Random Matrix Theory for Closed Quantum Dots with Weak Spin-Orbit Coupling

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To lowest order in the coupling strength, the spin-orbit coupling in quantum dots results in a spin-dependent Aharonov-Bohm flux. This flux decouples the spin-up and spin-down random matrix theory ensembles of the quantum dot. We employ this ensemble and find significant changes in the distribution of the Coulomb blockade peak height, in particular, a decrease of the width of the distribution. The puzzling disagreement between standard random matrix theory and the experimental distributions by Patel *et al.* [Phys. Rev. Lett. **81**, 5900 (1998)] might possibly be attributed to these spin-orbit effects.

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The spin-orbit coupling in a two-dimensional semiconductor quantum well mainly contributes through the Rashba [1] and Dresselhaus [2] terms, arising from the asymmetry of the confining potential and the lattice structure, respectively. It is much weaker than in three-dimensional semiconductors where it is induced mainly by impurities, which are absent in a high-mobility two-dimensional electron gas. The spin-orbit scattering is further suppressed if the two-dimensional system is confined to a quantum dot; estimates of the spin-flip rates were given in Ref. [3]. This fact is of great importance for future applications of quantum dots as spintronics devices. However, it was shown that spin-orbit scattering has a significant effect in the presence of an in-plane magnetic field [3–5], which explains [4] recent experiments [6].

In this Letter, we discuss another manifestation of the spin-orbit coupling in confined structures, which takes place even in the absence of appreciable spin-flip scattering. Aleiner and Fal'ko recently showed [7] that a weak spin-orbit coupling creates a spin-dependent Aharonov-Bohm flux. While this flux does not flip spins, it can change the random matrix ensemble of the quantum dot. For broken time-reversal symmetry, the spin-up and spin-down parts of the spectrum are completely uncorrelated and described by independent Gaussian unitary ensembles (GUE) [7]. The possibility of such an ensemble was raised by Alhassid [8], while the relation to the spin-orbit coupling was already suggested by Lyanda-Geller and Mirlin [9]. In the present Letter, we study the statistical distribution of the Coulomb blockade peak height in this ensemble and find the distribution to be narrowed. This might explain the discrepancy between a recent experiment by Patel *et al.* [10] and standard random matrix theory (RMT) [8,11] at low temperatures.

In Ref. [7], the free-electron Hamiltonian with Rashba and Dresselhaus spin-orbit terms was expanded to second order in the coordinates, under the assumption that $L_{1,2}/\lambda_{1,2} \ll 1$ ($L_{1,2}$: lateral dimensions of the two-dimensional quantum dot; $\lambda_{1,2}$: characteristic length scale of the spin-orbit coupling which is proportional to the inverse spin-orbit coupling strength). One obtains

$$\tilde{H} = \frac{1}{2m}\left(\vec{p} - e\vec{A} - \vec{a}_\perp \frac{\sigma_z}{2} - \vec{a}_\parallel\right)^2 + u(\vec{r}). \qquad (1)$$

Here, $u(\vec{r})$ is the (disordered) confining potential; $\vec{p} = \vec{P} - e\vec{A}$ is the kinetic momentum with the canonical momentum $\vec{P}$ and the vector potential $\vec{A} = B_z[\vec{r}\times\vec{n}_z]/2c$; $\vec{a}_\perp = [\vec{r}\times\vec{n}]/(2\lambda_1\lambda_2)$; $\vec{a}_\parallel = \frac{1}{6}\frac{[\vec{r}\times\vec{n}_z]}{\lambda_1\lambda_2}(\frac{x_1\sigma_1}{\lambda_1} + \frac{x_2\sigma_2}{\lambda_2})$; σ_i denote the Pauli matrices and B is the magnetic field in the direction [001] perpendicular to the lateral quantum dot. The coordinates x_1 and x_2 are along the directions [110] and [1$\bar{1}$0] and we neglected the Zeeman splitting term as we are interested in the behavior at relatively low magnetic fields. The term $\vec{a}_\parallel$ is responsible for spin flips, but it is of higher order in the spin-orbit coupling strength than $\vec{a}_\perp$. Thus, it is neglected in the following as we assume the spin-orbit coupling to be weak such that $\vec{a}_\perp$ dominates. The $\vec{a}_\perp$ term has exactly the same form as the vector potential $\vec{A}$ except for its spin dependence. As an electron collects an Aharonov-Bohm flux on a close path due to the vector potential $\vec{A}$, it also collects a spin-dependent flux due to $\vec{a}_\perp$. This spin-dependent flux translates to a spin-dependent effective magnetic field, so that the electrons feel a total magnetic field of strength $B_\sigma^{\text{eff}} = B + \frac{c}{e}\frac{1}{\lambda_1\lambda_2}\frac{\sigma}{2}$ with $\sigma = \pm\hbar$ for up-spin and down-spin, respectively. An increase of the flux changes the matrix elements and scrambles the eigenenergies and eigenvectors. In the absence of spin-orbit coupling, the flux is exactly the same for spin-up and spin-down electrons such that their eigenenergies and eigenvectors are degenerate. If the spin-orbit terms are present, but no external magnetic field is applied, the time-reversal symmetry is preserved, and the states are still Kramers degenerate (up-spin and down-spin see the same magnitude of magnetic field with opposite signs). However, when spin-orbit coupling and external magnetic field are present, electrons with different spin see different magnetic fields, and their eigenenergies and eigenvectors decorrelate. If the spin-dependent flux is large enough spin-up and spin-down eigenenergies and eigenvectors are distributed according to two *independent* GUEs [7].

Before we analyze this weak spin-orbit RMT ensemble, we study the decorrelation of the eigenenergies and eigenvectors due to a change in the magnetic field to determine how much flux is needed in order to have two uncorrelated ensembles. The additional flux translates to a change of the random matrix described by [12]

$$H = (H_1 + xH_2)/\sqrt{1 + x^2} \tag{2}$$

with RMT matrices H_1 and H_2 in the unitary ensemble. As the perturbation x increases from zero the eigenenergies $E_i(x)$ and eigenfunctions $\Psi_i(x)$ of H change. We analyze the decorrelations of the energies via the level diffusion correlator $C_E = \langle\langle\sqrt{[E_i(x) - E_i(0)]^2}\rangle\rangle/\Delta$, where $\langle\langle\cdots\rangle\rangle$ means averaging over different realizations and different levels i. This correlator has been shown to have a universal form [13]. The decorrelation of the eigenfunctions is measured by $C_\Gamma = \langle\langle|\langle\psi_i(x)|\psi_i(0)\rangle|^2\rangle\rangle$. It can be shown that this correlator also measures the correlations of the level tunneling rates and has a universal form as well [14]. The results are presented in Fig. 1 and show that the correlations in both quantities disappear at about the same value of $x\sqrt{N} \approx 1$, where N is the size of the random matrix. Hence we conclude that the decorrelation of the eigenvalues and the eigenfunctions (dot-lead coupling) occur together. Thus, the spin-orbit coupling leads to a crossover from two degenerate GUE spectra to an ensemble of two uncorrelated GUE spectra.

For the above RMT model, the crossover to two independent GUE ensembles occurs at $x\sqrt{N} \approx 1$. The corresponding flux difference needed to decorrelate the spectrum is given by the following relation [12]:

$$x\sqrt{N} = \chi\sqrt{g_T}\frac{\delta\Phi_{\rm eff}}{\Phi_0}, \tag{3}$$

where $\delta\Phi_{\rm eff}$ is the flux difference, Φ_0 is the quantum unit

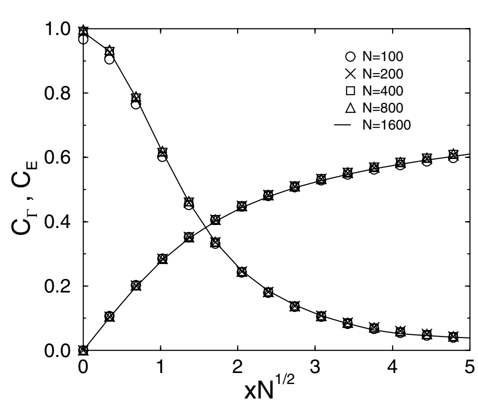

FIG. 1. Correlations of the level tunneling rates C_Γ and the rescaled spectral diffusion correlator C_E for the RMT model (2), as a function of $x\sqrt{N}$ for different matrix sizes N. The data collapse for different N indicates the universality.

of flux, g_T denotes the Thouless conductance, and χ is a nonuniversal sample-dependent constant of order unity. We thus realize that one needs about $1/\sqrt{g_T}$ flux quanta to cross over to two uncorrelated GUE ensembles.

Let us now estimate the strength of spin-orbit interaction required to create this amount of flux difference. As mentioned above, the difference in effective flux between the two spin sectors is $\delta\Phi_{\rm eff}/\Phi_0 = L_1L_2/(\lambda_1\lambda_2)$. The λ's are connected to the Rashba and Dresselhaus spin-orbit parameters γ and η via $1/\lambda_1\lambda_2 = 4(\gamma^2 - \eta^2)$. Independent estimates of γ and η are not available, but, in principle, can be obtained [7]. One can get an approximate value via the better known parameter $Q_{\rm SO}^2 = (\hbar v_F/E_F)^2(\gamma^2 + \eta^2)$:

$$\left|\frac{1}{\lambda_1\lambda_2}\right| \le Q_{\rm SO}^2\left(\frac{2E_F}{\hbar v_F}\right)^2 = Q_{\rm SO}^2 k_F^2, \tag{4}$$

$$\frac{\delta\Phi_{\rm eff}}{\Phi_0} = Q_{\rm SO}^2 k_F L_1 k_F L_2. \tag{5}$$

Typical experimental values are $k_F L_{1,2} \sim 50$, $g_T \sim$ 10–100, and estimates for $Q_{\rm SO}$ are in the range 4–16 $\times$ 10^{-3} [4]. Thus, we estimate $\sqrt{g_T}\delta\Phi_{\rm eff}/\Phi_0 = 0.1$–$6.4$; i.e., the right-hand side of Eq. (3) can be expected to be of order unity [15]. Hence, the spin-orbit effect is strong enough to decorrelate (or to start to decorrelate) the spin-up and spin-down sector, while being weak enough not to yield significant spin scattering. A strong spin scattering, which can be generated by the application of an in-plane magnetic field, would mix the two spin species and result in a single GUE.

We now analyze this situation where the weak spin-orbit coupling results in two uncorrelated GUE ensembles for spin-up and spin-down electrons and the results do not depend on the spin-orbit strength. Under the assumption of a constant Coulomb interaction [12] and applying the master equation for sequential tunneling through the quantum dot, the conductance of a closed quantum dot is given by [16] (for a review, see Ref. [8])

$$G = \frac{e^2}{k_B T}\sum_{i\sigma}\frac{\Gamma_{i\sigma}^L\Gamma_{i\sigma}^R}{\Gamma_{i\sigma}^L + \Gamma_{i\sigma}^R}P_{\rm eq}(N)P(E_{i\sigma}|N)[1 - f(E_{i\sigma})]. \tag{6}$$

Here $\Gamma_{i\sigma}^{L(R)}$ is the tunneling rate between the ith one-particle eigenlevel of the dot with spin σ and the left (right) lead, $E_{i\sigma}$ is the one-particle eigenenergy of this level, $P_{\rm eq}(N)$ denotes the equilibrium probability to find N electrons in the dot (we assume the typical experimental situation where the Coulomb blockade allows only N and $N+1$ electrons in the quantum dot), $P(E_{i\sigma}|N)$ is the canonical probability to have the ith level of the spin-σ sector occupied given the presence of N electrons in the dot, and $f(E_{i\sigma})$ is the Fermi function at an effective chemical potential μ which includes the charging energy. In Eq. (6), $\Gamma_{i\sigma}^{L(R)}$ is distributed according to the Porter-Thomas distribution for the GUE $P_2(\Gamma) = \frac{1}{\overline{\Gamma}}\exp(-\Gamma/\overline{\Gamma})$, which depends only on the mean value $\overline{\Gamma}$ of the distribution (we assume this mean value to be the same for the coupling to the left and right lead in the following).

At zero temperature, only one level (i_1, σ_1) contributes in Eq. (6) such that $\mu = E_{i_1\sigma_1}$, $P(E_{i\sigma}|N) = 1$, and $P_{\mathrm{eq}}(N) = 1/2$. Thus, the zero temperature average conductance is given by $\langle G\rangle = \frac{1}{12}\frac{\hbar\bar{\Gamma}}{k_B T}\frac{e^2}{\hbar}$ and the ratio of standard deviation to mean value becomes $\sigma(G)/\langle G\rangle = 2/\sqrt{5}$. Here, we have used $\langle\Gamma^R_{i\sigma}/(\Gamma^L_{i\sigma} + \Gamma^R_{i\sigma})\rangle = 1/3$ and $\langle(\Gamma^R_{i\sigma})^2/(\Gamma^L_{i\sigma} + \Gamma^R_{i\sigma})^2\rangle = 1/5$ for the GUE distribution. At low temperatures, there are a few realizations of the RMT eigenlevel distribution where a second level (i_2, σ_2) is within an interval of order $k_B T$ around the first level at the Fermi energy. Then, the second level also contributes to the conductance through the quantum dot. Neglecting the shift of the chemical potential due to the second level (i.e., keeping $\mu = E_{i_1\sigma_1}$), we calculated this two level situation. This gives the leading behavior in $k_B T/\Delta$ for Eq. (6):

$$\frac{\sigma(G)}{\langle G\rangle} = \frac{2}{\sqrt{5}}\left(1 + \left[\frac{781}{9}\ln 2 - \frac{127}{3}\ln 3 - \frac{409}{27}\right]\frac{k_B T}{\Delta}\right). \quad (7)$$

For general temperatures, we maximized numerically the conductance Eq. (6) with respect to μ and averaged over 100 000 RMT realizations of the eigenenergies and the dot-lead couplings. The results are shown in Fig. 2, in comparison to the experiment of Patel *et al.* In contrast to the standard RMT result [10], the RMT ensemble for weak spin-orbit coupling describes the width of the conductance distribution and its change with temperature reasonably well at low temperatures, without any adjustable parameter. Compared to the standard GUE, the width of the distribution is reduced at low temperatures because of the absence of level repulsion for levels with opposite spin. This results in a higher probability to find a close-by level (with opposite spin and independent tunneling rate), and leads to more RMT realizations in which two or more levels contribute to the low-temperature conductance. Having more independent channels for the conductance makes the probability distribution more Gaussian and decreases its width. At higher temperatures, the experimental results are not adequately described by spin-orbit effects alone. In the regime $k_B T \gtrsim \Delta$, however [17], one has to account for inelastic scattering Γ_{in}. Taking the limit $\Gamma_{\mathrm{in}} \to \infty$, we obtain the high-temperature asymptotic behavior $\frac{\sigma(G)}{\langle G\rangle} = \sqrt{\frac{1}{24}\frac{\Delta}{k_B T}}$ which gives reasonable results except for the quantum dot with diamond symbols. Note that upon reducing $k_B T/\Delta$, Γ_{in} will decrease, resulting in a crossover from the dot-dashed to the dashed line in Fig. 2. The inelastic scattering rates of [17] would imply that the dashed $\Gamma_{\mathrm{in}} = \infty$ line is approached in the range $k_B T = 1.5\Delta - 4\Delta$.

In Fig. 3, we compare the full probability distribution with the experimental one [10] at $k_B T = 0.1\Delta$ and $k_B T = 0.5\Delta$. Within the experimental statistical fluctuation, good agreement is achieved without any free parameters, much better than for the standard RMT [10]. This suggests that the spin-orbit strength is sufficient to fully decorrelate the spin-up and spin-down ensembles. With an estimate of the experimental Thouless conductance $g_T \approx 20$ obtained from $g_T \approx \sqrt{N}$, this means that a spin-orbit coupling strength $Q_{\mathrm{SO}} \gtrsim 10^{-2}$ is required in the quantum dot of Ref. [10] [where we set $\chi = 1$ in Eq. (3)]. In general, the crossover to the weak spin-orbit regime occurs at $Q^2_{\mathrm{SO}}(k_F L)^{5/2} \gtrsim 1$. Thus, the size of the dot and the parameter Q_{SO} which depends on the dot's specific asymmetry of the confining potential determine whether this quantum dot is in the weak spin-orbit regime. The size dependence might explain why earlier measurements by Chang *et al.* [18] using very small quantum dots showed agreement with the standard RMT without spin-orbit interaction. A similar agreement was found by Folk *et al.* [19], despite using similarly large

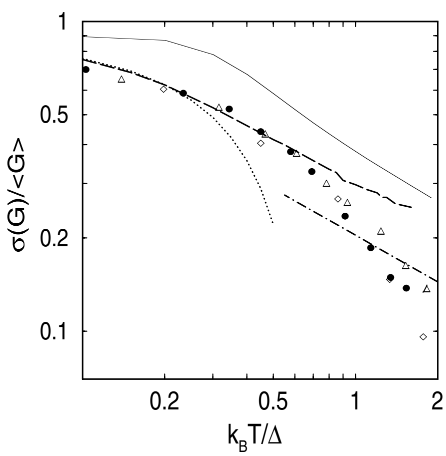

FIG. 2. Width of the conductance distribution $\sigma(G)/\langle G\rangle$ vs temperature. At low temperatures, the RMT ensemble for weak spin-orbit interaction [dashed line; dotted line: low temperature behavior Eq. (7)] well describes the experiment [10] (symbols correspond to slightly different quantum dots), in contrast to standard RMT (solid line) [10]. At higher temperatures, a further suppression is due to inelastic scattering processes (dot-dashed line: $\Gamma_{\mathrm{in}} = \infty$ high-T asymptote).

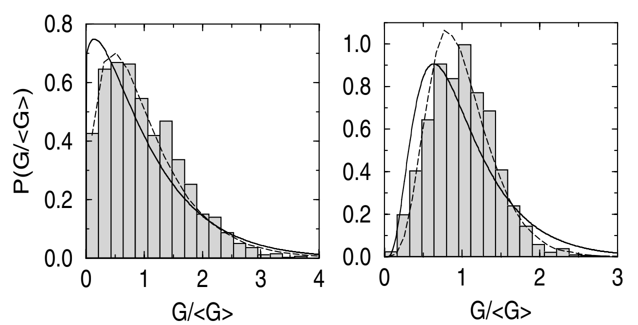

FIG. 3. RMT predictions with weak spin-orbit coupling (dashed line) for the probability distribution of the Coulomb blockade peak conductance for a quantum dot at $k_B T = 0.1\Delta$ (left) and $k_B T = 0.5\Delta$ (right), compared with the Patel *et al.* experiment [10] (histograms) and standard RMT theory [10] (solid line). There are no free parameters in these distributions.

quantum dots as in Ref. [10]. The contradictory results of Refs. [10,19] might be due to the better statistics of the latter experiment, or could be explained within the framework presented here: Possible differences in the confining potential (which might be, e.g., caused by differences in the realization of the two-dimensional electron gas and the gate voltage) translate into differences in Q_{SO}. Alternatively, it is possible that the spin-orbit effect in both samples is weak, and the deviations from RMT in [10] should be explained by another mechanism (e.g., exchange [20]).

In order to validate that the quantum dot is indeed in the weak spin-orbit coupling regime described here, we suggest to repeat the experiment with an in-plane magnetic field. A strong in-plane magnetic field should drive the system towards the strong spin-orbit scattering limit. In general, one would expect the spin-orbit scattering to suppress $\sigma(G)/\langle G\rangle$. However, in the case of weak spin-orbit coupling the in-plane magnetic field, which drives the system towards a single GUE, regenerates the level repulsion. Therefore, we predict $\sigma(G)/\langle G\rangle$ to increase upon applying an in-plane magnetic field at low temperatures. Another crucial test to the the weak-spin orbit scenario is the behavior in the absence of a magnetic field. In this case, the degeneracy is preserved but the spin-orbit coupling drives the system from the Gaussian orthogonal to the unitary ensemble [7]. One implication is a strong suppression of the magnetoconductance.

Finally we note that the disagreements between RMT predictions and the results of [10] cannot be attributed to dephasing. Had this been the case, this experiment would indicate an appreciable dephasing even at low temperatures, in contradiction with theoretical predictions [21]. However, recent measurements of the low-temperatures dephasing rates [22] are consistent with theory [17], and furthermore, it has been shown by Rupp and Alhassid [23] that dephasing alone cannot explain the results of [10]. Our calculation shows that the spin-orbit coupling without dephasing can describe the low-temperature part of [10] and that the inclusion of strong dephasing gives reasonable agreement for the high-temperature part.

In conclusion, we analyzed the effect of weak spin-orbit coupling on closed quantum dots in the presence of a perpendicular magnetic field which breaks the time-reversal symmetry. In this regime which can be realized for (some) quantum dots, the spin-orbit coupling does not lead to one nondegenerate GUE ensemble but to two independent GUEs for spin-up and spin-down electrons. This has important consequences, in particular, at low temperatures, as there is no level repulsion for levels with opposite spins. The statistical distribution of the conductance peak maximum shows a good agreement with recent experimental distributions by Patel *et al.* [10] but disagrees with experiments for similarly sized quantum dots [19]. The exchange interaction might yield similar changes in the statistical distribution [20], and it is unclear at present whether the complete explanation for the peak heights statistics behavior is given by the weak spin-orbit RMT. More experiments are needed to clarify the relative importance of the two effects and to explain the experimental contradiction mentioned above. If the spin-orbit effect is dominant, we predict an increase of the width of the distribution upon applying a strong in-plane magnetic field and a very low magnetoconductance. We further note that without spin degeneracy there will also be *no* δ-function-like contribution in the level-spacing distribution, in contrast to standard RMT.

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