Boolean Algebra

History

- In 1854 George Boole introduced systematic treatment of logic and developed for this purpose an algebraic system now called ***Boolean Algebra***.
- In 1938 C. E. Shannon introduced a two-valued Boolean Algebra called ***Switching Algebra***, in which he demonstrated that this algebra can represented by electrical switching.

Chapter Outline

- Boolean Algebra (Switching Algebra)
  - Definitions
  - Basic Axioms
  - Basic Theorems
  - Representation of Boolean Functions
- Combinational Circuit Analysis
- Combinational Circuit Synthesis

Boolean Algebra Definitions

- A boolean algebraic structure consists of
  - a set of elements (constants) B = {0,1}
  - binary operations { + , • }
  - and a unary operation { }
  - such that the following **axioms** hold:

1. closure: a + b is in B | a • b is in B
2. commutative: a + b = b + a | a • b = b • a
3. associative: a + (b + c) = (a + b) + c | a • (b • c) = (a • b) • c
4. Identity: a + 0 = a | a • 1 = a
   a + 1 = 1 | a • 0 = 0
5. distributive: a + (b • c) = (a + b) • (a + c) | a • (b + c) = (a • b) + (a • c)
6. complement: a + a' = 1 | a • a' = 0

Definitions

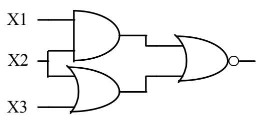
- Boolean Algebra : An algebraic structure defined with a set of elements B={0,1}, a set of binary operators (+ , . , ‘), and a number of unproved axioms.
- Symbolic variables such as X, Y , Z represent the elements. A variable can take the value “0” or “1” which corresponds to the condition of a logic signal.
- Algebraic operators :
  - Addition operator (+)
  - Multiplication operator (.)
  - Complement operator (‘)

Basic Axioms

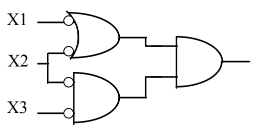
- A variable can take only one of two values{0,1}
  (A1) X= 0 if X ≠1 (A1’) X=1 if X≠0
- NOT operation (The complement Operation) :
  (A2) If X=0 then X’=1 (A2’) If X =1 then X’=0
- AND and OR operations (Multiplication and Addition) :
  (A3) 0 . 0 = 0 (A3’) 0 + 0 = 0
  (A4) 1 . 1 = 1 (A4’) 1 + 1 = 1
  (A5) 0 . 1 = 1 . 0 = 0 (A5’) 1 + 0 = 0 + 1 = 1

Generalized Demorgan's Theorem

- (T14)[F(X1,X2, ... , Xn,+, .)]'= F(X1',X2',...,Xn', . ,+)
- Example : F= (X1.X2)+(X2+X3)



F'= [(X1.X2)+(X2+X3)]'



F'= (X1'+X2').(X2'.X3')

Duality

- Duality : Every Boolean expression remains **<u>valid</u>** if the (AND, OR) operators and {0,1}elements are interchanged.
- Mathematical definition : F is a Boolean Function , FD the dual function is :
  FD(X1,X2, ... , Xn, + , . ,') def as F(X1,X2,....,Xn, . ,+ ,')
- FD $\neq$ F

Theorem

7. idempotency:
   X + X = X   X • X = X
8. involution:
   (X')' = X
9. uniting:
   X • Y + X • Y' = X   (X + Y) • (X + Y') = X
10. absorption:
    X + X • Y = X   X • (X + Y) = X
    (X + Y') • Y = X • Y   (X • Y') + Y = X + Y
11. factoring:
    (X + Y) • (X' + Z) = X • Z + X' • Y   X • Y + X' • Z = (X + Z) • (X' + Y)
12. consensus:
    (X • Y) + (Y • Z) + (X' • Z) = X • Y + X' • Z   (X + Y) • (Y + Z) • (X' + Z) = (X + Y) • (X' + Z)
13. de Morgan's:
    (X + Y + ...)' = X' • Y' • ...   (X • Y • ...)' = X' + Y' + ...
14. generalized de Morgan's:
    f'(X1,X2,...,Xn,0,1,+,•) = f(X1',X2',...,Xn',1,0,•,+)

Representation of Logic Functions

- Truth table with 2^n rows, n: the number of variables
- Definitions :
  - Literal : a variable or its complement
    Example : X , Y’
  - n- variable minterm : product term with n literals
    Example : X’.Y.Z
  - n- variable maxterm : sum term with n literals
    Example : X+Y’+Z

Truth Table

- Example : F(X, Y , Z)

Row	X	Y	Z	F	Minterms	Maxterms
0	0	0	0	0	X’.Y’.Z’	X +Y +Z
1	0	0	1	1	X’.Y’.Z	X +Y +Z’
2	0	1	0	0	X’.Y .Z’	X +Y’+Z
3	0	1	1	1	X’.Y .Z	X +Y’+Z’
4	1	0	0	0	X .Y’.Z’	X’+Y+Z
5	1	0	1	0	X .Y’.Z	X’+Y+Z’
6	1	1	0	1	X .Y .Z’	X’+Y’+Z
7	1	1	1	0	X .Y. Z	X’+Y’+Z’

Canonical Representation :

- Canonical Sum (Sum of Products- SOP):
  Sum of minterms corresponding to input combinations for which the function produces a 1 output.
  - Example :
    F = X’.Y’.Z + X’.Y.Z + X.Y.Z’
    $F_{X,Y,Z} = \Sigma(1, 3, 6)$
- Canonical Product (Product Of Sums-POS):
  Product of maxterms corresponding to input combinations for which the function produces a 0 output.
  - Example :
    F = (X+Y+Z).(X+Y’+Z).(X’+Y+Z).(X’+Y+Z’).(X’+Y’+Z’)
    $F_{X,Y,Z} = \Pi(0,2,4,5,7)$