Goodwill and Market Shares in Oligopoly

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INTRODUCTION

Economic literature treats price as the traditional competition variable. However, Schmalensee (1976, p. 493) argued that prices change infrequently, and that firms compete mainly through non-price variables. The most important non-price competition variables are advertising and product variation. Although there are many works dealing with non-price competition, they have not devoted much attention to the multi-period effect of these variables and the models have been mainly static, or a sequence of static models. (Further discussion of this argument can be found in Kydland (1977), in which dynamic dominant player models are discussed.)

In an earlier paper, Arrow and Nerlove (1962) pointed out that advertising expenditure should be treated in the same way as investment in a durable good. They assumed that there is a stock of goodwill that determines the current demand. This stock of goodwill summarizes the advertising in the past and, like capital stock, depreciates over time.

Arrow and Nerlove analysed the optimal advertising strategy for the monopolistic firm. In this paper we extend the Arrow-Nerlove model to oligopolistic competition, in which the firms compete among themselves via goodwill. The game we describe takes place over time and discounted profit is used as a criterion. The current advertising of any player affects future demand, and therefore the model has an intertemporal dependence structure.

Extending an assumption by Schmalensee (1978), in Section I we allow market shares to depend on the firm’s goodwill and not merely on current advertising. Using this definition of market share, we discuss in Section II the optimal advertising policy for a single firm. In Section III we prove the existence of a unique stationary equilibrium point for oligopolistic competition. Since we allow firms to have different cost functions, the equilibrium point is not a symmetric one; i.e. different firms have different market shares and different goodwill at the equilibrium point. In Section IV we discuss this asymmetric solution by explaining the relation between production cost and market shares. The impact of the number of firms on the advertising expenditure is discussed in Section V. We prove that the firms’ advertising expenses tend to decline as the number of firms increases. However, if one of the firms is a leading firm in the industry and has a market share above one-half, we show that this firm might increase its advertising as other firms enter into the industry. As an example, we discuss in Section VI the duopoly case and offer an explicit statement of the market shares at the equilibrium point. In Section VII we analyse an industry consisting of identical firms, and explore the influence of the interest rate and the depreciation rate on the firms’ goodwill and advertising policy.
I. THE MODEL

Consider a market in which there are $N$ non-identical sellers indexed by $i$. Let $G(t) = \{G_1(t), \ldots, G_N(t)\}$ be the firms’ stock of goodwill at time $t$, and let $a(t) = \{a_1(t), \ldots, a_N(t)\}$ be the advertising expenditures at time $t$. We assume that the stocks of goodwill accumulate according to the Nerlove–Arrow capital accumulation equation:

$$\frac{dG(t)}{dt} = \dot{G}(t) = a(t) - \delta G(t)$$

where $\delta$ is a constant proportional depreciation rate.

Let $S_i(t)$ denote the $i$th firm’s market share at time $t$. We assume that the firms’ market shares are functions of the firms’ goodwill. For every $G(t) = \{G_1(t), \ldots, G_N(t)\}$ such that $\sum_j G_j(t) > 0$ it is assumed that the firms’ market shares are:

$$S_i(t) = \frac{G_i^\alpha(t)}{\sum_{j=1}^{N} G_j^\alpha(t)}$$

where $\alpha$ is a non-negative constant. The larger $\alpha$ is, the more the demand is influenced by goodwill. If $\alpha = 0$ the consumers ignore the goodwill and choose a brand in a random way. In this case market shares will be $S_i = 1/N$ for every $i$ and sellers will not advertise.

This assumption is similar to that in Schmalensee (1978), in which it is assumed that $\phi$ is the probability that any buyer who becomes dissatisfied with any brand in period $t$ will purchase brand $i$ in period $t + 1$. Assuming no differences in the products’ quality and a large number of consumers, according to Schmalensee the market shares are

$$S_i = a_i^\alpha \sum_j a_j^\alpha.$$  

The difference between Schmalensee’s assumption and the assumption that is made in this work is that we assume that goodwill, and not the current advertising, determines the market shares.

For $\alpha > 1$ equation (1) implies that there are economies of scale in goodwill. Firms with twice as much goodwill as their rivals will have market shares more than double that of their rivals. Thus, like Schmalensee we assume that $0 < \alpha < 1$, which means that consumers respond positively to goodwill but that there are diminishing returns to goodwill. (In other words, $S_i(t)$ is assumed to be a concave function of $G_i(t)$.)

Given the market price $p$ and the market share at time $t$, the rate at which sales are made ($q_i(t)$) is

$$q_i(t) = S_i(t) \phi(p) \quad i = 1, \ldots, N$$

where $\phi(p)$ is the market demand function.
Let \( C_i(q_i(t)) \) be the \( i \)th firm's production cost. We assume in this work constant returns to scale, so \( C_i(q_i(t)) = m_i q_i(t) \) where \( m_i \) is a positive constant and \( m_i < p \) for every \( i \). Under these assumptions the single-firm instantaneous profit function can be written as

(3) \[ \pi_i(S_i(t), a_i(t)) = (p - m_i)\phi(p)S_i(t) - a_i(t) \quad i = 1, \ldots, N. \]

The advertising path that maximizes the expected present value of the firm's profits is a function of the expected goodwill path of the rivals. Therefore, in dealing with the multi-firm problem we will adopt the open-loop equilibrium concept. Each firm maximizes its discounted profits, given the goodwill paths of its competitors. These paths are also optimal and are correctly anticipated by each firm.

Under these assumptions, the aim of each firm is to maximize its discounted profit:

(4) \[ \max \int_0^\infty e^{-rt}\{(p - m_i)\phi(p)S_i(t) - a_i(t)\} \ dt \]

subject to

\[ \dot{G}_i(t) = a_i(t) - \delta G_i(t); \]
\[ G(0) = G_0 \quad \text{(the initial level of goodwill is given)}; \]
\[ G_j(t) \text{ are given for every } j \neq i; \]
\[ a_i(t) \geq 0 \]

where \( r \) is the discount rate.

The market can now be described as \( N \) non-identical firms, seeking to maximize their discounted profits (4). In order to find the equilibrium point we must investigate the single firm's optimal policy.

**II. THE SINGLE FIRM'S OPTIMAL POLICY**

**Proposition 1.** For every given \( 0 < S_i < 1 \), there is a unique optimal stock of goodwill \( \hat{G}_i \).

**Proof.** The maximization problem (4) can be solved directly using the optimal control theory. The current-value Hamiltonian associated with this problem is:

(5) \[ H(a_i, G_i, S_i, \lambda) = \{(p - m_i)\phi(p)S_i(t) - a_i(t)\} + \lambda(t)\{a_i(t) - \delta G_i(t)\}. \]

Since the Hamiltonian is a linear function of \( a_i \), it is well known that when \( \partial H / \partial a_i > 0 \) the optimal policy is \( a_i = \infty \), and when \( \partial H / \partial a_i < 0 \) the optimal policy is \( a_i = 0 \). So it remains to investigate the optimal policy for \( \partial H / \partial a_i = 0 \). The other equations that must be satisfied by the optimal policy are:

(6) \[ \frac{\partial H}{\partial G_i} = \left\{(p - m_i)\phi(p)\frac{\partial S_i}{\partial G_i}\right\} - \lambda \delta = -\dot{\lambda} + r\lambda \]

and

(7) \[ \dot{G}_i = a_i - \delta G_i. \]
The condition $\partial H/\partial a_i = 0$ implies that $\lambda(t) = 1$ and thus $\dot{\lambda}(t) = 0$. Equation (6) can therefore be written after simple manipulations as:

$$ (p - m_i) \phi(p) \frac{\partial S_i}{\partial G_i} = r + \delta. $$

Differentiating (1) yields

$$ \frac{\partial S_i}{\partial G_i} = \frac{\alpha G_i^{\alpha - 1} \sum_{j \neq i} G_j^\alpha}{\left( \sum_{j=1}^N G_j^\alpha \right)^2} = \frac{\alpha S_i(1 - S_i)}{G_i}. $$

Therefore, using equations (8) and (9) the optimal stock of goodwill $\hat{G}_i$ can be described as a function of the current market share $S_i(t)$:

$$ \hat{G}_i(S_i) = (p - m_i) \phi(p) \frac{\alpha S_i(1 - S_i)}{r + \delta}. $$

Since at time $t$ the market share $S_i(t)$ is given, the optimal advertising policy for the $i$th firm can be summarized as:

$$ a_i(t) = 0 \quad \text{if } G_i(t) > \hat{G}_i \{S_i(t)\}, $$

$$ a_i(t) = \delta \hat{G}_i \{S_i(t)\} \quad \text{if } G_i(t) = \hat{G}_i \{S_i(t)\}, $$

$$ a_i(t) = \infty \quad \text{if } G_i(t) < \hat{G}_i \{S_i(t)\}. $$

The optimal advertising policy derived from this model has the same characteristics as the optimal policy described by Arrow and Nerlove (1962, p. 135).

III. Market Equilibrium

Consider an industry in which there are $N$ non-identical firms, each of which is faced with the problem described in Section II. In this section we will prove the existence and uniqueness of an open-loop stationary Nash equilibrium point for this differential game.

Denote $(S^*, G^*)$ as the stationary Nash equilibrium point. There are then two conditions that must be satisfied:

(i) For every firm $i$ equation (10) must be satisfied; i.e. every firm has its optimal stock of goodwill according to its market share $S_i^*$:

$$ G_i^* = (p - m_i) \phi(p) \frac{\alpha S_i^*(1 - S_i^*)}{r + \delta} \quad i = 1, \ldots, N. $$

(ii) $(S^*, G^*)$ must satisfy the definition of market share:

$$ S_i^* = \frac{G_i^{*\alpha}}{\sum_{j=1}^N G_j^{*\alpha}} \quad i = 1, \ldots, N. $$

Every $(S^*, G^*)$ that satisfies equations (12) and (13) is a stationary equilibrium point. Each firm has its optimal stock of goodwill according to its market share.
and the stock of goodwill remains unchanged, as for each firm the optimal advertising policy is \( a_i^* = \delta G_i^* \) and therefore \( \dot{G}_i^* = 0 \).

**Proposition 2.** Under the assumptions of the model, there is a unique stationary equilibrium point \((S^*, G^*)\).

**Proof.** For every \( i \in \{1, \ldots, N\} \) define a constant \( k_i > 0 \) such that

\[
(14) \quad k_i = (p - m_i) \phi(p) \frac{\alpha}{r + \delta} \quad i = 1, \ldots, N.
\]

For every \( c \in \mathbb{R} \) we define a point \((\tilde{S}, \tilde{G}) \in \mathbb{R}^{2N}\) such that

\[
(\tilde{S}, \tilde{G}) = ((\tilde{S}_1, \tilde{G}_1), \ldots, (\tilde{S}_N, \tilde{G}_N))
\]

and

\[
(15) \quad \tilde{G}_i = k_i c \tilde{G}_i^\alpha (1 - c \tilde{G}_i^\alpha) \quad i = 1, \ldots, N
\]

\[
(16) \quad \tilde{S}_i = c \tilde{G}_i^\alpha \quad i = 1, \ldots, N.
\]

In order to prove that this is a good definition, we have to show that equation (15) and (16) define, for every \( i \), a unique pair \((\tilde{S}_i, \tilde{G}_i)\) that differs from \((0, 0)\).

For every \( G_i \neq 0 \), equation (15) can be written as

\[
(17) \quad G_i^{1 - \alpha} + k_i c^2 G_i^\alpha = k_i c.
\]

Since the left-hand expressions is increasing in \( G_i \) (accepting values below and above \( k_i c \)), there is a unique solution in \( G_i \). Moreover, it is clear that the solution of (17) is such that \( c \tilde{G}_i^\alpha < 1 \) and therefore \( \tilde{S}_i < 1 \).

Notice that for every \( i \) the pair \((\tilde{S}_i, \tilde{G}_i)\) satisfies equation (12); in order to prove the existence of an equilibrium point it remains to find such \( c^* \) that defines \((\tilde{S}, \tilde{G})\) that satisfies equation (13).

According to Appendix 1, \( \tilde{S}_i = 1, \ldots, N \) is a continuous monotonically increasing function of \( c \).

Let \( m(c) = \sum_{j=1}^N \tilde{S}_j(c) \); then \( m(c) \) is a continuous monotonically increasing function of \( c \) (as a final sum of such functions). For

\[
c_i = k_i^{-\alpha} \left( \frac{1}{N} \right)^{1-\alpha} \left( \frac{N-1}{N} \right)^{-\alpha}
\]

the appropriate \( \tilde{S}_i(c_i) \) equals \( 1/N \). Choosing \( c = \min \{c_1, \ldots, c_N\} \) assures us that for \( c < c \) we get \( \tilde{S}_i(c) < 1/N \) for every \( i \). Therefore for \( c < c \), we get

\[
(18) \quad m(c) = \sum_{j=1}^N \tilde{S}_j(c) < 1.
\]

On the other hand, choosing \( c = \max \{c_1, \ldots, c_N\} \) assures us that for \( c > c \), \( \tilde{S}_i(c) \geq 1/N \) for every \( i \); therefore for \( c > c \) we get \( m(c) \geq 1 \). From the continuity and monotonicity of \( m(c) \), it is clear (according to the intermediate value theorem) that there is a unique \( c^* \) such that \( m(c^*) = 1 \).

Denote the point that \( c^* \) defines as \((S^*, G^*)\). Since \((S^*, G^*)\) satisfies condition (12), we need only to show that it also satisfies condition (13) to prove that this is the equilibrium point.
Since \((S^*, G^*)\) satisfies equation (16), \(S^*_i = c^* G^*_i\) for every \(i\); therefore
\[
\sum_{j=1}^{N} S^*_j = c^* \sum_{j=1}^{N} G^*_j.
\]

However, since
\[
\sum_{j=1}^{N} S^*_j = m(c^*) = 1
\]
we can conclude that
\[
c^* \sum_{i=1}^{N} G^*_j = 1
\]
and therefore
\[
c^* = 1 / \sum_{j=1}^{N} G^*_j,
\]
which implies that
\[
S^*_i = c^* G^*_i = \frac{G^*_i}{\sum_{j=1}^{N} G^*_j} \quad i = 1, \ldots, N.
\]
Therefore condition (13) is met and \((S^*, G^*)\) is the unique equilibrium point. Q.E.D.

From equation (11) the advertising policy at the equilibrium point is
\[
a^*_i(t) = \delta G^*_i, \quad i = 1, \ldots, N.
\]
Therefore for each firm the advertising at the equilibrium point is bounded from above by \(\bar{a}_i = \delta(k_i/4)\) and the goodwill of each firm at the equilibrium point is bounded from above by \(\bar{G}_i = k_i/4\).

IV. The Production Cost, Advertising and Market Share

The equilibrium point discussed in Section II is not a symmetric one. Different firms get different market shares at the equilibrium point. Moreover, the advertising expenditures at the equilibrium point are not identical for all firms. The only difference between firms is the production cost. Therefore, although the market shares are assumed to be determined by the vector of goodwill, the cause of the asymmetric equilibrium must be differences in the cost function.

**Proposition 3.** There is a systematic connection between the production cost and the market shares at the equilibrium point. Firms that have a lower production cost will have a higher market share and will advertise more extensively.
**Proof.** Assume that there are two firms \((i, j)\) which have different production costs, for example \(m_1 > m_j\). Using the definition of \(k_i\) (equation (14)), this difference implies that \(k_j > k_i\).

Since at the equilibrium point

\[
k_i = \frac{G_i^*}{c^*G_i^{*\alpha}(1 - c^*G_i^{*\alpha})}
\]

for every \(i\) (see equation (15)), the inequality \(k_j > k_i\) implies that at the equilibrium point

\[
\frac{G_j^*}{c^*G_j^{*\alpha}(1 - c^*G_j^{*\alpha})} > \frac{G_i^*}{c^*G_i^{*\alpha}(1 - c^*G_i^{*\alpha})}
\]

which can be rewritten as

\[
G_j^{*\alpha - 1}(1 - c^*G_j^{*\alpha}) > G_i^{*\alpha - 1}(1 - c^*G_i^{*\alpha}).
\]

Since \(\alpha < 1\), inequality (20) holds iff \(G_j^* > G_i^*\), which directly implies that \(S^*_j > S^*_i\). Therefore firm \(j\) will have the higher market share and will advertise more extensively. Q.E.D.

This result is intuitively appealing: since the firm that has the lower production cost will profit more from any unit that it sells, it therefore has incentive to advertise more, in order to increase the number of units it sells.

**V. Goodwill and the Number of Firms**

In the previous sections we investigated the existence of an equilibrium point in an industry consisting of a fixed number of firms. In this section we explore the effect of an increase in the number of firms on the firms’ market share and advertising expenses.

For every \(c \in R\), equations (15) and (16) define a unique \(\bar{S}(c)\); this definition does not depend on the number of firms. The existence of an equilibrium point was proven by finding \(c^*\) such that \(m(c^*) = 1\) when \(m(c)\) is defined as \(m(c) = CEl S_j(c)\).

Since for every \(c\), an increase in the number of firms from \(N\) to \((N + 1)\) will cause an increase in \(m(c)\). Since \(m(c)\) is a monotonically increasing function of \(c\), \(m(c)\) is now one at a smaller value of \(c^*\) than before.

According to Appendix 1, \(S^*_j\) is a decreasing function of \(c^*\); thus we can conclude that as a result of an entry there is a reduction in the market shares of all the firms.

According to Appendix 2, this decrease in \(c^*\) will reduce the optimal stock of goodwill for at least \((N - 1)\) firms. Since at the equilibrium point the optimal advertising is \(a_i^* = \delta G_i^*\), the increase in the number of firms will be followed by a decrease in the advertising expenses by at least \((N - 1)\) firms. If prior to the entry \(S_i = \frac{1}{2}\) for every \(i\), then all the firms will reduce their advertising
expenses; but, as described in Appendix 2, if one of the firms is a leading firm in the industry and has a market share greater than one-half, this firm may increase its advertising expenditure as a response to the entry. However, as discussed above, this increase in the advertising expenditure is not big enough to keep the leading firm’s market share.

VI. THE DUOPOLY CASE

Consider an industry in which there are two firms. Let \( m_i \) \((i = 1, 2)\) denote their constant marginal cost. Assume that there is a constant market price \( p \), and the firms use advertising in order to compete with their rivals. According to proposition 1, for every given \( S_i \) there is a unique optimal stock of goodwill \( \hat{G}_i \) such that

\[
(21) \quad \hat{G}_i = k_i S_i (1 - S_i) \quad i = 1, 2
\]

where \( k_i \) is defined in (14).

**Proposition 4.** The market shares at the duopoly stationary equilibrium point are

\[
S_i^* = \frac{k_i^\alpha}{k_1^\alpha + k_2^\alpha} \quad i = 1, 2.
\]

**Proof.** According to (21), if \( S_i^* = k_i^\alpha / (k_1^\alpha + k_2^\alpha) \), then the firm’s optimal stock of goodwill must be equal to

\[
(22) \quad G_i^* = k_i S_i^* (1 - S_i^*) = k_i \left( \frac{k_1^\alpha k_2^\alpha}{k_1^\alpha + k_2^\alpha} \right) \quad i = 1, 2.
\]

In order to prove that \((S^*, G^*)\) is the equilibrium point, it is sufficient to show that

\[
S_i^* = \frac{G_i^\alpha}{G_1^\alpha + G_2^\alpha} \quad i = 1, 2.
\]

By using (22), we get

\[
(23) \quad \frac{G_i^\alpha}{G_1^\alpha + G_2^\alpha} = \frac{k_i^\alpha \left\{ \frac{k_1^\alpha k_2^\alpha}{(k_1^\alpha + k_2^\alpha)^2} \right\}^\alpha}{(k_1^\alpha + k_2^\alpha) \left\{ \frac{k_1^\alpha k_2^\alpha}{(k_1^\alpha + k_2^\alpha)^2} \right\}^\alpha} = \frac{k_i^\alpha}{k_1^\alpha + k_2^\alpha} = S_i^* \quad i = 1, 2.
\]

\((S^*, G^*)\) satisfies the two equilibrium conditions; therefore the market shares at the equilibrium point are

\[
S_i^* = \frac{k_i^\alpha}{k_1^\alpha + k_2^\alpha} \quad i = 1, 2 \quad \text{Q.E.D.}
\]
VII. Identical Firms

Using the first-order condition (8), it can be easily proved that if an industry consists of \( N \) identical firms the equilibrium point is a symmetric one; namely, the market shares at the equilibrium point are equal to \( 1/N \) and all the firms have the same stock of goodwill. Therefore, according to (12), the stock of goodwill at the equilibrium point is

\[ G^*_i = (p-m)\phi(p)\frac{1}{r+\delta} \frac{1}{n} \left(1 - \frac{1}{n}\right) \quad i = 1, \ldots, N. \]

By using this explicit expression of \( G^*_i \) we can investigate the effect of changes in \( r, \alpha, \delta \) and \( m \) on the firms' advertising and goodwill at the equilibrium point:

\[
\begin{align*}
\frac{\partial G^*_i}{\partial m} &= -(p-m)\phi(p)\frac{1}{r+\delta} \frac{1}{N} \left(1 - \frac{1}{N}\right) < 0 \\
\frac{\partial G^*_i}{\partial \alpha} &= (p-m)\phi(p)\frac{1}{N(r+\delta)} \left(1 - \frac{1}{N}\right) > 0 \\
\frac{\partial G^*_i}{\partial \delta} &= -(p-m)\phi(p)\frac{1}{(r+\delta)^2} \frac{1}{N} \left(1 - \frac{1}{N}\right) < 0 \\
\frac{\partial G^*_i}{\partial r} &= -(p-m)\phi(p)\frac{1}{(r+\delta)^2} \frac{1}{N} \left(1 - \frac{1}{N}\right) < 0
\end{align*}
\]

Using equations (25)–(28), we can conclude that, in the symmetric case:

(a) an increase in the production cost will be followed by a reduction in the firms’ goodwill and advertising expenditures;
(b) the stock of goodwill (and therefore the advertising) at the equilibrium point tends to decrease as the interest rate or the depreciation rate increases.

VIII. Summary

It is commonly accepted that the influence of some non-price competition variables on demand extend beyond the current period. As advertising is an important non-price competition variable, we have analysed some dynamic aspects of advertising in oligopoly. By using the concept of goodwill, as introduced by Arrow and Nerlove (i.e. accumulated advertising, subject to depreciation), we described an industry in which the firms’ goodwill determine their market shares; therefore firms can change their market shares by advertising. The paper generalizes existing works in two ways: The Arrow–Nerlove model is extended by allowing an oligopoly; and Schmalensee’s argument is extended by allowing market shares to depend on a measure of cumulated advertising expenditures with depreciation (i.e. goodwill), and not merely on current advertising.

We derived the optimal advertising policy for an individual firm in an oligopolistic market and proved that for each market share there is an optimal level of goodwill. Since the equilibrium point is not a symmetric one, we analysed the connection between production cost and market shares at the equilibrium point. We proved that firms that have a lower production cost will obtain a higher market share.
Analysing the equilibrium point, we found that, as the number of firms increases, the firms' goodwill and advertising expenditures at the equilibrium point tend to decrease. However, if one of the firms in this industry has a market share greater than one-half, this firm may increase its advertising when entry occurs.

By restricting the model to the case in which all the firms are identical, we proved that in the symmetric case the firms' goodwill and advertising expenditures tend to decrease as the interest rate, the depreciation rate or the production cost increases.

Finally, this work provides a plausible framework for analysing dynamic non-price competition in which the variables' impact extends beyond the current period. This is because goodwill can be generalized by a state variable that summarizes the current impact of the non-price variables in the previous periods.

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APPENDIX 1

\( \tilde{S}_i \text{ as an increasing function of } c \)

From equations (15) and (16) we obtain that

\[(A1) \quad k_i \tilde{S}_i (1 - \tilde{S}_i) - c^{-1/\alpha} \tilde{S}_i^{1/\alpha} = 0.\]

Therefore

\[(A2) \quad \frac{d \tilde{S}_i}{dc} = -\frac{\frac{1}{\alpha} c^{-(1/\alpha)-1} \tilde{S}_i^{1/\alpha}}{k_i(1-2\tilde{S}_i) - \frac{1}{\alpha} c^{-1/\alpha} \tilde{S}_i^{(1/\alpha)-1}}\]

The numerator in (A2) is positive. Thus in order to prove that \(d \tilde{S}_i / dc > 0\), it is sufficient to prove that the denominator is negative.

Using (A1), we can conclude that

\[k_i(1-2\tilde{S}_i) - \frac{1}{\alpha} c^{-1/\alpha} \tilde{S}_i^{(1/\alpha)-1} = k_i(1-2\tilde{S}_i) - \frac{1}{\alpha} k_i(1-\tilde{S}_i)\]

\[= k_i(1-\tilde{S}_i) \left( 1 - \frac{1}{\alpha} \right) - k_i \tilde{S}_i < 0\]

which is negative since \(\alpha < 1\) and \(\tilde{S}_i < 1\).

APPENDIX 2

*Goodwill and a reduction in c*

For every \(c \in R\), \(\tilde{G}_i\) is defined by equation (17) as

\[(A3) \quad \tilde{G}_i^{1-\alpha} + k_i c^2 \tilde{G}_i^\alpha = k_i c.\]
Differentiating (A3) yields
\[ \frac{\partial \hat{G}_i}{\partial c} = \frac{2k_i c \hat{G}_i^\alpha - k_i}{(1 - \alpha) \hat{G}_i^\alpha + \alpha k_i c^2 \hat{G}_i^\alpha - 1}. \]  

The denominator in (A4) is positive; therefore the sign of \( \frac{\partial \hat{G}_i}{\partial c} \) is the opposite sign of the numerator. Since \( \hat{S}_i = c \hat{G}_i^\alpha \), the numerator can be written as \( (2k_i \hat{S}_i - k_i) \), which will be positive for \( \hat{S}_i > \frac{1}{2} \) and negative for \( \hat{S}_i < \frac{1}{2} \). Therefore
\[ \frac{\partial \hat{G}_i}{\partial c} > 0 \text{ for } \hat{S}_i < \frac{1}{2} \]
\[ \frac{\partial \hat{G}_i}{\partial c} < 0 \text{ for } \hat{S}_i > \frac{1}{2} \]

If we compare \( \hat{G}_i(c_1) \) and \( \hat{G}_i(c_2) \) when \( c_1 > c_2 \), then
\[ \hat{G}_i(c_1) - \hat{G}_i(c_2) = \int_{c_2}^{c_1} \frac{\partial \hat{G}_i(c)}{\partial c} \, dc. \]

Using Appendix 1, it is clear that if \( \hat{S}_i(c_1) \leq \frac{1}{2} \), then for every \( c_2 \leq c \leq c_1 \), \( \hat{S}(c) \leq \frac{1}{2} \). Thus \( \frac{\partial \hat{G}_i(c)}{\partial c} > 0 \) for every \( c_2 \leq c \leq c_1 \), which implies (using (A6)) that \( \hat{G}_i(c_1) > \hat{G}_i(c_2) \). On the other hand, if \( \hat{S}_i(c_1) > \frac{1}{2} \), a reduction of \( c \) can be big enough so \( \hat{S}_i(c) < \frac{1}{2} \) for some \( c_2 < c < c_1 \). Therefore, since the sign of \( \frac{\partial \hat{G}_i(c)}{\partial c} \) can be changed as \( \hat{S}_i(c) \) becomes less than one-half, we cannot conclude which of the two is bigger, \( \hat{G}_i(c_1) \) or \( \hat{G}_i(c_2) \).

Since at the equilibrium point \( \sum_{i=1}^{N} \hat{S}_i(c^*) = 1 \), we know that there is at most one firm whose market share is greater than one-half; therefore we can conclude that, for at least \((N - 1)\) firms, a reduction of \( c \) causes a reduction of the optimal stock of goodwill.

REFERENCES


