Solutions Algebra B 1

Moed B

2) Let G be a group whose order is p^n with $n \geq 3$. Assume that the center of G, denoted by $Z(G)$, there are p^{n-2} elements.
a) How many elements are there in a conjugacy class of some $a \in G$ which is not in $Z(G)$?
b) How many conjugacy classes the group G has?

Solution: Denote by $C_a = \{g \in G : gag^{-1} = a\}$, and let k denote its order. Then $Z(G) \subset C_a$. Hence p^{n-2} divides k, and k divides p^n which is the order of the group G. Hence $k = p^{n-2}, p^{n-1}$ or p^n. Since a is not in $Z(G)$, then $Z(G)$ is a proper subgroup of C_a. Hence $k > p^{n-2}$. If $k = p^n$ then $C_a = G$ which means that a commutes with all elements of G and hence a would be in $Z(G)$, which is not the case. Hence $k = p^{n-1}$. Since the number of elements in a conjugacy class is equal to $o(G)/o(C_a)$ we obtain that this number of elements is equal to $p^n/p^{n-1} = p$.
b) We proved in class the identity

$$o(G) = o(Z(G)) + \sum_a \frac{o(G)}{o(C_a)}$$

where the sum is over all representatives of conjugacy classes which contain more than one element. From part **a)** it follows that $o(G)/o(C_a) = p$. If the number of conjugacy classes which contain more than one element is l, the above identity is the same as $p^n = p^{n-2} + lp$. Hence, the number of conjugacy classes G has is $l + o(Z(G)) = p^{n-1} - p^{n-3} + p^{n-2}$.

3) Let G be a finite group, and let $f : G \mapsto G$ be a homomorphism without a fixed point (that is, if $f(x) = x$ for some $x \in G$, then $x = e$). Also, assume that $f(f(x)) = x$ for all $x \in G$.
a) Prove that for every $g \in G$ there is a $h \in G$ such that $g = hf(h)^{-1}$.
b) Prove that $f(g) = g^{-1}$ for all $g \in G$.
c) Prove that G is an abelian group.

Solution: a) Define a map $F : G \mapsto G$ by $F(g) = gf(g)^{-1}$. We prove that it is one to one. Indeed, suppose that $F(g) = F(h)$. Then $gf(g)^{-1} = hf(h)^{-1}$ which implies $h^{-1}g = f(h)^{-1}f(g) = f(h^{-1}g)$. Since f has no fix point, it follows that $h^{-1}g = e$ or $g = h$. Hence F is one to one. Any one to one function from a finite set to itself is onto. Hence, given $g \in G$ there is $h \in G$ such that $g = hf(h)^{-1}$.
b) Given $g \in G$ there is an $h \in G$ such that $g = hf(h)^{-1}$. Hence, using $f(f(h)) = h$ we obtain,

$$f(g) = f(hf(h)^{-1}) = f(h)f(f(h^{-1})) = f(h)h^{-1} = (hf(h)^{-1})^{-1} = g^{-1}$$

c) For $g, h \in G$ we have

$$h^{-1}g^{-1} = (gh)^{-1} = f(gh) = f(g)f(h) = g^{-1}h^{-1}$$

Hence $gh = hg$ and G is abelian.

4) a) Let G be a group and H a proper subgroup of G. Prove that $< G - H >= G$.
b) Prove that a group whose order is 300, is not a simple group.

Solution: a) It is enough to prove that if $h \in H$ then $h \in < G - H >$. Since H is a proper subgroup of G, there is an element $u \in G - H$. Then $u^{-1}h \in G - H$, for if $u^{-1}h \in H$ then it would follow from the fact that $h \in H$ that $u^{-1} \in H$ and hence $u \in H$ which is not the case. For the same reason $u(u^{-1}h) \in G - H$. But this last element is h and hence $h \in G - H$.
b) Assume that G is simple. Since $|G| = 300 = 2^2 \cdot 3 \cdot 5^2$ it follows that G has a 5 Sylow group of order 25. Let r_5 denote the number of such groups. Thus r_5 divides 12 and $r_5 \equiv 1$ mod 5. If $r_5 = 1$ then there is a unique Sylow subgroup and hence it is normal. This is a contradiction. Hence $r_5 = 6$. Let X denote the set of all 5 Sylow subgroups of G. From the above $|X| = 6$. The group G acts on X by $\varphi(g)P = gPg^{-1}$ where $g \in G$ and $P \in X$. Thus, φ produces a homomorphism from G into S_6. The kernel of φ is a normal subgroup G. Since G is simple then the kernel is G or $\{e\}$. If it G, then for all $P \in X$ and all $g \in G$ we have $gPg^{-1} = P$. Thus P is normal which is impossible. Hence the kernel of φ is $\{e\}$ but then φ is an isomorphism from G onto a subgroup of S_6. Thus 300 must divide $720 = 6! = |S_6|$, and this is a contradiction.