C > ||f - q||_C$, the set C' is not empty. Hence C' contains all $z \in C$ with $|f(z)| = ||f||_C$. For $z \in C'$ we have $|f(z)| > |f(z) - q(z)|$, and hence $|f(z)| = |q(z)|$. This proves (ii). Also $||f||_C = \sup_{z \in C'} |f(z)| = \sup_{z \in C'} |q(z)| = ||q||_C$. ∎

Lemma 5.5: *Let $r \in |K^\times|$, and let $b_1, \ldots, b_N \in K$ such that $|b_1| = \cdots = |b_N| = r$. Put*

$$\begin{aligned} C &= \{z \in K \mid |z| = r, |z - b_\nu| = r,\ 1 \le \nu \le N\} \\ &= \{z \in K \mid |z| = r\} \smallsetminus \bigcup_{\nu=1}^{N} \{z \in K \mid |z - b_\nu| < r\}, \end{aligned}$$

Let q be a rational function with no poles in C. Let $\{d_1, \ldots, d_n\} \subseteq C$ contain all the zeroes of q in C. Then

(a) $|q(z)| = ||q||_C$, *if* $z \in C$ *and* $|z - d_i| \geq r$, *for* $i = 1, \ldots, n$*;*

(b) $||q||_{\{z \mid \ |z-d_i|<r\}} = ||q||_C$, *for* $i = 1, \ldots, n$.

Proof: It suffices to show that there are $k \in \mathbb{N}$ and $p, \rho \in |K^\times|$ such that $p < r$ and:

(i) if $z \in C$ and $|z - d_i| \geq r$, for $i = 1, \ldots, n$, then $|q(z)| = \rho$;

(ii) $|q(z)| \leq \rho$ for all $z \in C$;

(iii) For each $1 \leq i \leq n$, if $z \in C$ and $p < |z - d_i| < r$, then $|\frac{z-d_i}{r}|^k \rho \leq |q(z)| \leq \rho$.

Observe that if this assertion is true for two rational functions q_1, q_2, then it also holds for their product $q_1 q_2$. Thus we may assume that either $q(z) = z - a$, where $a \in K$ or $q(z) = \frac{1}{z-a}$, where $a \notin C$.

Futhermore, we may assume that $\{d_1, \ldots, d_n\}$ is the set of all zeroes of q in C. (We could have assumed this from the beginning, but this "more general" setup was necessary for the preceding reduction from q to its factors: The set of zeroes of $q_1 q_2$ may properly contain the set of zeroes of q_1.) More precisely, let k, p, ρ such that (i), (ii) and (iii) hold, and let $d_{n+1}, \ldots, d_{n'} \in C$. Let

$$p' = \max(p, |d_i - d_j| \mid 1 \leq i, j \leq n', \ |d_i - d_j| < r).$$

Then the corresponding assertions, say (i′), (ii′), and (iii′), hold for $d_1, \ldots, d_{n+1}, \ldots, d_{n'}$ with k, p', ρ. Indeed, (i′) is weaker than (i), and (ii′) does not depend on $d_1, \ldots, d_{n'}$. Fix $1 \leq j \leq n'$ and $z \in C$ such that $p' < |z - d_j| < r$. If there is no $1 \leq i \leq n$ such that $|z - d_i| < r$, then $|q(z)| = \rho$ by (i). If there is $1 \leq i \leq n$ such that $|z - d_i| < r$, then $|d_i - d_j| < r$, and hence $|d_i - d_j| \leq p'$, by the definition of p', whence $|z - d_j| = |(z - d_i) + (d_i - d_j)| = |z - d_j|$. As $p \leq p'$, condition (iii′) for j follows from (iii) for i.

Let $q(z) = z - a$. Let $a \in K$, and let $z \in C$. Recall that $|z| = r$.

(1) If $|a| > r$, then $|z - a| = |a|$. (In this case $n = 0$.)

(2) If $|a| < r$, then $|z - a| = r$. (In this case $n = 0$.)

(3) If $|a| = r$, but $a \notin C$, then there is ν such that $|a - b_\nu| < r$. As $|z - b_\nu| = r$, we have $|z - a| = |(z - b_\nu) - (a - b_\nu)| = r$. (In this case $n = 0$.)

(4) If $|a| = r$ and $a \in C$, then $n = 1$ and $a = d_1$, because a is the only zero of q. If $|z - d_1| \geq r$, then $|z - a| = |z - d_1| = r$ (because $|z| = |d_1| = r$). If $|z - d_1| < r$, then $|z - a| = \frac{|z-d_1|}{r} r$.

In case (1) put $\rho = |a|$, otherwise $\rho = r$. Let $k = 1$, and let p be arbitrary. Then (i),(ii),(iii) hold.

If $q(z) = \frac{1}{z-a}$, where $a \notin C$, then the assertion follows from cases (1),(2),(3) above. ∎

LEMMA 5.6: *Let F be an affinoid that contains $D = \{z \in K \mid |z| < 1\}$. Let $0 \neq f \in \mathcal{O}(F)$. Then f has finitely many zeroes in D. Furthermore, $f(z) = g(z)\prod_{i=1}^{m}(z - c_i)$, where $c_1, \ldots, c_m$ are the zeroes of f in D, and $g \in \mathcal{O}(F)$ has no zeroes in D. Moreover, $||g||_D = |g(z)|$ for all $z \in D$.*

Proof: In this proof let D_r denote the closed disk of radius r around 0, and U_r the circle of radius r around 0. Put $\rho = ||f||_D$ ($\leq ||f||_F$). Then $\rho > 0$ by Lemma 4.10. Let $q \in \mathcal{O}(F)$ be a rational function such that $||f - q||_D < \frac{\rho}{2}$. (E.g., $||f - q||_F < \frac{\rho}{2}$.)

If $0 < r_0 < 1$ is sufficiently large, $||f||_{D_{r_0}} \geq \frac{\rho}{2}$; this, together with $||f - q||_{D_{r_0}} < \frac{\rho}{2}$, gives $||q||_{D_{r_0}} \geq \frac{\rho}{2}$ (there is $z \in D_{r_0}$ such that $|f(z)| \geq \frac{\rho}{2}$; of course, $|f(z) - q(z)| < \frac{\rho}{2}$, so $|q(z)| \geq \frac{\rho}{2}$). In particular, $q \neq 0$ has only finitely many zeroes. Provided that r_0 is sufficiently large, we may assume that $q(z)$ has no zeroes in $\{z \in K \mid r_0 < |z| < 1\}$.

Let $r_0 < r < 1$, and let $z \in D$ such that $|z| = r$. We have

$$|f(z) - q(z)| \leq ||f - q||_D < \frac{\rho}{2} \leq ||q||_{D_{r_0}} \leq ||q||_{D_r}.$$

But $||q||_{D_r} = ||q||_{U_r}$, and, by Lemma 5.5 or Proposition 4.4, $||q||_{U_r} = |q(z)|$. Thus $|f(z) - q(z)| < |q(z)|$, and hence $|f(z)| = |q(z)| > 0$.

In particular, all the zeroes of f in D are in D_{r_0}. By Lemma 5.3 there are $c_1, \ldots, c_m \in D_{r_0}$ and $g' \in \mathcal{O}(D_{r_0})$ with no zeroes such that $\mathrm{res}_{D_{r_0}} f(z) = g'(z)\prod_{i=1}^{m}(z - c_i)$. (Observe that this g' is unique.) By the Factorization Lemma and by induction on i we can write $f(z) = g(z)\prod_{i=1}^{m}(z - c_i)$, where $g \in \mathcal{O}(F)$. By the uniqueness of g' we have $\mathrm{res}_{D_{r_0}} g(z) = g'(z)$. Thus g has no zeroes in D_{r_0}, and hence also in D (by the first statement of this paragraph).

Let $z \in D$. Let $|z| < r < 1$. By Lemma 5.3, $|g(z)| = ||g||_{D_r}$. Hence $|g(z)| = \lim_{r \to 1^-} ||g||_{D_r} = ||g||_D$. ∎

LEMMA 5.7: *Let C be as in Lemma 5.5, and let F be an affinoid that contains C. Let $0 \neq f \in \mathcal{O}(F)$.*

(i) *f has finitely many zeroes in C. More precisely, $f(z) = g(z)\prod_{i=1}^m (z - c_i)$, where $c_1, \ldots, c_m$ are the zeroes of f, and $g \in \mathcal{O}(F)$ has no zeroes in C.*

(ii) *If f has no zeroes in C, then $|f(z)| = ||f||_C$ for all $z \in C$.*

Proof: Since C contains a closed disk, by Lemma 4.10, $||f||_C > 0$. Let $q \in \mathcal{O}(F)$ be a rational function such that $||f - q||_C < ||f||_C$. Then $q \neq 0$. By Remark 5.4, $||q||_C = ||f||_C$. Let $d_1, \ldots, d_n$ be the zeroes of q in C. Put

$$D_i = \{z \in C \mid |z - d_i| < r\},\ 1 \leq i \leq n, \quad \text{and} \quad G = C \smallsetminus \bigcup_{i=1}^n D_i.$$

By Lemma 5.5, $|q(z)| = ||q||_C$ for every $z \in G$.

It follows that for every $z \in G$ we have $|f(z) - q(z)| \leq ||f - q||_C < ||f||_C = ||q||_C = |q(z)|$, and hence $|f(z)| = |q(z)| = ||q||_C$. In particular, $f(z)$ has no zeroes in G. Thus all the zeroes of f are in the open disks $D_1, \ldots, D_n$. By Lemma 5.6 their number is finite, and we get the required factorization.

(ii) Let

$$\rho = ||f||_C = ||q||_C = ||q||_{D_i},\ \text{for } i = 1, \ldots, n$$

(the equalities follow from Remark 5.4 and Lemma 5.5, respectively). It suffices to show that $|f(z)| = \rho$ for every $z \in C$. For $z \in G$ this is written above. For $z \in D_i$, by Lemma 5.3, (present D_i as the increasing union of closed disks) $|f(z)| = ||f||_{D_i}$. As

$$||f - q||_{D_i} \leq ||f - q||_C < ||f||_C = \rho = ||q||_{D_i},$$

by Remark 5.4, $||f||_{D_i} = ||q||_{D_i}$. Thus $|f(z)| = ||q||_{D_i} = \rho$. ∎

LEMMA 5.8: *Let $r_1, r_2 \in |K^\times|$, where $r_1 < r_2$. Put*

$$C = \{z \in K \mid r_1 < |z| < r_2\}.$$

Let F be an affinoid that contains C.

(i) *Let* $f \in \mathcal{O}(F)$. *If* $f \neq 0$, *then* f *has a finite number of zeroes in* C. *Furthermore,* $f(z) = g(z)\prod_{i=1}^{m}(z - c_i)$, *where* $c_1, \ldots, c_m$ *are zeroes of* f *in* C, *and* $g \in \mathcal{O}(F)$ *has no zeroes in* C.

(ii) *If* $g \in \mathcal{O}(F)$ *has no zeroes in* C, *there is* $\theta > 0$ *such that* $|g(z)| > \theta$ *for all* $z \in C$.

Proof: For each $r_1 < r < r_2$ let $U_r = \{z \in K \mid |z| = r\}$. Put $\theta = \inf\{||f||_{U_r} \mid r_1 < r < r_2\}$. We claim that $\theta > 0$.

Indeed, for all $r_1 < r_1' \leq r_2' < r_2$ let $F' = \{z \in K \mid r_1' \leq |z| \leq r_2'\}$. By Example 4.8, there are $a_n \in K$ such that

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z_n,$$

where $|a_n|(r_2')^n \to 0$ as $n \to \infty$ and $|a_n|(r_1')^n \to 0$ as $n \to -\infty$. By Example 4.8(b), these a_n are unique. This implies that a_n do not depend on r_1', r_2'. As $f \neq 0$, there is $k \in \mathbb{Z}$ such that $a_k \neq 0$.

If $r_1 < r_1' = r = r_2' < r_2$, then $F' = U_r$. By Example 4.8, $||f||_{U_r} = \max_n |a_n| r^n$. Hence $||f||_{U_r} \geq |a_k| r^k \geq |a_k| \cdot \min(r_1^k, r_2^k)$. It follows that $\theta > 0$.

Let $q \in \mathcal{O}(F)$ be a rational function such that $||f - q||_F < \theta$. Then $q \neq 0$, and hence q has only finitely many zeroes in C. Let $r_1 < r < r_2$ such that $r \neq |d|$ for each zero $d \in C$ of q, and let $z \in U_r$. Then q has no zero in U_r, and hence by Lemma 5.5, $|q(z)| = ||q||_{U_r}$. Furthermore, $||f - q||_{U_r} \leq ||f - q||_F < \theta \leq ||f||_{U_r}$. Hence by Remark 5.4, $||f||_{U_r} = ||q||_{U_r}$. Thus for every $z \in U_r$

$$|f(z) - q(z)| < \theta \leq ||f||_{U_r} = ||q||_{U_r} = |q(z)|,$$

and hence $|f(z)| = |q(z)| = ||q||_{U_r} = ||f||_{U_r}$.

Therefore $|f(z)| \geq \theta$ for all $z \in C$ except for finitely many U_r's on which f has zeroes. In particular, this prove (ii). Now apply Lemma 5.7 (to each U_r instead of C there). ∎

Proof of Theorem 5.1: (a) By Lemma 3.17, F is the union of certain sets $C_1, \ldots, C_n$. By induction, $f = f_0 \prod_{i=1}^{k}(z - c_i)$, where $c_1, \ldots, c_k \in \bigcup_{i=1}^{n-1} C_i$ and $f_0 \in \mathcal{O}(F)$ has no zeroes in $\bigcup_{i=1}^{n-1} C_i$. By Lemmas 5.3, 5.5, 5.7, $f_0 = g\prod_{i=k+1}^{m}(z - c_i)$, where $c_{k+1}, \ldots, c_m \in C_n$ and $g \in \mathcal{O}(F)$ has no zeroes in C_n.

(b) Implication (iii) $\Rightarrow$ (ii) is trivial. By the preceding lemmas, (ii) $\Rightarrow$ (iii). To deduce (iii) $\Rightarrow$ (i), approximate f by rational functions with no zeroes on F, so that their inverses are rational functions on F; they converge to f^{-1}.

(c) First notice that F is an integral domain: Let $f, g \in \mathcal{O}(F) \smallsetminus \{0\}$. By (i) they have only finitely many zeroes in F. Since F is an infinite set, there is $c \in F$ such that $f(c), g(c) \neq 0$. Hence $fg \neq 0$. (One could also use Lemma 4.10, which proves that $\mathcal{O}(F) \subseteq \mathcal{O}(D)$ for some closed disk D. As $\mathcal{O}(D)$ is an integral domain, so is $\mathcal{O}(F)$.)

Consider the obvious homomorphism (actually, an embedding) $K[z] \to \mathcal{O}(F)$. Let $J \leq \mathcal{O}(F)$ be an ideal. Let $\{f_i\}_{i \in I}$ be a set of its generators. By (i) and (ii) each f_i is, up to an element of $\mathcal{O}(F)^\times$, a polynomial in z. Thus we may assume that $f_i \in K[z]$. Let J_0 be the ideal of $K[z]$ generated by the f_i; then $J = J_0\mathcal{O}(F)$. As $K[z]$ is a PID, the ideal J_0 is generated by some $f \in K[z]$. Hence $J = f\mathcal{O}(F)$. ∎

6. Affinoid algebras

In this section let $(k, |\ |)$ be a complete non-archimedean valued field. Let K be the completion of the algebraic closure of k. (Then K is algebraically closed.)

Definition 6.1: Formal power series. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The elements of $\mathbb{N}_0^n$ are n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$. For an n-tuple of indeterminates $z = (z_1, \ldots, z_n)$ and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. (Thus $z^\alpha z^\beta = z^{\alpha+\beta}$.)

Let R be a commutative ring with 1. Then

$$R[[z_1, \ldots, z_n]] = \{\sum_\alpha a_\alpha z^\alpha \mid\ a_\alpha \in R\}$$

is an R-algebra, **the ring of formal power series in $z_1, \ldots, z_n$ over R.** ∎

Lemma 6.2: *Let R be a commutative ring with 1.*

(a) $R[[z_1, \ldots, z_n]] = R[[z_1, \ldots, z_{n-1}]][[z_n]]$.

(b) *If R is an integral domain, then so is $R[[z_1, \ldots, z_n]]$.*

Proof: (b) Suppose $f = \sum_\alpha a_\alpha z^\alpha, g = \sum_\beta b_\beta z^\beta \neq 0$. Choose smallest α, β, in the lexicographical order on $\mathbb{N}_0^n$, such that $a_\alpha, b_\beta \neq 0$. Then the coefficient of $z^{\alpha+\beta}$ in fg is $a_\alpha b_\beta \neq 0$.) ∎

Assume that $(R, ||\ ||)$ is a normed Banach $(k, |\ |)$-algebra. Then

$$R^o = \{r \in R \mid\ ||r|| \leq 1\}$$

is a subring of R (in fact, a k^o-algebra) and

$$R^{oo} = \{r \in R \mid\ ||r|| < 1\}$$

an ideal in R^o. Let $\bar{R} = R^o/R^{oo}$. This is an $\bar{k}$-algebra.

Definition 6.3: Standard affinoid algebra. For $\alpha \in \mathbb{N}_0^n$ put $|\alpha| = \max_i(\alpha_i)$. (This has got nothing to do with the absolute value on k.) Put

$$T_n(R) = R\langle z_1, \ldots, z_n\rangle = \{\sum_\alpha a_\alpha z^\alpha \mid\ a_\alpha \in R,\ \lim_{|\alpha|\to\infty} a_\alpha = 0\}.$$

This is a subalgebra of $R[[z_1,\ldots,z_n]]$. Put

$$||\sum_\alpha a_\alpha z^\alpha|| = \max_\alpha ||a_\alpha||.$$

This is a norm (of an algebra over k):

(a) $||f|| = 0$ if and only if $f = 0$.

(b) $||f+g|| \le ||f|| + ||g||$. In fact, $||f+g|| \le \max(||f||, ||g||)$.

(c) $||cf|| = |c|||f||$, for $c \in k$ and $f \in T_n$.

(d) $||fg|| \le ||f|| \cdot ||g||$.

It follows that

$$T_n^o = \{\sum_\alpha a_\alpha z^\alpha \mid\ a_\alpha \in R^o,\ \lim_{|\alpha|\to\infty} a_\alpha = 0\}.$$

is a subring of T_n and

$$T_n^{oo} = \{\sum_\alpha a_\alpha z^\alpha \mid\ a_\alpha \in R^{oo},\ \lim_{|\alpha|\to\infty} a_\alpha = 0\}.$$

is an ideal in T_n^o.

Remark 6.4: We have $T_n^o/T_n^{oo} \cong \bar{R}[\bar{z}_1,\ldots,\bar{z}_n]$, the ring of polynomials in n variables. Indeed, the map $T_n^{(0)} \to \bar{R}[\bar{z}_1,\ldots,\bar{z}_n]$ given by $\sum_\alpha a_\alpha z^\alpha \mapsto \sum_\alpha \overline{a_\alpha}\bar{z}^\alpha$ is well defined and its kernel is precisely T_n^{oo}. ∎

EXERCISE 6.5: *Let R be a Banach algebra over k.*

(a) *T_n is complete, that is, a Banach algebra.*

(b) *$T_n(R) = T_{n-1}(R)\langle z_n\rangle$ (and the norm on $T_n(R)$ is the norm coming from the right handed side). (This is the main reason that we consider a general ring R instead of a complete field k.)*

PROPOSITION 6.6: *Let $R = k$ be a field. Then $\bar{R} = \bar{k}$ is the residue field.*

(a) *$||fg|| = ||f|| \cdot ||g||$ for all $f, g \in T_n$.*

(b) *T_n is an integral domain.*

(c) *$f = \sum a_\alpha z^\alpha$ of T_n is invertible if and only if $|a_0| > |a_\alpha|$ for each $\alpha \neq 0$. (Here $0 = (0,\ldots,0) \in \mathbb{N}_0^n$.)*

Proof: (a) We may assume that $f, g \neq 0$. Multiplying them by suitable elements of k we may assume that $||f|| = ||g|| = 1$. In particular their images $\bar{f}, \bar{g}$ in $\bar{k}[\bar{z}_1,\ldots,\bar{z}_n]$ are not 0. As $\bar{k}[\bar{z}_1, \ldots, \bar{z}_n]$ is an integral domain also the image $\bar{f}\bar{g}$ of fg is not 0, that is, $||fg|| = 1$.

(b) If $f, g \neq 0$, then $||f||, ||g|| \neq 0$, and hence $||fg|| = ||f \cdot ||g|| \neq 0$, whence $fg \neq 0$.

(c) Suppose that $|a_0| > |a_\alpha|$ for all $\alpha \neq 0$. Dividing by a_0 we may assume that $a_0 = 1$. Then f may be written as $f = 1 - h$, where $||h|| < 1$. It is easy to see that $g = \sum_{n=0}^{\infty} h^n \in T_n$ satisfies $fg = 1$. Hence f is invertible.

Conversely, suppose that f is invertible. Then $||f|| \neq 0$. Dividing by $||f||$ we may assume that $||f|| = 1$. In particular, $f \in T_n^o$. Its residue $\bar{f} = \sum \bar{a}_\alpha \bar{z}^\alpha$ is invertible in $\bar{k}[\bar{z}_1, \ldots, \bar{z}_n]$. Therefore $\bar{a}_\alpha = 0$ for each $\alpha \neq 0$. Thus $|a_\alpha| < 1 = ||f||$. It follows that $|a_0| = 1$. ∎

In what follows we could take $R = k\{z_1, \ldots, z_{n-1}\rangle$ and $z = z_n$, so that $R\{z\} = T_n(k)$.

Definition 6.7: For $g = \sum_{n=0}^{\infty} a_n z^n \neq 0$ in $R\{z\}$ define the **pseudodegree** of g to be the integer $d = \max(n : ||a_n|| = ||g||)$. Call a_d the **pseudoleading coefficient** of g. Call g **regular**, if $a_d \in R^\times$ and $||ca_d|| = ||c|| \cdot |a_d||$ for all $c \in R$. ∎

Remark 6.8: Let g be regular of pseudodegree d and let $0 \neq q \in R\{z\}$ of pseudodegree l. Then qg is of pseudodegree $d + l \geq d$ and $||qg|| = ||q|| \cdot ||g||$.

Indeed, let $g = \sum_{n=0}^{\infty} a_n z^n$ and $q = \sum_{n=0}^{\infty} c_n z^n$ and let l be the pseudodegree of q. Then $||qg|| \leq ||q|| \cdot ||g||$, but, by Remark ? (if $||a|| < ||b||$ then $||a + b|| = ||b||$), the norm of the coefficient of z^{d+l} in qg is $||c_l a_d|| = ||c_l|| \cdot ||a_d|| = ||q|| \cdot ||g||$. ∎

THEOREM 6.9 (Weierstrass Division Theorem): *Let $f \in R\{z\}$ and let $g \in R\{z\}$ be regular of pseudodegree d. Then there are unique $q \in R\{z\}$ and $r \in R[z]$ such that $f = qg + r$ and $\deg r < d$. Moreover,*

$$(1) \qquad ||qg|| = ||q|| \cdot ||g|| \leq ||f|| \qquad \text{and} \qquad ||r|| \leq ||f||.$$

Proof: Write g as $g = \sum_{n=0}^{\infty} a_n z^n \in R\{z\}$.

PART I: *Estimates (1).* Assume that $f = qg + r$, where $\deg r < d$. If $q = 0$, then (1) is clear. Assume that $q \neq 0$. By Remark 6.8, $||qg|| = ||q|| \cdot ||g||$ and qg is of pseudodegree $m \geq d$. In particular, $||q|| \cdot ||g||$ is the norm of the coefficient of z^m in

qg. This coefficient is also the coefficient of z^m in $f = qg + r$, since $\deg r < d \le m$. Therefore $||q|| \cdot ||g|| \le ||f||$. It follows that $||r|| = ||f - qg|| \le \max(||f||, ||qg||) \le ||f||$.

PART II: *Uniqueness.* Assume that $f = qg + r = q'g + r'$, where $\deg r, \deg r' < d$. Then $0 = (q - q')g + (r - r')$. By Part I, $||q - q'|| = ||r - r'|| = 0$. Hence $q = q'$ and $r = r'$.

PART III: *Existence if g is a polynomial of degree d.* Write f as $\sum_{n=0}^{\infty} b_n z^n$. For each $m \ge 0$ let $f_m = \sum_{n=0}^{m} b_n z^n \in R[z]$. As g is regular of pseudodegree d, its leading coefficient is invertible. Euclid's algorithm for polynomials over R produces $q_m, r_m \in R[z]$ such that $f_m = q_m g + r_m$ and $\deg r_m < \deg g$. Thus for all k, m we have $f_m - f_k = (q_m - q_k)g + (r_m - r_k)$. By Part I, $||q_m - q_k|| \cdot ||g||, ||r_m - r_k|| \le ||f_m - f_k||$. Thus $\{q_m\}_{m=0}^{\infty}$ and $\{r_m\}_{m=0}^{\infty}$ are Cauchy sequences in $R\{z\}$, and hence they converge to $q \in R\{z\}$ and $r \in R[z]$ with $\deg r < d$. Clearly $f = qg + r$.

PART IV: *Existence for arbitrary g.* If $g = \sum_{n=0}^{\infty} a_n z^n$, put $g_0 = \sum_{n=0}^{d} a_n z^n \in R[z]$. Then $||g - g_0|| < ||g||$. By Part III with g_0 and f there are $q_0 \in R\{z\}$ and $r_0 \in R[z]$ such that $f = q_0 g_0 + r_0$ and $\deg r_0 < d$. By Part I, $||q_0|| \le \frac{||f||}{||g||}$ and $||r_0|| \le ||f||$. Thus $f = q_0 g + r_0 + f_1$, where $f_1 = -q_0(g - g_0)$, and $||f_1|| \le \frac{||g - g_0||}{||g||} \cdot ||f||$.

Put $f_0 = f$. By induction we get, for each $k \ge 0$, elements $f_k, q_k \in R\{z\}$ and $r_k \in R[z]$ such that $\deg r < d$ and

$$f_k = q_k g + r_k + f_{k+1}, \quad ||q_k|| \le \frac{||f_k||}{||g||}, \ ||r_k|| \le ||f_k||, \quad \text{and} \quad ||f_{k+1}|| \le \frac{||g - g_0||}{||g||}||f_k||.$$

It follows that $||f_k|| \to 0$, whence also $||q_k||, ||r_k|| \to 0$. Therefore $q = \sum_{k=0}^{\infty} q_k \in R\{z\}$ and $r = \sum_{k=0}^{\infty} r_k \in R[z]$. Clearly $f = qg + r$ and $\deg r < d$. ∎

THEOREM 6.10 (Weierstrass Preparation Theorem): *Let $f \in T_n(k)$ have norm 1. Then there exists a norm-preserving k-algebra automorphism σ of $T_n(k)$ such that $\sigma(f)$ is regular in z_n.*

Proof: Let $e_1, \ldots, e_{n-1} \in \mathbb{N}$. Define σ by

$$z_1 \mapsto z_1 + z_n^{e_1}, \ldots, z_{n-1} \mapsto z_{n-1} + z_n^{e_{n-1}}, z_n \mapsto z_n.$$

that is, if $g = \sum_\alpha a_\alpha z^\alpha$, then $\sigma(g) = \sum_\alpha a_\alpha \sigma(z^\alpha)$, where

$$\sigma(z^\alpha) = (z_1 + z_n^{e_1}) \cdots (z_{n-1} + z_n^{e_{n-1}}) z_n.$$

This is a well defined continuous homomorphism $T_n(k) \to T_n(k)$. Indeed, $||\sigma(z^\alpha)|| \leq ||z^\alpha||$. Hence for each $g = \sum_\alpha a_\alpha z^\alpha \in T_n(k)$ the series $\sum_\alpha a_\alpha \sigma(z^\alpha)$ converges, whence $\sigma(g) \in T_n(k)$. Moreover, $||\sigma(g)|| \leq ||g||$. The inverse of σ is given by replacing $+$ with $-$ in the definition of σ.

We claim that $\sigma(f)$ is regular in z_n for suitable $e_1, \ldots, e_{n-1} \in \mathbb{N}$.

Indeed, write $f = \sum_\alpha c_\alpha z^\alpha$. The set $\Lambda = \{\alpha \in \mathbb{N}_0^n \mid \overline{c_\alpha} \neq 0\}$ is finite. We have

$$\begin{aligned} \overline{\sigma(f)} &= \sum_{\alpha \in \Lambda} \overline{c_\alpha} (z_1 + z_n^{e_1})^{\alpha_1} \cdots (z_{n-1} + z_n^{e_{n-1}})^{\alpha_{n-1}} z_n^{\alpha_n} \\ &= \sum_{\alpha \in \Lambda} \overline{c_\alpha} \big(z_n^{e_1\alpha_1 + \cdots + e_{n-1}\alpha_{n-1} + \alpha_n} + \ldots \big) \end{aligned}$$

where the other monomials with coefficient $\overline{c_\alpha}$ are of degree in z_n strictly smaller than $e_1\alpha_1 + \cdots + e_{n-1}\alpha_{n-1} + \alpha_n$. Thus if the degrees $e_1\alpha_1 + \cdots + e_{n-1}\alpha_{n-1} + \alpha_n$ of the 'leading' monomials are distinct for distinct $\alpha \in \Lambda$, these monomials will not cancel each other, and one of them will be with the maximal degree.

To achieve it, take $e_i = e^i$ with $e > \alpha_j$ for all j and all $\alpha \in \Lambda$. (The above degrees are then e-adic expansions of natural numbers; the sequences of digits in these expansions are distinct, hence the numbers are distinct.) ∎

THEOREM 6.13: *The ring T_n is noetherian (every ideal of T_n is finitely generated).*

Proof: By induction on n. Suppose T_{n-1} is noetherian. Then so is the ring of polynomials $T_{n-1}[z_n]$. Let I be a non-zero ideal of T_n. Then there is $f \in I$ such that $||f|| = 1$. By the Preparation we may assume that f is regular in z_n, say, of degree d. By the Division each $g \in I$ is of the form $g = qf + r$, where $q \in T_n$ and $r \in T_{n-1}[z_n] \cap I$. Thus I is generated by f and the finitely many generators of the ideal $T_{n-1}[z_n] \cap I$ of $T_{n-1}[z_n]$.

∎

LEMMA 6.14: *Let $f \in T_n$ be regular in z_n of pseudodegree d. Then $f = qg$, where $g \in (T_n)^\times$ and $g \in T_{n-1}[z_n]$ is monic of degree d and norm 1 (and hence also regular in z_n of degree d).*

Proof: The Division gives $q \in T_n$ and $r \in T_{n-1}[z_n]$ such that $z_n^d = fq + r$; moreover $\deg_{z_n} r < d$ and $||r|| \leq ||z_n^d|| = 1$. Hence $z_n^d - r$ is also regular of degree d, and so we may perform another division: $f = q'(z_n^d - r) + r'$. This gives $f = qq'f + r'$. But also $f = 1f + 0$. The uniqueness of division by f gives $qq' = 1$ and $r' = 0$. Thus $f = q'g$, where q' is a unit and $g = z_n^d - r \in T_{n-1}[z_n]$ is monic with norm 1. ∎

LEMMA 6.15: *Let $f, g \in T_{n-1}[z_n]$, and g be monic of norm 1. Then $g|f$ in $T_{n-1}[z_n]$ if and only if $g|f$ in T_n.*

Proof: The division with reminder in $T_{n-1}[z_n]$ gives $f = qg + r$, with $q, r \in T_{n-1}[z_n]$ and $\deg r < d$. But $q \in T_n$ and g is regular in z_n. Thus if $g|f$ in T_n, by the uniqueness of the division in T_n we must have $r = 0$. Therefore $g|f$ in $T_{n-1}[z_n]$. The converse is trivial. ∎

LEMMA 6.16: *Let $g \in T_{n-1}[z]$ be monic of norm 1. Then g is irreducible in $T_{n-1}[z_n]$ if and only if g is irreducible in T_n.*

Proof: An element of a ring is invertible if and only if it divides 1 in that ring, Thus by Lemma 6.15, a monic polynomial of norm 1 in $T_{n-1}[z_n]$ is invertible in $T_{n-1}[z_n]$ if and only if it is invertible in T_n.

Suppose g is reducible in $T_{n-1}[z_n]$, that is, $g = g_1 g_2$, where $g_1, g_2 \in T_{n-1}[z_n]$ are not invertible. Wlog g_1, g_2 are monic, whence $||g_1||, ||g_2|| \geq 1$. But $||g_1|| \cdot ||g_2|| = ||g|| = 1$, so $||g_1|| = ||g_2|| = 1$. By the preceding paragraph g_1, g_2 are not invertible in T_n. Thus g is reducible in T_n.

Conversely, suppose g is reducible in T_n, that is, $g = g_1 g_2$, where $g_1, g_2 \in T_n$ are not invertible. We may assume that $||g_1|| = ||g_2|| = 1$. By Exercise 6.12, g_1, g_2 are regular in z_n. By Lemma 6.14 we may assume that g_1 is monic in $T_{n-1}[z_n]$. Division with remainder in $T_{n-1}[z_n]$ gives $g = g_1 q + r$ with $q, r \in T_{n-1}[z_n]$ and $\deg r < \deg g_1$. By the uniqueness of division in T_n we have $q = g_2$ and $r = 0$. Thus $g_2 \in T_{n-1}[z_n]$. As $g = g_1 g_2$, also g_2 is monic. By the first paragraph of this proof g_1, g_2 are not invertible in $T_{n-1}[z_n]$. Thus g is reducible in $T_{n-1}[z_n]$. ∎

THEOREM 6.17: *The ring T_n is a unique factorization domain.*

Proof: By induction on n. Suppose T_{n-1} is a UFD. Then so is the ring of polynomials $T_{n-1}[z_n]$ [Lang, Algebra, Theorem IV.2.3].

Let $0 \neq f \in T_n$. We want to show that f is a product of irreducibles, unique up to invertibles. Without loss of generality $||f|| = 1$. By the Preparation we may assume that f is regular in z_n, say, of pseudodegree d. By Lemma 6.14 we may assume that $f \in T_{n-1}[z_n]$ is monic of degree d and norm 1.

Write $f = g_1 \cdots g_r$, where $g_i \in T_{n-1}[z_n]$ are irreducible. Then their leading coefficients must be invertible. So wlog they are monic. Thus $||g_i|| \geq 1$. As $f = g_1 \cdots g_r$, we have $||g_i|| = 1$. By Lemma 6.16, the g_i are irreducible in T_n.

To show the uniqueness of the product, let $g \in T_n$ be irreducible, $g|f$ in T_n. By Lemma 6.14 we may assume that $g \in T_{n-1}[z_n]$ is monic of norm 1. By Lemma 6.15, $g|f$ in $T_{n-1}[z_n]$. Thus there is i such that $g|g_i$ in $T_{n-1}[z_n]$. Therefore $g = g_i$. ∎

THEOREM 6.18: *Let I be an ideal of T_n. Then there exist an integer $d \leq n$ and a norm preserving k-automorphism σ of T_n such that the composition $T_d \to T_n \xrightarrow{\sigma} T_n \to T/I$ is a finite injective morphism.*

Proof: (a) By induction on n. The assertion is clear for $n = 0$. Assume $n \geq 1$. If $I = 0$, take $d = n$ and let σ be the identity. So assume that $I \neq 0$.

By the Preparation there is a norm-preserving k-automorphism ρ of T_n such that $\rho^{-1}(I)$ contains some f regular of degree m in z_n. Put $J = \rho^{-1}(I) \cap T_{n-1}$. The canonical morphism $\bar{\lambda}$: $T_{n-1}/J \to T_n/\rho^{-1}(I)$ is injective. The division by f in T_n shows that $T_n/\rho^{-1}(I) = T_{n-1}\{z_n\}/\rho^{-1}(I)$ is a finite T_{n-1}/J-module, generated by $1, z_n, \ldots, z_n^{m-1}$. Thus $\bar{\lambda}$ is finite. The map $\bar{\rho}$: $T_n/\rho^{-1}(I) \to T_n/I$ induced from ρ is an isomorphism.

By the induction hypothesis there is d and a norm-preserving k-automorphism τ of T_{n-1} such that $T_d \to T_{n-1} \xrightarrow{\tau} T_{n-1} \to T_{n-1}/J$ is a finite injective morphism. Extend

τ to an automorphism of T_n by $\tau(z_n) = z_n$.

$$
\begin{array}{ccccccc}
T_n & \xrightarrow{\tau} & T_n & \xrightarrow{\mathrm{id}} & T_n & \xrightarrow{\rho} & T_n \\
\uparrow & & \uparrow & & \downarrow & & \downarrow \\
T_{n-1} & \xrightarrow{\tau} & T_{n-1} & & & & \\
\uparrow & & \downarrow & & & & \\
T_d & \xrightarrow{\bar{\tau}} & T_{n-1}/J & \xrightarrow{\bar{\lambda}} & T_n/\rho^{-1}(I) & \xrightarrow{\bar{\rho}} & T_n/I
\end{array}
$$

Then $\bar{\rho}\bar{\lambda}\bar{\tau}\colon T_d \to T_n/I$ is an injective finite morphism. Hence $\sigma = \rho\tau$ has the required property. ∎

COROLLARY 6.19: *Let* $\mathfrak{m}$ *be a maximal ideal of* T_n. *Then the field* $T_n/\mathfrak{m}$ *is a finite extension of* k.

Proof: By Theorem 6.18 there is a subring T_d of $T_n/\mathfrak{m}$ over which $T_n/\mathfrak{m}$ is finite. As $T_n/\mathfrak{m}$ is a field, so is T_d [AM, Prop. 5.7]. It follows that $d = 0$ (for instance, z_1 is not invertible in T_d) and hence $T_n/\mathfrak{m}$ is a finite extension of $T_0 = k$. ∎

Definition 6.20: An **affinoid algebra** A over k is a k-algebra which is finite over T_n, for some n. That is, there is a ring homomorphism $T_n \to A$ such that via it A is a finite T_n-module. By Theorem 6.18 we may assume that $T_n \to A$ is injective. (A composition of finite homomorphisms is finite.) ∎

THEOREM 6.21: *An affinoid algebra is a noetherian ring.*

Proof: By definition, an affinoid algebra is a finitely generated extension of some T_n, which is noetherian by Theorem 6.13. Hence A is noetherian. ∎

COROLLARY 6.22: *Let* A *be an affinoid algebra, and suppose* A *is a Banach algebra with respect to some norm on* A. *Let* $I \leq A$ *be an ideal. Then*

(a) *I is closed with respect to the norm.*

(b) *The norm on A induces a norm on A/I such that A/I is a Banach algebra with respect to it.*

Proof: (a) This is Theorem 2.5.

(b) Put $E = A/I$. Define norm on E by $||e||_E = \inf\{||f|| \mid \varphi(f) = e\}$. We check that this is a norm: Suppose $||e||_E = 0$. Then there is $\{f_i\}_{i=0}^{\infty} \subseteq A$ such that $\varphi(f_i) = e$ and $||f_i|| \to 0$. Thus $f_0 - f_i \in I$ and $f - f_i \to f_0$. But I is closed by (a), hence $f_0 \in I$. Thus $e = \varphi(f_0) = 0$.

Clearly $||\alpha e|| = \alpha| \cdot ||e||_E$, for every $\alpha \in k$. Let $e, e' \in E$. Let $f, f' \in A$ such that $\varphi(f) = e, \varphi(f') = e'$. Then $||ee'||_E \leq ||ff'|| \leq ||f|| \cdot ||f'||$. Taking infimum on the right handed side, $||ee'||_E \leq ||e|| \cdot ||e'||$.

In particular ($e = e' = 1$), $||1||_E \geq 1$. But $||1||_E \leq ||1|| = 1$. So $||1||_E = 1$.

To show that $||e + e'|| \leq \max(||e||, ||e'||)$, use that for $A, B \subseteq [0, \infty)$ we have $\inf_{a \in A, b \in B} \max(a, b) = \max(\inf(A), \inf(B))$. ∎

Exercise 6.23: *Let $g \in T_{n-1}[z_n]$ be monic of norm 1. Then $T_{n-1}[z_n]/gT_{n-1}[z_n] \to T_n/gT_n$ is an isomorphism.*

Theorem 6.24: *Let E be an affinoid algebra. Then $E \cong T_n/I$ for some n and for some ideal $I \leq E$.*

Proof: (a) By the definition there exists a finite homomorphism $\varphi\colon T_d \to E$. Thus $E = T_d[e_{d+1}, \ldots, e_n]$, (by abuse of notation we write T_d instead of $\varphi(T_d)$) and each e_i is integral over T_d, that is, satisfies some monic $g_i(X) \in T_d[X]$.

Fix i. Say, $g_i = X^m + a_1X^{m-1} + \ldots + a_m$, with $a_j \in T_d$. We may assume that $\max ||a_j|| \leq 1$, otherwise replace e_i by αe_i, where $\alpha \in k^{\times}$ with $|\alpha|$ sufficiently small. (Then αe_i satisfies $X^m + \alpha a_1X^{m-1} + \ldots + \alpha^m a_m$.)

Claim: *We can extend φ to a homomorphism $\varphi\colon T_n \to E$ such that $\varphi(z_i) = e_i$.* Indeed, by induction on i suppose we have already extended φ to $\varphi\colon T_{i-1} \to E$. Extend it to $\varphi\colon T_{i-1}[z_i] \to E$ by $\varphi(z_i) = e_i$. Then $g_i(z_i) \in T_{i-1}[z_i]$ and $\varphi(g_i(z_i)) = 0$. Hence φ factors into $T_{i-1}[z_i] \to T_{i-1}[z_i]/g_iT_{i-1}[z_i] \to E$. By the preceding paragraph, $||g_i|| = 1$. By Exercise 6.23 we may replace the first map by $T_i \to T_i/g_iT_i$ and thus extend φ to T_i.

As the image of φ contains the generators of E over T_d, φ is surjective. Let $I = \ker(\varphi)$; then $E \cong T_n/I$. It is easy to see that E is complete. ∎

Theorem 6.25: *Let $(A_i, || \, ||_i)$, for $i = 1, 2$, be two affinoid algebras, which are Banach k-algebras w.r.t. their respective norms. Let $u\colon A_1 \to A_2$ be a homomorphism of k-algebras. Then u is continuous. In particular, all norms on an affinoid algebra which make it into a Banach k-algebra are equivalent.*

Proof: By Corollary 2.3 we have to show that the graph $\{(x, u(x)) \mid x \in A_1\}$ is closed in $A_1 \times A_2$. That is, if $(x_i, u(x_i)) \to (x, y) \in A_1 \times A_2$, then $y = u(x)$. Replacing x_i by $x_i - x$ and y by $y - u(x)$ we have to prove: if $\lim x_i = 0$ and $\lim u(x_i) = y \in A_2$, then $y = 0$.

Let $I_2 \leq A_2$ be an ideal such that $\dim_k A_2/I_2 < \infty$. Let $I_1 = \mathrm{Ker}(A_1 \to A_2 \to A_2/I_2)$. Then

$$\begin{array}{ccc} A_1 & \xrightarrow{u} & A_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ A_1/I_1 & \xrightarrow{\bar{u}} & A_2/I_2 \end{array}$$

commutes, with $\bar{u}$ an embedding. So also $\dim_k A_1/I_1 < \infty$.

By Theorem 6.18, A_i/I_iA_i are affinoid algebras and by Corollary 6.22, they are Banach algebras, wrt the induced norms. The norm of A_2/I_2A_2 restricts via $\bar{u}$ to another norm on A_1/I_iA_1. By Theorem 2.14 these two norms are equivalent. Thus $\bar{u}$ is continuous. Therefore $\pi_2 \circ u = \bar{u} \circ \pi_1$ is continuous. Thus $\pi_2(y) = 0$, that is, $y \in I_2$.

It remains to show that $\bigcap_{\dim_k A/I < \infty} I = 0$.

Let $M \leq A$ be a maximal ideal. By Theorem 6.24 there is an epimorphism $\pi\colon T_n \to A$; As $\pi^{-1}(M) \leq T_n$ is maximal and $T_n/\pi^{-1}(M) \cong A/M$, by Corollary 6.19, $\dim_k A/M < \infty$. Moreover, $\dim_k A/M^n < \infty$ for every $n \geq 1$. (Indeed, by induction on n, using the short exact sequence $0 \to M^{n-1}/M^n \to A/M^n \to A/M^{n-1} \to 0$, it suffices to show that $\dim_k M^{n-1}/M^n < \infty$. As A is noetherian, the A-ideal M^{n-1} is a finite A-module; hence M^{n-1}/M^n is a finite A/M-module. But A/M is a finite k-module, so M^{n-1}/M^n is a finite k-module.)

Assume there is $0 \neq y \in \bigcap_M \bigcap_n M^n$. Put $J = \{a \in A \mid ay = 0\}$. This is a a proper ideal of A. Hence there is a maximal $M \leq A$ such that $J \subseteq M$. Thus every $s \in A \smallsetminus M$ satisfies $sy \neq 0$. This means that $\frac{y}{1} \in A_M$ is not zero. Furthermore,

$\frac{y}{1} \in M^n A_M = (MA_M)^n$. But by Krull's Theorem, (in noetherian ring A we have $\bigcap_n \mathrm{rad}(A)^n = 0$) $\bigcap_n (MA_M)^n = 0$. A contradiction. ∎

7. Affinoid spaces

Definition 7.1: An **affinoid space** is the set $X = \mathrm{Sp}(A)$ of the maximal ideals of an affinoid algebra A. For each $x \in X$ the field A/x is a finite extension of k by Corollary 6.19. The valuation $|\ |$ of k uniquely extends to A/x. For $f \in A$ put $f(x)$ to be the image of f in A/x under the quotient map $A \to A/x$. Define topology on A: generated by $\{x \in X \mid |f(x)| \le 1\}$. Put $||f||_{\mathrm{sp}} = \sup_{x\in X} |f(x)|$. Define

$$A^o = \{f \in A \mid \ ||f||_{\mathrm{sp}} \le 1\} \qquad A^{oo} = \{f \in A \mid \ ||f||_{\mathrm{sp}} < 1\}.$$

∎

LEMMA 7.2: *Let $||\ ||$ be a norm on A. Then $||f||_{\mathrm{sp}} \le ||f||$ for every $f \in A$.*

Proof: It suffices to prove: $|f(x)| \le ||f||$ for every $f \in A$ and every $x \in X$. Fixing x, it suffices to prove: $|f(x)| \le ||g||$ for every $g \in A$ such that $f(x) = g(x)$. That is, $|a| \le ||a||$ for every $a \in A/x$, where $||\ ||$ is the induced norm on A/x.

There is $C > 0$ such that $C|b| \le ||b||$ for every $b \in A/x$. In particular, $C|a|^m = C|a^m| \le ||a^m|| \le ||a||^m$. Thus $C^{1/m}|a| \le ||a||$. Taking limit, $|a| \le ||a||$. ∎

Remark 7.3: The map $||\ ||_{\mathrm{sp}}$ is a semi-norm, called the **spectral semi-norm**. It is a norm if and only if the intersection of all maximal ideals of A is 0. ∎

Example 7.4: Let A be an affinoid algebra. Let $\tilde{k}$ be an algebraic closure of k. Every $x \in \mathrm{Sp}(A)$ defines a homomorphism (necessarily continuous, by Theorem 6.25) u: $A \to \tilde{k}$, whose image is a finite extension A/x of k. Two such homomorphisms u_1, u_2 are equivalent if they have the same kernel, i.e., there is a k-isomorphism θ: $u_1(A) \to u_2(A)$ such that $u_2 = \theta \circ u_1$. Thus elements of $\mathrm{Sp}(A)$ correspond to equivalence classes of k-algebra homomorphisms u: $A \to \tilde{k}$ with image finite over k. (If $k = K$ is algebraically closed, each equivalence class contains a unique homomorphism.)

In particular, for $A = T_n$, each such u: $T_n \to \tilde{k}$ defines $(x_1, \dots, x_n) \in \tilde{k}^n$ by $x_i = u(z_i)$. The continuity of u implies that $|x_i| \le 1$ (for every $a \in \tilde{k}$ with $|a| < 1$ the Cauchy series $\sum_{j=1}^\infty a^j z_i^j$ is mapped into a Cauchy series $\sum_{j=1}^\infty a^j x_i^j$, so $|a| \cdot |x_i| < 1$). Conversely, every such $(x_1, \dots, x_n) \in \tilde{k}^n$ defines a homomorphism u: $T_n \to \tilde{k}$ with image finite over k. Thus $\mathrm{Sp}(T_n) = D_n = \{x = (x_1, \ldots, x_n) \in \tilde{k}^n \mid |x_i| \le 1\}/\cong$. If $k = K$ is algebraically closed, then $\mathrm{Sp}(T_n) = D_n = \{x = (x_1, \ldots, x_n) \in \tilde{k}^n \mid |x_i| \le 1\}$. ∎

Lemma 7.5: *The spectral norm on T_n coincides with the standard norm. Moreover, for every $f \in T_n$ there is $x \in \mathrm{Sp}(T_n)$ such that $||f|| = |f(x)|$.*

Proof: By Lemma 7.2, $||f||_{\mathrm{sp}} \le ||f||$ for every $f \in T_n$. So we only have to prove the second assertion. Wlog $||f|| = 1$. Hence $\bar{f} \in \bar{k}[z_1, \ldots, z_n]$ is not zero. So there are $\bar{x}_1, \ldots, \bar{x}_n$ in the algebraic closure of $\bar{k}$ such that $\bar{f}(\bar{x}_1, \ldots, \bar{x}_n) \ne 0$. Lift them to $x_1, \ldots, x_n \in \bar{k}$ with $|x_i| \le 1$. (For instance, first lift $\bar{x}_i$ to $x_i \in K^o$, where K is the completion of $\bar{k}$, and then, as $\bar{k}$ is dense in K, replace x_i by a sufficiently close element of $\bar{k}$.) There is a finite extension l of k such that $x_1, \ldots, x_n \in l^o$. The k-map $T_n \to l$ defined by $z_i \mapsto x_i$ is a continuous epimorphism. Its kernel $x \in \mathrm{Sp}(T_n)$ satisfies $|f(x)| = 1$. ∎

Exercise 7.6: *Let A be an affinoid algebra. Let $f \in A$. TFAE:*

(a) $\inf\{|f(x)| \mid x \in \mathrm{Sp}(A)\} > 0$;

(b) $f(x) \ne 0$ *for all* $x \in \mathrm{Sp}(A)$;

(c) $f \in A^\times$;

Example 7.7: Let $k = K$ be algebraically closed. We have defined a connected affinoid in $\mathbb{P}$ as the complement F of a union of disjoint disks in $\mathbb{P}$. We now show that $\mathcal{O}(F)$ is an affinoid algebra and that $\mathrm{Sp}(\mathcal{O}(F)) = F$.

To make notation easier assume that $\infty \in F$. Thus $F^c = \bigcup_{i=1}^n \{a \in \mathbb{P} \mid |a - a_i| < |\pi_i|\}$, with $a_i, \pi_i \in K$. Define φ: $F \to (K^o)^n$ by

$$\varphi(a) = (\frac{\pi_1}{a - a_1}, \ldots, \frac{\pi_n}{a - a_n}).$$

It is an injection and

$$\begin{aligned}\varphi(F) &= \{(x_1, \ldots, x_n) \in (K^o)^n \mid \frac{\pi_i}{x_i} + a_i = \frac{\pi_j}{x_j} + a_j \text{ for } i \ne j\} \\ &= \{(x_1, \ldots, x_n) \in (K^o)^n \mid \pi_i x_j - \pi_j x_i + (a_i - a_j) x_i x_j = 0 \text{ for } i \ne j\} \\ &= \{(x_1, \ldots, x_n) \in (K^o)^n \mid \frac{\pi_i}{a_i - a_j} x_j + \frac{\pi_j}{a_j - a_i} x_i + x_i x_j = 0 \text{ for } i \ne j\}\end{aligned}$$

Let I be the ideal of T_n generated by

$$E_{ij} = \frac{\pi_i}{a_i - a_j} z_j + \frac{\pi_j}{a_j - a_i} z_i + z_i z_j \in T_n = K\langle z_1, \ldots, z_n \rangle, \text{ for } i \neq j\}.$$

and put $A = T_n/I$. Then A is an affinoid algebra and $\mathrm{Sp}(A)$ can be identified with $\varphi(F)$. We show that there is an isomorphism $\psi: A \to \mathcal{O}(F)$ such that $\varphi = \mathrm{Sp}(\psi)$.

Since $\mathcal{O}(F)$ is a Banach algebra with respect to the 'supremum' norm $|| \, ||_F$ and $||\frac{\pi_i}{z-a_i}||_F \leq 1$, the map $z_i \mapsto \frac{\pi_i}{z-a_i}$ extends to a unique homomorphism $\hat{\psi}: T_n \to \mathcal{O}(F)$ such that $||\hat{\psi}(f)||_F \leq ||f||$ for every $f \in T_n$. Obviously $\hat{\psi}(E_{ij}) = 0$, hence $\hat{\psi}$ induces a homomorphism $\psi A \to \mathcal{O}(F)$ such that $||\psi(f)||_F \leq ||f||_A$ for every $f \in A$ (in the infimum norm on A). Using the E_{ij} it is easy to see that every $f \in T_n$ is of the form $f = f_0 + a + \sum_{i=1}^{n} \sum_{m=1}^{\infty} a_{i,m} z_i^m$, where $f_0 \in I$ and $a, a_{i,m} \in K$ with $\lim_m a_{i,m} = 0$. By the Mittag-Leffler decomposition in $\mathcal{O}(F)$ we see that $\hat{\psi}$ is surjective, its kernel is I, and for every $g \in \mathcal{O}(F)$ there is a preimage $f \in T_n$ such that $||f|| = ||g||_F$. Thus ψ is an isometric isomorphism. ∎

The above identification allows to give a different proof of ?

Theorem 7.8: *Let F be a connected affinoid in $\mathbb{P}$. Then $\mathcal{O}(F)$ is a principal ideal domain. In particular, every $0 \neq f \in \mathcal{O}(F)$ has only finitely many zeroes.*

8. Spectral norm

LEMMA 8.1: *Let K be an algebraically closed complete field. Let $P(X) = X^n + a_1X^{n-1} + \cdots + a_n \in K[X]$ and let $\alpha_1, \ldots, \alpha_n \in K$ be its roots. Then* $\max_j |\alpha_j| = \max_i |a_i|^{1/i}$.

Proof: We have

$$P(X) = X^n + a_1X^{n-1} + \cdots + a_n = (X - \alpha_1)\cdots(X - \alpha_n).$$

Wlog $|\alpha_1| \geq |\alpha_i|$ for all i. Substitute $X = \alpha_1 Y$. Then $\alpha_1^{-n}P(\alpha_1 Y)$ is

$$Y^n + \frac{a_1}{\alpha_1}Y^{n-1} + \cdots + \frac{a_n}{\alpha_1^n} = (Y-1)(Y - \frac{\alpha_2}{\alpha_1})\cdots(Y - \frac{\alpha_n}{\alpha_1}).$$

The right handed side is in $K^o[Y]$. Hence $|\frac{a_i}{\alpha_1^i}| \leq 1$ for each i. We must have $|\frac{a_i}{\alpha_1^i}| = 1$ for some i, otherwise modulo K^{oo} the left handed side of the above displayed equation would be Y^n and the right handed side would have root 1, a contradiction. ∎

PROPOSITION 8.2: *Let A be an affinoid algebra without zero-divisors and let $T_d \to A$ be a finite monomorphism. Then every $f \in A$ satisfies a monic irreducible $P = X^n + a_1X^{n-1} + \cdots + a_n \in T_d[X]$. We have $||f||_{\mathrm{sp}} = \max_i ||a_i||_{\mathrm{sp}}^{1/i}$ and there is $x \in \mathrm{Sp}(A)$ with $|f(x)| = \max_i ||a_i||_{\mathrm{sp}}^{1/i}$. (We can write $||\ ||$ instead of $||\ ||_{\mathrm{sp}}$, by Lemma 7.5.)*

Proof: The map $T_d \to A$ is an inclusion of integral domains. Let $P(X)$ be the monic irreducible polynomial of $f \in A$ over the quotient field of T_d. But T_d is a unique factorization domain, hence integrally closed, [L, Prop. VII.1.7], hence $P(X) \in T_d[X]$ [L, Cor. VII.1.6]. Division with remainder gives that $T_d[f] \cong T_d[X]/(P(X))$.

Let $x \in \mathrm{Sp}(A)$ (a maximal ideal of A). As A/T_d is integral, $y = x \cap T_d$ is a maximal ideal of T_d [AM, 5.8], that is, $y \in \mathrm{Sp}(T_d)$. Thus $k \subseteq T_d/y \subseteq A/x$. There is a complete algebraically closed field K such that $A/x \subseteq K$. As $P(f) = 0$, $f(x)$ is a root of $X^n + a_1(y)X^{n-1} + \cdots + a_n(y) \in K[X]$. By Lemma 8.1,

$$|f(x)| \leq \max_i |a_i(y)|^{1/i} \leq \max_i ||a_i||_{\mathrm{sp}}^{1/i}.$$

In particular, $||f||_{\mathrm{sp}} \leq \max_i ||a_i||_{\mathrm{sp}}^{1/i}$. So we only have to find $x \in \mathrm{Sp}(A)$ such that $|f(x)| \geq \max_i ||a_i||_{\mathrm{sp}}^{1/i}$.

Choose i which attains the maximum on the right handed side. By Lemma 7.5 there is $y \in \mathrm{Sp}(T_d)$ with $|a_i(y)| = ||a_i||_{\mathrm{sp}}$. Let K be a complete algebraically closed field such that $T_d/y \subseteq K$. By Lemma 8.1 there is a root $\lambda \in K$ of $X^n + a_1(y)X^{n-1} + \cdots + a_n(y) \in K[X]$ such that $|\lambda| \geq |a_i(y)|^{1/i}$. So it suffices to find $x \in \mathrm{Sp}(A)$ such that $f(x) = \lambda$.

As $T_d[f] \cong T_d[X]/(P(X))$, we may extend the homomorphism $T_d \to T_d/y$ to u: $T_d[f] \to K$ such that $u(f) = \lambda$. The image $u(T_d[f]) = T_d/y[\lambda]$ is a field, because T_d/y is a field. Hence $\mathrm{Ker}(u)$ is a maximal ideal of $T_d[f]$. As A is integral over T_d and hence also over $T_d[f]$, there is $x \in \mathrm{Sp}(A)$ lying over $\mathrm{Ker}(u)$ [AM, 5.10 and 5.8]. Then $f(x) = \lambda$. ∎

EXERCISE 8.3: *Let u: $A \to B$ be an epimorphism of affinoid algebras. Then* $||u(f)||_{\mathrm{sp}} \leq ||f||_{\mathrm{sp}}$ *for every $f \in A$.*

Proof: Let $y \in \mathrm{Sp}(B)$. Then $x = u^{-1}(y) \in \mathrm{Sp}(A)$ and $(u(f))(y) = f(x)$. Therefore $||u(f)||_{\mathrm{sp}} = \sup_{x \in u^{-1}(\mathrm{Sp}(A))} |f(x)| \leq \sup_{x \in \mathrm{Sp}(A)} |f(x)| = ||f||_{\mathrm{sp}}$. ∎

Let A be a commutative ring with unity. Recall that the **nilradical** $\mathrm{nil}(A) = \{f \in A \mid (\exists n \in \mathbb{N}) f^n = 0\}$ is an ideal of A. It is the intersection of all prime ideals of A, and hence the intersection of all minimal prime ideals of A. If A is noetherian, there are only finitely many minimal prime ideals of A. Always $\mathrm{nil}(A) \subseteq \mathrm{rad(A)}$, the intersection of the maximal ideals of A. We say that A is **reduced** if $\mathrm{nil}(A) = 0$.

COROLLARY 8.4: *Let A be an affinoid algebra. Then* $\mathrm{nil}(A) = \mathrm{rad(A)}$. *If A is reduced, then* $||\ ||_{\mathrm{sp}}$ *is a norm.*

Proof: We have $\mathrm{rad(A)} = \{\mathrm{f} \in \mathrm{A} \mid ||\mathrm{f}||_{\mathrm{sp}} = 0\}$. So the second assertion follows from the first one.

Let $f \in \mathrm{rad(A)}$, that is, $||f||_{\mathrm{sp}} = 0$.

Suppose first that A has no zero-divisors. By Theorem 6.18 there exists a finite monomorphism $T_d \to A$. By Proposition 8.2, f satisfies a monic irreducible $P(X) \in T_d[X]$ whose coefficients, except for the leading one, are 0. Thus $P = X$, and hence $f = 0$. Therefore $\mathrm{rad(A)} = 0$.

In the general case let $\mathcal{P}$ be a prime ideal of A. Then $A/\mathcal{P}$ is an affinoid algebra with no zero-divisors. Let $\bar{f}$ be the image of f in $A/\mathcal{P}$. By Exercise 8.3, $||\bar{f}||_{\mathrm{sp}} \le ||f||_{\mathrm{sp}} = 0$. Hence by the previous case $\bar{f} = 0$. Thus $f \in \mathcal{P}$. Therefore $f \in \bigcap \mathcal{P} = \mathrm{nil}(A)$. ∎

PROPOSITION 8.5: *Let A be an affinoid algebra. Let $\varphi\colon T_d \to A$ be a finite monomorphism. Then*

(a) $\varphi(T_d^0) \subseteq A^o$.

(b) *A^o is integral over T_d^o.*

Proof: (a) By a home exercise, $||f||_{\mathrm{sp}} = ||\varphi(f)||_{\mathrm{sp}}$. Thus $\varphi(T_d^0) \subseteq A^o$.

(b) Let $f \in A^o$. We want to find a monic $P(X) \in T_d^o[X]$ such that $P(f) = 0$.

If A has no zero divisors, the irreducible polynomial $P(X)$ of f over T_d has coefficients in T_d^o, by Proposition 8.2.

In the general case let $\mathcal{P}_1, \ldots, \mathcal{P}_s$ be the minimal prime ideals in A.

Fix $1 \le i \le s$. Let $A_i = A/\mathcal{P}_i$, let $\pi_i\colon A \to A_i$ be the quotient map, and put $f_i = \pi_i(f)$. Then A_i is without zero-divisors. By Exercise 8.3, $f_i \in A_i^o$. Let $Q_i = \mathrm{Ker}(\pi_i \circ \varphi)$. Then φ induces a finite monomorphism $\bar{\varphi}\colon T_d/Q_i \to A_i$. By Theorem 6.18, there is $c \le d$ and a norm-preserving automorphism σ of T_d such that $\bar{\sigma}\colon T_c \to T_d \xrightarrow{\sigma} T_d \to T_d/Q_i$ is a finite monomorphism. The composition $\bar{\varphi}\bar{\sigma}$ a finite monomorphism $T_c \to A_i$.

$$
\begin{array}{ccccc}
T_d & \xrightarrow{\ \sigma\ } & T_d & \xrightarrow{\ \varphi\ } & A \\
\uparrow & & \downarrow & & \downarrow{\scriptstyle \pi_i} \\
T_c & \xrightarrow{\ \bar{\sigma}\ } & T_d/Q_i & \xrightarrow{\ \bar{\varphi}\ } & A_i
\end{array}
$$

By the above special case there is a monic $\hat{P}_i(X) \in T_c^o[X]$ such that $\hat{P}_i(f_i) = 0$. Since the spectral norms on T_c, T_d are the standard norms and σ preserves the latter, $P_i(X) = \sigma(\hat{P}_i) \in T_d^o[X]$. Moreover, P_i is monic and $P_i(f_i) = 0$. Thus $P_i(f) \in \mathcal{P}_i$.

Put $P(X) = \prod_{i=1}^s P_i(X)$. Then $P \in T_d^o[X]$ is monic and $P(f) \in \bigcap_i \mathcal{P}_i$. Therefore $P(f)$ is nilpotent. So for a suitable $m \ge 1$ we have $P^m(f) = 0$. ∎

COROLLARY 8.6: *Let A be an affinoid algebra with a norm $||\ ||$ which makes it a Banach algebra. Then $A^o = \{f \in A \mid \sup_{n \ge 0} ||f^n|| < \infty\}$.*

Proof: Let $f \in A$.

Suppose $N = \sup_{n\geq 0} ||f^n|| < \infty$. Let $x \in \mathrm{Sp}(A)$. Then for every $n \geq 1$, $|f(x)^n = |f^n(x)| \leq ||f^n||_{\mathrm{sp}} \leq ||f^n|| \leq N$, whence $|f(x)| \leq 1$. Therefore $||f||_{\mathrm{sp}} \leq 1$, whence $f \in A^o$.

Conversely, suppose $f \in A^o$. There is a finite monomorphism $T_d \to A$. By Theorem 8.5(b), f is integral over T_d^o. Thus $f^n = \sum_{i=0}^{n-1} a_i f^i$ with $a_i \in T_d^o$. By induction, $f^m = \sum_{i=0}^{n-1} b_i f^i$ where $b_i \in T_d^o$. As $T_d \to A$ is continuous (Theorem 6.25), there is $C > 0$ such that $||b_i|| \leq C||b_i||_{T_d}$. But $||b_i||_{T_d} = ||b_i||_{\mathrm{sp}}$, by Lemma 7.5, and $||b_i||_{\mathrm{sp}} \leq 1$, hence $||b_i|| \leq C$. Thus $||f^m|| \leq \max_{i=0}^{n-1} C||f^i||$ is bounded. ∎

Corollary 8.7: *Let A be an affinoid algebra with a norm $||\ ||$ which makes it a Banach algebra. Then $||f||_{\mathrm{sp}} = \lim_{n\to\infty} ||f^n||^{1/n}$.*

Proof: By a home exercise, $||f||_{\mathrm{sp}}^n = ||f^n||_{\mathrm{sp}}$. Hence by Lemma 7.2, $||f||_{\mathrm{sp}}^n = ||f^n||_{\mathrm{sp}} \leq ||f^n||$, whence $||f||_{\mathrm{sp}} \leq ||f^n||^{1/n}$. It now suffices to show that $\limsup ||f^n||^{1/n} \leq ||f||_{\mathrm{sp}}$.

Choose $a \in k$ such that $|a| > 1$. Let $s \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that $||f||_{\mathrm{sp}} \leq |a|^{\frac{s}{m}}$. Then $||f||_{\mathrm{sp}}^m \leq |a|^s$, hence $||\frac{1}{a^s} f^m||_{\mathrm{sp}} \leq 1$, whence by Corollary 8.6 there is $C' > 0$ such that $||\frac{1}{a^{sq}} f^{mq}|| \leq C'$ for every $q \in \mathbb{N}$. In particular, if $n \in \mathbb{N}$, write it as $n = mq + r$ with $q, r \in \mathbb{N}$ and $0 \leq r < m$. Then $sq = \frac{s}{m} n - \frac{sr}{m}$, and hence

$$||f^n|| \leq ||f^{mq}|| \cdot ||f^r|| \leq C'|a|^{sq}||f^r|| \leq C' \frac{||f^r||}{|a|^{\frac{sr}{m}}} \big(|a|^{\frac{s}{m}}\big)^n$$

Let C be the maximum of $C' \frac{||f^r||}{|a|^{\frac{sr}{m}}}$ over the finitely many choices of r, s. Then $||f^n|| \leq C\big(|a|^{\frac{s}{m}}\big)^n$. Thus $\limsup ||f^n||^{1/n} \leq |a|^{\frac{s}{m}}$. ∎

Exercise 8.8: *Let $\varphi\colon A \to B$ be a homomorphism of affinoid algebras over k. Put $C = A\langle X_1, \ldots, X_s\rangle$. Let $b_1, \ldots, b_s \in B$. Then there exists a homomorphism of k-algebras $\psi\colon C \to B$ extending φ such that $\psi(X_i) = b_i$ for each i if and only if $||b_i||_{\mathrm{sp}} \leq 1$ for each i. If ψ exists, it is unique and continuous.*

Lemma 8.9: *Let T be an integral domain, E its quotient field, V a vector space over E, and $A, B \subseteq V$ finitely generated T-modules. Let A_E, B_E be the E-vector spaces generated by A, B. If $A_E \subseteq B_E$, then there is $0 \neq t \in T$ such that $tA \subseteq B$.*

Proof: Suppose that $A = \sum_{i=1}^m T\alpha_i$ and $B = \sum_{j=1}^n T\beta_j$. For each i there are $t_{ij}, 0 \neq$

$t'_{ij} \in T$ such that $\alpha_i = \sum_{j=1}^n \frac{t'_{ij}}{t_{ij}} \beta_j$. Put $t = \prod_i \prod_j t_{ij}$. Then $0 \neq t \in T$ and and $\frac{t}{t_{ij}} \in T$ for all i, j. Hence $t\alpha_i = \sum_j t'_{ij} \frac{t}{t_{ij}} \beta_j \in \sum_{j=1}^n T\beta_j = B$, for all i, whence $tA \subseteq B$. ∎

LEMMA 8.10: *Let A be an affinoid algebra without zero-divisors and let $T_d \to A$ be a finite morphism. Then $||f\alpha||_{\mathrm{sp}} = ||f|| \cdot ||\alpha||_{\mathrm{sp}}$ for all $f \in T_d$ and $\alpha \in A$. (Recall that $||f||_{\mathrm{sp}} = ||f||$ and the norm on T_d is multiplicative.)*

Proof: Let $X^n + a_1 X^{n-1} + \cdots + a_n \in T_d[X]$ be the irreducible polynomial of $\alpha \in A$ over the quotient field E of T_d. Then $X^n + a_1 X^{n-1} + \cdots + a_n \in T_d[X]$ is the irreducible polynomial of $f\alpha$ over E. Hence by Proposition 8.2

$$||f\alpha||_{\mathrm{sp}} = \max_i ||f_i^i a_i||^{1/i} = \max_i ||f_i^i||^{1/i} \cdot ||a_i||^{1/i} = ||f|| \max_i ||a_i||^{1/i} = ||f|| ||\alpha||_{\mathrm{sp}}.$$

∎

LEMMA 8.11: *Let l/k be a finite extension of complete fields, and let $q \in \mathbb{N}$. Then $T' = l\langle z_1^{1/q}, \ldots, z_d^{1/q}\rangle$ is a finite extension of $T_d = k\langle z_1, \ldots, z_d\rangle$.*

Proof: Let $\beta_1, \ldots, \beta_m$ be a basis of l over k. We show that

$$T' = \sum_{i=1}^m \sum_{j=1}^n \sum_{\mu_1=0}^{q-1} \cdots \sum_{\mu_n=0}^{q-1} T_d(\beta_i z_1^{\mu_1/q} \cdots z_n^{\mu_n/q}).$$

Let $f = \sum_\alpha a_\alpha \big(z_1^{1/q}\big)^{\alpha_1} \cdots \big(z_n^{1/q}\big)^{\alpha_n} \in T'$, with $a_\alpha \in l$ such that $a_\alpha \to 0$. Then each $a_\alpha \in l$ can be uniquely written as

$$a_\alpha = \sum_{i=1}^m a_{\alpha,i} \beta_i, \qquad a_{\alpha,i} \in k.$$

We have seen that $a_\alpha \to 0$ implies $a_{\alpha,i} \to 0$ for each i. Therefore $f = \sum_{i=1}^m f_i \beta_i$, where

$$f_i = \sum_\alpha a_{\alpha,i} \big(z_1^{1/q}\big)^{\alpha_1} \cdots \big(z_n^{1/q}\big)^{\alpha_n}, \quad i = 1, \ldots, m$$

are well defined elements of T'. But

$$\begin{aligned} f_i &= \sum_{0 \le \mu_1, \ldots, \mu_n < q} \sum_{\substack{\alpha \\ \alpha_j \equiv \mu_j \pmod q}} a_{\alpha,i} \big(z_1^{1/q}\big)^{\alpha_1} \cdots \big(z_n^{1/q}\big)^{\alpha_n} \\ &= \sum_{0 \le \mu_1, \ldots, \mu_n < q} \left(\sum_{\substack{\alpha \\ \alpha_j \equiv \mu_j \pmod q}} a_{\alpha,i} \big(z_1^{1/q}\big)^{\alpha_1 - \mu_1} \cdots \big(z_n^{1/q}\big)^{\alpha_n - \mu_n} \right) z_1^{\mu_1/q} \cdots z_n^{\mu_n/q} \end{aligned}$$

and the series in the brackets are elements of T_d. ∎

LEMMA 8.12: *Let k be a complete field of characteristic $p > 0$ and assume that $[k : k^p] < \infty$. Let q be a power of p. Let $T = k\langle z_1, \ldots, z_d\rangle$ and $T' = k^{1/q}\langle z_1^{1/q}, \ldots, z_d^{1/q}\rangle$. Then $T' = T^{1/q}$.*

Proof: (The equality takes places in some algebraically closed field K containing T' and hence also T.)

Let $i \in \mathbb{N}$. The isomorphism $k \to k^{p^i}$ given by $a \mapsto a^{p^i}$ maps $k^p \subseteq k$ onto $k^{p^{i+1}} \to k^{p^i}$, hence $[k^{p^{i+1}} : k^{p^i}] < \infty$. Therefore $[k : k^q] < \infty$. Apply the inverse of the isomorphism $k \to k^q$ to get that $[k^{1/q} : k] < \infty$.

CLAIM: $T' \subseteq T^{1/q}$. Let $f = \sum_\alpha a_\alpha z_1^{\alpha_1/q} \cdots z_n^{\alpha_n/q} \in T'$. Then $a_\alpha \in k^{1/q}$ and $a_\alpha \to 0$. Therefore $a_\alpha^q \in k$ and $a_\alpha^q \to 0$. It follows that $f^q = \sum_\alpha^q a_\alpha^q z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in T$.

CLAIM: $T^{1/q} \subseteq T'$. Let $f \in T^{1/q}$. Then $f^q \in T$, hence $f^q = \sum_\alpha a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, with $a_\alpha \in k$ and $a_\alpha \to 0$. Then $a_\alpha^{1/q} \in k^{1/q}$ and $a_\alpha^{1/q} \to 0$. Put $g := \sum_\alpha a_\alpha^{1/q} z_1^{\alpha_1/q} \cdots z_n^{\alpha_n/q} \in T'$. Then $g^q = f$. Hence $f \in T^{1/q}$. ∎

THEOREM 8.13: *The spectral norm on a reduced affinoid algebra A is equivalent to any norm which makes A a Banach algebra.*

Proof: Let $|| \ ||$ be a norm on A such that A is a Banach k-algebra. We have to show that there is $C > 0$ such that $|| \ || \le C|| \ ||_{\mathrm{sp}}$. Since all Banach norms on an affinoid algebra are equivalent, we actually have to show that A is complete with respect to $|| \ ||_{\mathrm{sp}}$.

PART A: *Reduction to an integral domain.* Let $\mathcal{P}_1, \ldots, \mathcal{P}_s$ be the minimal prime ideals of A. Each $A_i = A/\mathcal{P}_i$ is a Banach algebra with respect to the norm $|| \ ||_i$ induced from A (Corolllary 6.22). Assume that each A_i satisfies the assertion of the theorem. Then so does $\hat{A} = A_1 \times \cdots \times A_s$ with respect to the Banach norm $|| \ ||_{\hat{A}}$ given by $||(a_1, \ldots, a_s)||_{\hat{A}} = \max_i ||a_i||_i$. Indeed,

$$\mathrm{Sp}(\hat{A}) = \bigcup_{i=1}^{s} \{A_1 \times \cdots \times A_{i-1} \times x \times A_{i+1} \times \cdots \times A_s \mid x \in \mathrm{Sp}(A_i)\},$$

and hence $||(a_1, \ldots, a_s)||_{\mathrm{sp}} = \max_i ||a_i||_{\mathrm{sp}}$. If $||a_i||_i \le C_i||a_i||_{\mathrm{sp}}$, then $||(a_1, \ldots, a_s)||_{\hat{A}} \le C||(a_1, \ldots, a_s)||_{\mathrm{sp}}$, where $C = \max_i C_i$.

As A is reduced, the canonical map $\iota\colon A \to \hat{A}$ is injective. Its image $\iota(A)$ is a closed A-submodule of $\hat{A}$ (Theorem 2.5), and hence Banach with respect to $|| \; ||_{\hat{A}}$. Therefore it induces a Banach norm $|| \; ||_\iota$ on A by $||f||_\iota = ||\iota(f)||_{\hat{A}}$. By Theorem 6.25, $|| \; ||_\iota$ and $|| \; ||$ are equivalent. On the other hand, the restriction of the spectral norm on $\hat{A}$ to A (via ι) is the spectral norm on A. (Indeed, every maximal ideal of $\hat{A}$ restricts to a maximal ideal of A, and every maximal ideal of A contains some $\mathcal{P}_i$ and hence extends to a maximal ideal of $\hat{A}$.) Therefore the assertion for A follows from the assertion for $\hat{A}$.

By Theorem 6.18 there is a finite monomorphism $T_d \to A$.

Part B: *Reduction to: the quotient field $Q(A)$ of A is a normal extension of the quotient field $Q(T_d)$ of T_d.* Let L be a finite normal extension of $Q(T_d)$ containing $Q(A)$. There are finitely many $b_1, \dots, b_m \in L$ such that $L = Q(A)[b_1, \dots, b_m]$. Multiplying them by a suitable element of A we may assume that $b_1, \dots, b_m$ are integral over A. Then $B = A[b_1, \dots, b_m]$ is finite over A, and hence also over T_d, and the qotient field of B is L. If we can show that B is complete with respect to its spectral norm $|| \; ||$, then A is complete with respect to $|| \; ||$, by Theorem 2.5. By a home exercise, the restriction of $|| \; ||$ to A is the spectral norm on A.

Part C: *Reduction to: the quotient field $Q(A)$ of A is a separable extension of the quotient field $Q(T_d)$ of T_d.* If $\mathrm{char}(k) = 0$, there is nothing to prove. If $\mathrm{char}(k) = p > 0$, we prove the theorem only in the case $[k : k^p] < \infty$. Let M be the maximal purely inseparable extension of $Q(T_d)$ in $Q(A)$. As $Q(A)/Q(T_d)$ is normal, $Q(A)/M$ is separable [L, V.6.11].

There are $\beta_1, \dots, \beta_m \in M$ such that $M = Q(T_d)[\beta_1, \dots, \beta_s]$. Each β_i is purely inseparable over $Q(T_d)$ and hence there is a power q_i of the characteristic p such that $\beta_i^{q_i} \in Q(T_d)$. Take $q = \max_i q_i$. Then q is a power of p and $M^q \subseteq Q(T_d)$, that is, $M \subseteq Q(T_d)^{1/q}$. By an exercise (to be written down later) $Q(T_d)^{1/q} = Q(T')$, where

$$T' = k^{1/q}\langle z_1^{1/q}, \dots, z_d^{1/q}\rangle.$$

is a finite extension of $T_d = k\langle z_1, \dots, z_d\rangle$. Let A' be the compositum of T' and A (that is, the smallest ring containing both T' and A) in the algebraic closure of $Q(T')$. Then

A' is finite over T' (is generated by the finitely many generators of A over T_d) and hence over T_d, whence also over A. We have the following commutative diagrams of rings and their quotient fields

$$\begin{array}{ccc} & A & \longrightarrow & A' \\ \nearrow & & & \uparrow \\ T_d & \longrightarrow & & T' \end{array} \qquad \begin{array}{ccccc} & & Q(A) & \longrightarrow & Q(A') \\ & & \uparrow & & \uparrow \\ Q(T_d) & \longrightarrow & M & \longrightarrow & Q(T') \end{array}$$

As $Q(A)/M$ is separable and $Q(A')$ is the compositum of $Q(A)$ and $Q(T')$, the extension $Q(A')/Q(T')$ is separable. If we can show that A' is complete with respect to its spectral norm $|| \ ||$, then A is complete with respect to $|| \ ||$, by Theorem 2.5. By a home exercise, the restriction of $|| \ ||$ to A is the spectral norm on A.

PART D: *A basis of $Q(A)$ over $Q(T_d)$.* Choose a basis $e_1, \ldots, e_r$ of $Q(A)$ over $Q(T_d)$. By Lemma 8.9 we may multiply each e_i by some $0 \neq f_i \in T_d$ to assume that $f_i(T_d e_i) \subseteq A$, that is, $f_i e_i \in A$. Replace e_i by $f_i e_i$ to assume that $e_1, \ldots, e_r \in A$.

Notice that $\sum_{i=1}^s T_d e_i$ is a free T_d-module, contained in A. The standard norm $|| \ ||$ on T_d induces the 'maximum' norm on $\sum_{i=1}^s f_i e_i$ by $||\sum_{i=1}^s T_d e_i|| = \max_i ||f_i||$. It is easy to see that $\sum_{i=1}^s T_d e_i$ is complete with respect to this norm.

PART E: *The restriction of the spectral norm of A to $\sum_{i=1}^s T_d e_i$ is equivalent to the above maximum norm.* To prove this, we will be using the trace Tr: $Q(A) \to Q(T_d)$ [L, ?]. This is a $Q(T_d)$-linear operator, defined as follows: If the irreducible polynomial of $\alpha \in Q(A)$ over $Q(T_d)$ is $X^n + a_1 X^{n-1} + \cdots + a_n$, then n divides $r = [Q(A) : Q(T_d)]$ and

$$\mathrm{Tr}(\alpha) = -\frac{r}{n} a_1.$$

In particular, if $\alpha \in A$, then by Proposition 8.2, $a_1, \ldots, a_n \in T_d$, and

$$(1) \qquad ||\alpha||_{\mathrm{sp}} = \max_i ||a_i||^{1/i} \geq ||a_1|| \geq ||\overbrace{a_1 + \cdots + a_1}^{\frac{r}{n} \text{ times}}|| = ||\mathrm{Tr}(\alpha)||.$$

Furthermore, as $Q(A)/Q(T_d)$ is separable, there is a basis $e_1^*, \ldots, e_r^*$ of $Q(A)$ over $Q(T_d)$. such that $\mathrm{Tr}(e_j^* e_i) = \delta_{ij}$. As in Part D, for each j there is $0 \neq g_j \in T_d$ such that

$g_je_j^* \in A$. Replace e_j^* by $g_je_j^*$ to assume that

(2) $\qquad e_1^*, \ldots, e_r^* \in A$ is a basis of $Q(A)$ over $Q(T_d)$ and $\mathrm{Tr}(e_j^* e_i) = \delta_{ij} g_j \in T_d$.

Let $f_1, \ldots, f_r \in T_d$. Then

$$\mathrm{Tr}(e_j^* \sum_{i=1}^r f_i e_i) = \sum_{i=1}^r f_i \mathrm{Tr}(e_j^* e_i) = \sum_{i=1}^r f_i g_j \delta_{ij} = g_j f_j,$$

hence by (1)

$$||g_j|| \cdot ||f_j|| = ||g_j f_j||_{\mathrm{sp}} = ||\mathrm{Tr}(e_j^* \sum_{i=1}^r f_i e_i)||_{\mathrm{sp}} \le ||e_j^* \sum_{i=1}^r f_i e_i||_{\mathrm{sp}} \le ||e_j^*||_{\mathrm{sp}} \cdot ||\sum_{i=1}^r f_i e_i||_{\mathrm{sp}}$$

whence

$$||\sum_{i=1}^r f_i e_i|| = \max_j ||f_j|| \le \max_j \Big(\frac{||e_j^*||_{\mathrm{sp}}}{||g_j||}\Big) ||\sum_{i=1}^r f_i e_i||_{\mathrm{sp}}.$$

On the other hand,

$$||\sum_{i=1}^r f_i e_i||_{\mathrm{sp}} \le \max_i ||f_i||_{\mathrm{sp}} ||e_i||_{\mathrm{sp}} \le (\max_i ||e_i||_{\mathrm{sp}}) \max ||f_i|| = (\max_i ||e_i||_{\mathrm{sp}}) || \cdot ||\sum_{i=1}^r f_i e_i||.$$

Hence the two norms on $\sum_{i=1}^s T_d e_i$ are equivalent.

Part F: *End of the proof.* Obviously, $\sum_{i=1}^s T_d e_i$ is complete with respect to the maximum norm. By the preceding part, $\sum_{i=1}^s T_d e_i$ is complete with respect to the spectral norm of A.

By Lemma 8.9, there is $0 \ne f \in T_d$ such that $fA \subseteq \sum_{i=1}^s T_d e_i$. Therefore the T_d-submodule fA of A is complete (Theorem 2.5). But by Lemma 8.10, $||f\alpha||_{\mathrm{sp}} = ||f||_{\mathrm{sp}} \cdot ||\alpha||_{\mathrm{sp}} = ||f|| \cdot ||\alpha||_{\mathrm{sp}}$ for every $\alpha \in A$. Hence A is complete with respect to $|| \;||_{\mathrm{sp}}$.
∎

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