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Time, Irreversibility and Unstable Systems In Quantum Physics

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Abstract

The recently developed quantum theory utilizing the ideas and results of Lax and Phillips for the description of scattering and resonances, or unstable systems, is reviewed. The framework for the construction of the Lax-Phillips theory is given by a functional space which is the direct integral over time of the usual quantum mechanical Hilbert spaces, defined at each t. It has been shown that quantum scattering theory can be formulated in this way. The theory of Lax and Phillips, however, also obtains a simple relation between the poles of the S-matrix and the spectrum of the generator of the semigroup corresponding to the reduced motion of the resonant state. It is shown in this work that in order to obtain such a relation in the quantum mechanical case, the evolution operator must act as an integral operator on the time variable. The structure required appears naturally in the Liouville space formulation of the evolution of the state of the system. The resulting S matrix is a function, as an integral operator, of $t-t'$ (i.e., homogeneous), and the semigroup is contractive. A physical interpretation of this structure may be introduced, from which we obtain a quantitative description of the expected age of a created system, and the expected time of decay of an unstable system. The superselection rule which distinguishes between the unstable system and its decay products is realized in this way. It is also shown that from this point of view, one has a natural mechanism for the dynamical mixing of the quantum mechanical states as observed by means of time translation invariant operators. In particular, this provides a model for certain types of irreversible processes, as well as for the measurement process for closed as well as open systems.

1. Introduction

The unstable quantum system is an important example of irreversible phenomena in nature. Such systems, ranging from excited atomic states to short-lived elementary particles, are characterized by what is generally observed to be an irreversible evolution. These phenomena raise the question of explanation of such processes from first principles. Moreover, since most of the decay processes are observed experimentally to obey an exponential decay law, one expects this behavior to follow from very general assumptions.

In the following, we give some historical background of efforts to describe the unstable system. In the next section we review the main ideas of Lax-Phillips scattering theory and show how it can emerge from the quantum theory. In Section 3, we show that a general law of evolution leads to an S-matrix of Lax-Phillips form for which the singularities are associated with the spectrum of the generator of a semigroup which describes the evolution of the unstable states. In Section 4, we discuss the connection of Lax-Phillips theory to some aspects of measurement theory. In Section 5, using related ideas, we show that mixing of states can occur (even in closed systems) by a similar mechanism in the Liouville space formulation of both classical and quantum mechanics.

The description of irreversible evolution in the quantum theory has been described by the addition of non-Hermitian terms to the Hamiltonian, such that it has complex eigenvalues, and the induced evolution is non-unitary. Structures of this type were originally introduced by Gamow [1] who studied the effect of assigning complex eigenvalues to the energy spectrum, and hence introduced a kind of generalized eigenvector. Wu and Yang [2] parameterized the K-meson decay in this way. In this method, the non-Hermitian terms in the Hamiltonian are introduced phenomenologically, and may only indirectly be associated with some known interaction terms. In addition, the interpretation of the generalized eigenvectors, and their relation to some wavefunction which describes a definite state of the system, are not at all clear.

Weisskopf and Wigner [3], in a well known paper of 1930, have introduced an alternative approach to the decay problem. According to their approach, the evolution takes place in a Hilbert space which is a direct sum of two subspaces: the subspace of the decaying states and that of decay products. These two subspaces are stable under the "free" evolution induced by H_0, but are combined linearly under the full evolution induced by $H = H_0 + V$. In this Hilbert space, the evolution is unitary, and hence its generator, i.e., the Hamiltonian, is self-adjoint. The decay is described as the probability flow from the subspace of the decaying states to its complement, the subspace of the decay products. Weisskopf and Wigner considered the most simple model, in which there is only one decaying state ψ, and a continuum of states corrsponding to the decay channel. They studied perturbatively what has become known as the *survival amplitude*

$$A(t) = (\psi, e^{-iHt}\psi)\,, \tag{1.1}$$

which is the probability amplitude for the system to remain in the discrete state until time t. Horwitz and Marchand extended this model for systems in which the unstable state subspace is two-dimensional, applied it to the decay of the K^0-meson [4], and then extended the treatment to the most general case [5]. They formulated the decay problem as the evaluation of the full unitary evolution projected into the subspace of the unstable states, and unified the mathematical treatment of this problem with that of scattering resonances. In the following we will describe this approach, and pose critical problems, motivating the development of a more general theory.

The Hilbert space which corresponds to an unstable system $\mathcal{H}$, may be represented as the direct sum of two Hilbert spaces which correspond to the state space of the decaying system, and the decay products. Let us denote the projection operators on these two subspaces P and $\bar{P}$, such that $P + \bar{P} = 1$. For the decay problem, the basic quantity is the *reduced motion*

$$U'(t) \;=\; PU(t)P\,. \tag{1.2}$$

where $U(t) \;=\; e^{-iHt}$, which governs the time evolution of the subspace $P\mathcal{H}$ of the unstable states. From this one can derive the decay law of the unstable states. If $\{\phi_i\}$ is an orthonormal basis of $P\mathcal{H}$, the probability that an unstable state ϕ, which exists at time $t = 0$, is in the subspace $P\mathcal{H}$ of unstable states at time t is given by

$$p(t) \;=\; \sum_i |(\phi_i, U(t)\phi)|^2\,. \tag{1.3}$$

Another way of writing this quantity is

$$p(t) \;=\; Tr_P(U'^{\dagger}(t)U'(t)P_\phi)\,, \tag{1.4}$$

where P_ϕ is the projection on the subspace spanned by the initial state $\phi \in P\mathcal{H}$.

The total evolution operator $U(t) \;=\; e^{-iHt}$, and the resolvent $R(z) \;=\; (z - H)^{-1}$ are related to each other by the (inverse) Laplace transform

$$U(t) \;=\; \frac{1}{2\pi i}\oint R(z)e^{-izt}dz\,, \tag{1.5}$$

where the integration contour is around the spectrum of H. If we project this operator into the subspace $P\mathcal{H}$, we can obtain a similar relation which expresses the reduced motion $U'(t)$ in terms of the *reduced resolvent* $R'(z) \;=\; PR(z)P$:

$$U'(t) \;=\; \frac{1}{2\pi i}\oint R'(z)e^{-izt}dz\,. \tag{1.6}$$

Using this relation, one can derive an evolution equation for the reduced motion, which permits us to examine the behavior of this function in different time regimes. For this purpose, let us use the identity $zR(z) \;=\; 1 + HR(z)$ and project it on the two (orthogonal) subspaces $P\mathcal{H}$ and $\bar{P}\mathcal{H}$. Doing so, we obtain the coupled system

$$\begin{aligned} zR'(z) - P \;&=\; PHR'(z) + PH\bar{P}R(z)P\,, \\ z\bar{P}R(z)P \;&=\; \bar{P}HR'(z) + \bar{P}H\bar{P}R(z)P\,, \end{aligned} \tag{1.7}$$

and eliminating $\bar{P}R(z)P$,

$$zR'(z) - P \;=\; PHR'(z) + PH\bar{P}(z - \bar{P}H\bar{P})^{-1}\bar{P}HR'(z)\,. \tag{1.8}$$

Taking the inverse Laplace transform of (1.8), and using the convolution theorem and (1.6), one obtains the following integro-differential equation for the reduced motion:

$$i\frac{d}{dt}U'(t) \,=\, PHPU'(t) - i\int_0^t d\tau PH\bar{P}e^{-i\bar{P}H\bar{P}\tau}\bar{P}HPU'(t-\tau)\,. \tag{1.9}$$

The latter equation is called the *Master Equation for the reduced motion*, and applies not only to decay systems but in general for evolutions obtained from a unitary group $\{U(t)\}$ by projection into a subspace [6]. This equation is particularly useful for small times where a solution by iteration is highly convergent [7]. In the limit of very small times, the second term of (1.9) is negligible and $U'(t)$ evolves unitarily. That the decay rate vanishes for $t \to 0$ can also be seen directly from (1.4) (provided that the Hamiltonian is defined on $P\mathcal{H}$):

$$\begin{aligned}\frac{d}{dt}p(t) &= \frac{d}{dt}Tr_P(PU^\dagger(t)PU(t)P_\phi)|_{t=0}\\ &= Tr_P(iPHU^\dagger(t)PU(t)P_\phi - iPU^\dagger(t)PHU(t)P_\phi)|_{t=0}\\ &= iTr_P(PHP_\phi - PHP_\phi) \,=\, 0\,.\end{aligned} \tag{1.10}$$

This reflects the so-called $O(t^2)$ short-time behavior, which leads to the prediction of special quantum effects such as the quantum Zeno effect [8].

If the term $\bar{P}H\bar{P}$ vanishes, one obtains an algebraically soluble model [9][10] which has been very helpful in investigating the analytic properties of the theory of unstable systems. The generalized states (elements of a Gel'fand triple [11]), providing exact exponential decay, have been studied in detail in this model [12][13].

It is not difficult to see that an irreversible evolution must be described by a semigroup [14] (for the reversible case this is a group induced by a unitary transformation). On the other hand, it can be shown generally, that the reduced motion, as described above, cannot generate a semigroup [15].

An operator family $U'(t)$ on the subspace $\mathcal{K}$ is called *of positive type and contractive*, if the following conditions hold:

$$U'^\dagger(-t) \,=\, U'(t), \quad U'(0) \,=\, I_{\mathcal{K}}\,. \tag{1.11}$$

According to the theory of extensions of Hilbert spaces [16], one may construct a Hilbert space $\mathcal{H}$ (which is an extension of $\mathcal{K}$), and a continuous group $U(t)$ of unitary operators on it, such that $U'(t)$ is the contraction of $U(t)$ to the subspace $\mathcal{K}$. Such an extension is unique if one requires $\mathcal{H}$ to be spanned by $U(t)\mathcal{K}$.

The physical interpretation of this theorem is, that knowing the subspace of unstable states and the contracted evolution in it, one can construct uniquely the Hilbert space of the unstable states and the decay products, and the generator of the evolution in this space which is the Hamiltonian.

If we require $U'(t)$ to be strongly contractive, i.e.

$$s - \lim_{t\to\infty} U'(t) \,=\, 0\,, \tag{1.12}$$

it can be shown [15][16] that $U'(t)$ may be a semigroup only if the spectrum of the Hamiltonian H is all the real line $\mathcal{R}$. When $\mathcal{K}$ is finite-dimensional one can prove further that if $U'(t)$ is a semigroup, there are states in $\mathcal{K}$ with infinite energy and for states with finite energy the decay rate vanishes at $t = 0$. These conditions cannot all be achieved for Schrödinger systems in which the Hamiltonian is semi-bounded and the subspace contains only finite energy states, hence this theorem is, in fact, a no-go theorem which proves that an exact semigroup law cannot be obtained for ordinary quantum-mechanical systems.

We have briefly discussed Weisskopf-Wigner theory, and seen the difficulties which arise. According to this theory, the decay law is not exponential, and the experimentally observed exponential law is approximated only for times not too short and not too long. Moreover, we have seen that the description of the decay process in the framework of contraction of unitary evolution, in the usual quantum mechanical Hilbert space, to a subspace, does not lead to a semigroup behavior, and therefore this is not an irreversible process in the full sense (i.e. there are regeneration effects). The $O(t^2)$ short time behavior for the contracted evolution, also indicates deviations from the semigroup law, which lead to effects such as the Quantum Zeno effect, which may occur in some physical systems; one would like to have a more general theory in which this effect may or may not occur, according to the dynamical laws.

Techniques of analytic continuation of the reduced resolvent to the second Riemann sheet, resulting in an exact exponential decay, lead to the concept of generalized vectors. The physical interpretation of these generalized states, and their relation to vectors of the Hilbert space which describe well-defined states of the system, are not at all clear. In particular, there are many ways of representing the space in which the pole is a (generalized) eigenvalue, for example, defining the space by analytic continuation through the cut in the complex energy plane, or by analytic continuation of dilations [12]. One would like to have a theory in which the states associated with the unstable system have a consistent physical interpretation.

There is, furthermore, another, perhaps more fundamental problem associated with the general method of Wigner and Weisskopf; this is that the expression (1.1) for the survival amplitude implicitly assumes the existence of a linear superposition

$$e^{-iHt}\psi \,=\, A(t)\psi \,+\, \chi(t)\,, \tag{1.13}$$

where $\chi(t)$ represents the decayed system and $(\psi, \chi(t)) = 0$. In general this linear superposition does not correspond to any physical situation in our experience; a short-lived particle, for example, is seen as either the particle before the decay, or the decay products at a certain time, which can not be predicted. This linear superposition does not correspond to the object that we see experimentally in such a process.

These features are essentially related to the attempt to describe an unstable system in a framework more suitable to the description of reversible phenomena. In what follows we will show another approach to irreversible phenomena which attempts to solve these difficulties.

2. The Quantum Unstable System: A Direct Integral Space Description

In the previous section, we have briefly discussed the traditional methods of dealing with unstable states in the framework of quantum theory, and pointed out some of the difficulties in such methods. There is, however, within the scope of quantum theory, a framework more suitable to the description of deterministic, irreversible phenomena. It is based on the use of a direct integral Hilbert space, which we describe in the following.

The characterization of a system undergoing an irreversible process cannot, in principle, be specified at a given instant of time. In fact, the physical quantities describing such processes involve time measurements (that is, measurements of the time at which certain defined phenomena occur). Therefore, the information about the decay which is to be deduced from the state is associated with its distribution in time which is an essential property of the system, just as the location or momentum of a quantal particle. The time variable is, from this point of view, an internal degree of freedom of the system, which provides a framework for the description of interactions which can influence the structure of the state. A more detailed discussion of interpretation is given in a later section.

The dynamical evolution of the system involves a change in its internal structure, including its distribution in t along with other observables characterizing the state. This evolution, parameterized by the laboratory time τ (which is not a dynamical variable), is defined on a Hilbert space with t in its measure space (e.g., with Lebesgue measure), with norm given by

$$\int \|\psi_t^\tau\|^2 dt \;=\; \|\psi^\tau\|^2$$

where the norm in the integral is taken as the norm in $\mathcal{H}_t$, a member of a family of auxiliary Hilbert spaces (all isomorphic), defined for each t.

The theory of Lax and Phillips [17], designed for systems of hyperbolic differential equations describing the scattering of, e.g., electromagnetic or acoustic waves, and the Floquet theory [18] for periodic time dependent quantum mechanical problems are examples of such a structure. Piron [14] has shown that methods of this type are applicable to the general time dependent quantum mechanical problem. Recently, Flesia and Piron [19] have shown that scattering problems in quantum theory can be put in the form of Lax-Phillips theory (Horwitz and Piron [20] have discussed its applicability to the problem of the unstable system) by forming a direct integral of the quantum mechanical Hilbert spaces $\mathcal{H}_t$ over t in order to construct a larger space $\bar{\mathcal{H}}$ which includes t in its measure space. Their approach to this kind of structure is as follows.

It is pointed out in the paper of Horwitz and Piron [20] that it has been known since 1952 [21] that physical systems cannot be always described by a single Hilbert space, but, in general, by a direct sum of Hilbert spaces for which the matrix elements of any observable between states belonging to different components vanish, i.e., there is a superselection rule between these components. There may be many superselection rules in physics. In particular, in the description of the state of a system, the time plays the role of a continuous superselection rule, and the system, therefore, may be described by a family of Hilbert spaces indexed by t [22]. It is useful to consider this family of Hilbert spaces as a functional space [23]. The Lebesgue measure is chosen for the weights of the orthogonal direct integral. Therefore, this functional space is a Hilbert space, which we shall call the direct integral space, of the form

$$\bar{\mathcal{H}} = \int_{\oplus} \mathcal{H}_t dt = L^2(-\infty, \infty; \mathcal{H}) . \tag{2.1}$$

Each component in the direct integral (2.1) is a copy of a single Hilbert space $\mathcal{H}$ (which is isomorphic to the Schrödinger quantum-mechanical Hilbert space). Scalar products in $\bar{\mathcal{H}}$ have the form

$$(f, g) = \int (f_t, g_t)_{\mathcal{H}} dt , \tag{2.2}$$

and the norm squared is

$$\|f\|^2_{\bar{\mathcal{H}}} = \int \|f_t\|^2_{\mathcal{H}} dt . \tag{2.3}$$

It is instructive to see how this method arises from a procedure in classical mechanics. Consider Hamilton's equation of motion

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} , \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q} , \tag{2.4}$$

where the Hamiltonian $H(p, q, t)$ depends on time. Defining the time as a new variable and the energy E of outside sources as its conjugate momentum, the new Hamiltonian [24]

$$K(q, p, t, E) = E + H(q, p, t) , \tag{2.5}$$

leads to the equivalent equations

$$\frac{dq}{d\tau} = \frac{\partial H}{\partial p} , \qquad \frac{dp}{d\tau} = -\frac{\partial H}{\partial q} ,$$

$$\frac{dt}{d\tau} = 1 , \qquad \frac{dE}{d\tau} = -\frac{\partial H}{\partial t} , \tag{2.6}$$

in which K is independent of the time τ, and, the time t and the new time τ have the same rate. Thus, the procedure of lifting the time to be a dynamical variable is used to transform a time-dependent problem to a time-independent one.

We have claimed before that the observation of irreversible processes involves time-measurements. Consequently, one would like to have a *time-operator*, which would corresponds to the measured time. It is well known [25] that the canonical quantization rule $[A, B] = i$, implies that the spectrum of both A and B is the whole real line. Therefore, the desired time operator T, whose canonical conjugate is the generator of translations in time, i.e., the generator of the evolution, cannot be defined in the framework of single Hilbert space where the generator of the evolution is the Hamiltonian which is semi-bounded. On the other hand, in the direct integral space, such a time operator can be defined naturally as

$$(T\psi)_t = t\psi_t . \tag{2.7}$$

In order to complete the construction of the theory in the framework of the direct integral space, one should specify the evolution law. In this section, we shall define the evolution of a system described by $\psi \in \bar{\mathcal{H}}$, as the ordinary Hilbert space unitary evolution combined with translation along the t-axis, i.e. [19],

$$\psi^{\tau}_{t+\tau} \;=\; W_t(\tau)\psi^0_t\,. \tag{2.8}$$

Since $W_t(\tau)$ represents an evolution, it follows that

$$W_{t+\tau_1}(\tau_2)W_t(\tau_1) \;=\; W_t(\tau_1+\tau_2)\,. \tag{2.9}$$

Eq. (2.8) is a representation (in fact, the t-representation) of a unitary evolution in $\bar{\mathcal{H}}$, which we will denote by $U(\tau)$, i.e.,

$$\psi^{\tau}_{t+\tau} \;=\; (U(\tau)\psi)_{t+\tau} \;=\; W_t(\tau)\psi^0_t\,. \tag{2.10}$$

The $U(\tau)$ are then unitary operators:

$$\begin{aligned}(U(\tau)\psi, U(\tau)\phi)_{\bar{\mathcal{H}}} \;&=\; \int (W_t(\tau)\psi_t, W_t(\tau)\phi_t)_{\mathcal{H}}dt \;=\\ &=\; \int (\psi_t,\phi_t)_{\mathcal{H}}dt \;=\; (\psi,\phi)_{\bar{\mathcal{H}}}\,.\end{aligned} \tag{2.11}$$

Furthermore, they form a one-parameter group, i.e.,

$$\begin{aligned}(U(\tau_1)U(\tau_2)\phi)_{t+\tau_1+\tau_2} \;&=\; W_{t+\tau_1}(\tau_2)W_t(\tau_1)\phi_t \;=\\ &=\; W_t(\tau_1+\tau_2)\phi_t \;=\; (U(\tau_1+\tau_2)\phi)_{t+\tau_1+\tau_2}\,.\end{aligned} \tag{2.12}$$

Since the $U(\tau)$ form a one-parameter group of unitary operators, it follows that if its action is continuous, it has a self-adjoint generator, K,

$$K \;=\; s-\lim_{\tau\to 0}\frac{1}{\tau}(U(\tau)-I)\,. \tag{2.13}$$

It has been shown by Piron [14] that if K, $-i\partial_t$, and $K+i\partial_t$ have a common dense domain on which they are essentially self-adjoint, then H, defined as the self-adjoint extension of $K+i\partial_t$, is a decomposable operator on $\bar{\mathcal{H}}$, i.e., $(H\psi)_t \;=\; H_t\psi_t$. Moreover, if $\|W_t(\tau)\phi_t\|_{\bar{\mathcal{H}}}$ is measurable in t and τ, K is unitarily equivalent to $-i\partial_t$, and hence its spectrum is absolutely continuous over all the real axis. The unitary transformation providing this equivalence is of the form

$$\big(R(t_0)\phi\big)_t \;=\; W_t\big(t_0-t\big)\phi_t \;=\; W^{-1}_{t_0}(t-t_0)\phi_t\,. \tag{2.14}$$

This result follows from the application of the definition of the operators,

$$\Big(R(t_0)e^{-iK\tau}R^{-1}(t_0)\phi\Big)_t \;=\; W^{-1}_{t_0}(t-t_0)\big(e^{-iK\tau}R^{-1}(t_0)\phi\big)_t \;=$$

$$= W_{t_0}^{-1}(t-t_0)W_{t-\tau}(\tau)\big(R^{-1}(t_0)\phi\big)_{t-\tau} \;=\; W_{t_0}^{-1}(t-t_0)W_{t-\tau}(\tau)W_{t_0}(t-t_0)\phi_{t-\tau}\,.$$

But according to the composition law (2.9), this is

$$\Big(R(t_0)e^{-iK\tau}R^{-1}(t_0)\phi\Big)_t \;=\; \phi_{t-\tau}\,, \tag{2.15}$$

so that

$$R(t_0)KR^{-1}(t_0) \;=\; -i\partial_t\,. \tag{2.16}$$

and therefore the spectrum of K is all the real line.

Misra, Courbage and Prigogine [26] have shown that the existence of an evolution generator with unbounded spectrum is necessary for the existence of an entropy operator M, with simple properties, i.e., that the rate of change of the entropy D, is compatible with the entropy itself, i.e., $[D,M]=0$. This result follows by constructing the expectation value of the evolution generator in states defined as $e^{iMs}\psi$, where s is an arbitrary parameter. Since

$$\frac{d}{ds}(e^{iMs}\psi, He^{iMs}\psi) \;=\; i(e^{iMs}\psi,[H,M]e^{iMs}\psi)\,, \tag{2.17}$$

and $D = i[H,M]$, the expression on the right hand side is just $(e^{iMs}\psi, D\, e^{iMs}\psi)$; from the commutation relation $[D,M]=0$, it follows that it is independent of s. Integrating (2.17) over s, one finds

$$(e^{iMs}\psi, He^{iMs}\psi) \;=\; (\psi, D\psi)s \,+\, \text{const.} \tag{2.18}$$

Hence, taking s to any arbitrary value, we see that H must be unbounded (in particular, from below), unless $(\psi, D\psi)=0$, in which case the entropy is constant.

If, on the other hand, the spectrum of H is unbounded from below (and absolutely continuous), then there exists a time operator. In this case, the theory can be put in correspondence with a theory of evolution in a larger Hilbert space. As will be discussed in the next chapter, the formulation of quantum dynamical problems in the Liouville space [27] forms a natural framework for this type of structure.

In the following, we will describe how Flesia and Piron[19] applied Lax-Phillips formalism to quantum systems with the help of the direct integral space which we have just discussed.

Lax-Phillips theory [17] assumes the existence of a one-parameter unitary group of evolution on a Hilbert space $\bar{\mathcal{H}}$, and incoming and outgoing subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$ such that

$$\begin{aligned} U(\tau)\mathcal{D}_+ \;&\subset\; \mathcal{D}_+\,,\ \text{for all}\,\tau>0 \\ U(\tau)\mathcal{D}_- \;&\subset\; \mathcal{D}_-\,,\ \text{for all}\,\tau<0 \\ \bigcap_\tau U(\tau)\mathcal{D}_\pm\},&= \{0\} \\ \overline{\bigcup_\tau U(\tau)\mathcal{D}_\pm} \;&=\; \bar{\mathcal{H}} \end{aligned} \tag{2.19}$$

where τ is the evolution parameter identified with the laboratory time. It follows from a theorem of Sinai [28] that $\bar{\mathcal{H}}$ can be foliated in such a way that it can be represented as a family of (auxiliary) Hilbert spaces in the form $L^2(-\infty, +\infty; \mathcal{H}_t)$, over Lebesgue measure in t , and all the $\mathcal{H}_t$ are isomorphic (we therefore sometimes refer to these spaces simply as $\mathcal{H}$). The scalar product in $\bar{\mathcal{H}}$ is given by

$$(f\,,\,g)\;=\;\int_{-\infty}^{\infty}(f_t\,,\,g_t)_{\mathcal{H}_t}dt\,. \tag{2.20}$$

Lax and Phillips show that there are unitary operators W_+^{-1}, W_-^{-1} which map the elements of $\bar{\mathcal{H}}$ into representations, called the outgoing and incoming translation representations, for which the evolution is translation in t. The subspaces $\mathcal{D}_+, \mathcal{D}_-$ correspond to the sets of functions with, in these representations, support in semi-infinite segments of the positive and negative t-axis respectively. They define the S matrix abstractly as the map from the incoming translation representation to the outgoing one, i.e., $S = W_+^{-1}W_-$. This map is defined up to unitary transformations on the auxiliary spaces $\{\mathcal{H}_t\}$, and refers to the equivalence classes for which the incoming and outgoing representations have the property that the evolution is represented by translation.

In the quantum theory, one constructs the space $\bar{\mathcal{H}}$ by taking the direct integral of the quantum mechanical Hilbert spaces over all values of the time t with Lebesgue measure. The form of the theory adopted by Flesia and Piron [19] distinguishes the elements of these equivalence classes, and constructs an S-matrix which maps the auxiliary space in the incoming translation representation to the auxiliary space of the outgoing one. In the model that they use to illustrate this structure, this map corresponds to a pre-asymptotic form of the S-matrix of the usual scattering theory. Their model assumes that the subspaces $\mathcal{D}_+, \mathcal{D}_-$ are represented in the "free" representation, for which the free evolution is translation, by $L^2(-\infty, \rho_-; \mathcal{H})$, $L^2(\rho_+, \infty; \mathcal{H})$, respectively. In the limit in which the interval between the two semi-infinite regions of support tends to infinity, their S-matrix becomes the usual S-matrix.

Under these conditions they identify the wave operators $W_\pm^{-1}$ with the operators $R(\rho_\pm)$, defined, according to (2.14), as

$$(R(\rho_\pm)\phi)_t\;=\;(W_\pm^{-1}\phi)_t\;=\;W_t(\rho_\pm - t)\phi_t\;=\;W_{\rho_\pm}^{-1}(t-\rho_\pm)\phi_t\,. \tag{2.21}$$

Therefore the S-matrix becomes (we denote by FP the Flesia - Piron form)

$$(S^{FP}(\rho_-, \rho_+)\phi)_t\;=\;(R(\rho_+)R(\rho_-)^{-1}\phi)_t\;=\;W_{\rho_-}(\rho_+)\phi_t\,. \tag{2.22}$$

When the model for the evolution is taken to be the interaction picture evolution, i.e.,

$$W_t(\tau)\;=\;e^{iH_0(t+\tau)}e^{-iH\tau}e^{-iH_0t}\,, \tag{2.23}$$

the S-matrix takes the form

$$S^{FP}(\rho_-, \rho_+)\;=\;e^{iH_0\rho_+}e^{-iH\rho_+}e^{iH\rho_-}e^{-iH_0\rho_-}\,. \tag{2.24}$$

In the limit $\rho_+ \to \infty$,$\rho_- \to -\infty$, this becomes $\Omega_+^\dagger\Omega_-$ (in the sense of a bilinear form on a dense set) which defines the S-matrix of the usual scattering theory.

Lax and Phillips define the operator

$$\mathcal{Z}(\tau) = P_+ U(\tau) P_- \tag{2.25}$$

on $\bar{\mathcal{H}}$, where $P_\pm$ is the projection on the orthogonal complement of $\mathcal{D}_\pm$. This operator vanishes on $\mathcal{D}_\pm$ and maps the subspace

$$\mathcal{K} = \bar{\mathcal{H}} \ominus (\mathcal{D}_+ \oplus \mathcal{D}_-)\,, \tag{2.26}$$

into itself. These mappings form a semigroup [17], i.e., for $\tau_1, \tau_2 \geq 0$,

$$\mathcal{Z}(\tau_1)\mathcal{Z}(\tau_2) = \mathcal{Z}(\tau_1 + \tau_2)\,, \tag{2.27}$$

and this semigroup is strongly contractive, i.e., for each $\phi \in \mathcal{K}$ and any ϵ, there exists a τ_ϕ such that

$$||\mathcal{Z}(\tau)\phi||_{\bar{\mathcal{H}}} < \epsilon \tag{2.28}$$

for $\tau > \tau_\phi$. It can be shown that under the conditions (2.19), $\mathcal{Z}(\tau)$ is just the unitary evolution $U(\tau)$ projected into the subspace $\mathcal{K}$. Since the states which lie in the subspaces $\mathcal{D}_\pm$, in the case of scattering, describe the incoming and outgoing waves which are not influenced by the interaction, the states which lie in $\mathcal{K}$ describe the unstable states, i.e., resonances of the scattering. From this point of view, the Lax-Phillips semigroup is analogous to the reduced motion discussed in the previous section. The direct integral space method, solves, therefore, the problem of deriving an exact semigroup law for the reduced motion and provides a realization of the unitary dilation of Nagy and Foias [16] (note that, since the generator of the evolution in $\bar{\mathcal{H}}$ has absolutely continuous spectrum $(-\infty, \infty)$, this result is not in contradiction with the no-go theorem discussed above).

Lax and Phillips prove that the S-matrix is a multiplicative operator in the spectral representation of K (which is the Fourier transform of the translation representation), i.e.,

$$(S\psi)_\sigma = S(\sigma)\psi_\sigma$$

and the eigenvalues of the generator of the semigroup $\mathcal{Z}(\tau)$ correspond to the singularities of the analytic continuation of $S(\sigma)$. The eigenstates corresponding to these eigenvalues are analogous to the generalized eigenstates found in the framework of Wigner and Weisskopf, as discussed in Section 1. Thus, the S-matrix contains all the information about the unstable states . It can be seen, however, from Eq. (2.22) (and, explicitly, in (2.24), that the S matrix obtained from a model in which the evolution is given in the form (2.8) has no t-dependence, and hence its spectal representation is trivial. We discuss this problem and its resolution in the next section.

3. Construction of non-trivial S-matrix

As we have seen , the Lax-Phillips theory is a natural framework for describing irreversible processes, and it may be applied to the quantum theory, using the direct integral space method, as suggested by Flesia and Piron.

There is, however a fundamental problem in their construction. As mentioned at the end of the previous section, the relation between the Lax-Phillips semigroup (which corresponds to the reduced motion) and the S-matrix, is established in terms of the spectral representation of the generator of the evolution, i.e., the Fourier transform of the translation representation . However, the operator S^{FP} of Eq. (2.22) does not depend on t and therefore the Fourier transform in terms of which the relation to the Lax-Phillips semigroup can be established, has only a trivial structure as a function of σ

$$\tilde{S}^{FP}(\sigma) = \left(e^{iH_0\rho_+}e^{-iH\rho_+}e^{iH\rho_-}e^{-iH_0\rho_-}\right)\delta(\sigma)\,. \tag{3.1}$$

Hence one can not obtain, under these conditions, a description of a decaying system.

In fact, from the point of view taken by Lax and Phillips, such an S-matrix, relating elements of an equivalence class (corresponding to unitary maps of the auxiliary spaces), corresponds to no scattering. To see this, we prove in the following that, in this case, in the *outgoing* translation representation , $\mathcal{D}_-$ is represented by $L^2(-\infty, \rho_-; \mathcal{H})$, i.e., that in this representation $\mathcal{D}_-$ has definite support property, and therefore, by definition, this representation is also *incoming.* Up to an isomorphic mapping of the auxiliary spaces (used by Flesia and Piron [19][20] to represent the scattering process), the Lax-Phillips S-matrix which relates these incoming and outgoing translations is therefore trivial.

Let $\psi \in \mathcal{D}_-$; then in the free representation $\psi \in L^2(-\infty, \rho_-; \mathcal{H})$. The outgoing representor of ψ is then given by

$$\psi_t^{out} = (W_+^{-1}\psi)_t = (R(\rho_+)\psi)_t = W_t(\rho_+ - t)\psi_t\,. \tag{3.2}$$

Since ψ_t vanishes for $t > \rho_-$, so does $(W_+^{-1}\psi)_t$, and hence the set $\mathcal{D}_-^{out}$ of incoming states in the outgoing representation satisfies

$$\mathcal{D}_-^{out} \subset L^2(-\infty, \rho_-; \mathcal{H})\,. \tag{3.3}$$

The opposite direction of the demonstration is similar. Let us assume now that $\psi^{out} \in L^2(-\infty, \rho_-; \mathcal{H})$, and consider ϕ defined by $\phi = W_+\psi^{out}$, i.e.,

$$\phi_t = W_{\rho_+}(t - \rho_+)\psi_t^{out}\,. \tag{3.4}$$

Since ψ_t^{out} vanishes when $t > \rho_-$, ϕ_t has the same property, and hence $\phi \in \mathcal{D}_-$ and $\psi^{out} \in W_+^{-1}\mathcal{D}_-$ or $L^2(-\infty, \rho_-; \mathcal{H}) \subset \mathcal{D}_-^{out}$. We conclude from this result and (3.3) that

$$\mathcal{D}_-^{out} = L^2(-\infty, \rho_-; \mathcal{H})\,, \tag{3.5}$$

and hence the outgoing representation coincides with the incoming one.

In fact, for any system where the evolution may be written as $\psi_{t+\tau}^{\tau} = W_t(\tau)\psi_t$, the incoming and outgoing representations coincide. In order to see this, we construct the incoming (outgoing) translation representations using the free representation. Let us assume that there is a representation in which both $\mathcal{D}_\pm$ have definite support properties, but the evolution is not necessarily just translation. The free representation , which in the absence of interaction is both incoming and outgoing, for example, should have this property . Let us define a new representation in the following way. Take f_0 to be in the free representation , and define its outgoing image

$$f_+ \,=\, W_+^{-1} f_0 \,=\, s - \lim_{\tau\to\infty} U_0(-\tau)U(\tau)f_0\,. \tag{3.6}$$

First we show that it belongs to an outgoing representation . Since $U(\tau)$ is the full evolution in the free representation , f_+ evolves according to

$$f_+(\tau) \,=\, W_+^{-1}U(\tau)W_+ f.$$

However,

$$W_+^{-1}U(\tau) \,=\, s - \lim_{\tilde\tau\to\infty} U_0(-\tilde\tau)U(\tilde\tau)U(\tau) \,=$$

$$=\, s - \lim_{\tilde\tau+\tau\to\infty} U_0(\tau)U_0(-\tilde\tau-\tau)U(\tilde\tau+\tau) \,=\, U_0(\tau)W_+^{-1}\,, \tag{3.7}$$

and therefore $f_+(\tau) \,=\, U_0(\tau)f_+(0)$, i.e., the evolution is translation .

We now prove the second condition for the outgoing representation , $\mathcal{D}_+^{out} \,=\, \mathcal{D}_+^{free} \,=\, L^2(\rho_+,\infty;\mathcal{H})$. This follows when the evolution is $\psi_{t+\tau}^\tau \,=\, W_t(\tau)\psi_t$, since in this case, for any positive τ,

$$\big(U_0(-\tau)U(\tau)\psi\big)_t \,=\, \big(U(\tau)\psi\big)_{t+\tau} \,=\, W_t(\tau)\psi_t\,, \tag{3.8}$$

i.e., there is only a unitary change in the little space $\mathcal{H}$ with no translation, which implies $U_0(-\tau)U(\tau)L^2(\rho_+,\infty;\mathcal{H}) \,=\, L^2(\rho_+,\infty;\mathcal{H})$, and therefore it is true in the limit $\tau\to\infty$. In a similar way we define

$$W_- \,=\, s - \lim_{\tau\to-\infty} U_0(-\tau)U(\tau) \tag{3.9}$$

to build the incoming representation .

Let us now further assume that the evolution is of the form $\psi_{t+\tau}^\tau \,=\, W_t(\tau)\psi_t$, and that there is given a free representation such that,

$$\mathcal{D}_+ \,=\, L^2(\rho_+,\infty;\mathcal{H}) \qquad \mathcal{D}_- \,=\, L^2(-\infty,\rho_-;\mathcal{H})\,, \tag{3.10}$$

and check the properties of the subspace $\mathcal{D}_-^{out} \,=\, W_+^{-1}\mathcal{D}_-$. Let $\{\psi_t\} \,\in\, \mathcal{D}_-$, and τ be some positive number. Then, one has

$$(U(\tau)\psi)_{t+\tau} \,=\, W_t(\tau)\psi_t\,. \tag{3.11}$$

The right hand side vanishes for $t > \rho_-$ and therefore $(U(\tau)\psi)_t \,=\, 0$ for $t > \rho_- + \tau$. We therefore have

$$(U_0(-\tau)U(\tau)\psi)_t^{free} \,=\, 0 \quad \text{for} \qquad t > \rho_-\,. \tag{3.12}$$

Since this is true for every positive τ, it is true in the limit $\tau\to\infty$, and therefore

$$\{\psi_t\}^{out} \,\in\, L^2(-\infty,\rho_-;H)\,, \tag{3.13}$$

$\{\psi_t\} \in \mathcal{D}_-$.

The inverse direction is the same type of argument, and it follows that

$$\mathcal{D}_-^{out} = L^2(-\infty, \rho_-; \mathcal{H}) . \tag{3.14}$$

Therefore, as before, the outgoing translation representation is also incoming and the S-matrix is the identity.

The assumption of an evolution law which is pointwise on the time axis of the measure space of $\bar{\mathcal{H}}$ therefore does not lead to the full structure of Lax-Phillips theory, i.e., an S-matrix which has a non-trivial Fourier transform. This result is highly significant for the theory of unstable systems. Although an axiomatization of the structure of such systems has not yet been carried out, it is clear, as we have pointed out above, that they cannot be characterized at a particular moment in time. The second law of thermodynamics, in particular, is a statement about time evolution which is intrinsic to irreversible processes; it is stronger than the condition for the unitary evolution of reversible systems, for which the only requirement is conservation of probability (and once differentiability for the element of the equivalence class of L^2 functions which must satisfy the Schrödinger equation). In the latter case, models are based on correspondence with a classical analog with Hamiltonian dynamics. For the irreversible system , one might expect a classical analog for the dynamical evolution, which contains some representations of correlation in time [29], such as a Boltzmann type equation, encompassing, in particular, the thermodynamics of the second law.

Although the generalization of Lax-Phillips theory by Flesia and Piron [19] provides a new point of view for scattering theory, we see that to extend the theory further to include a description of the evolution of an unstable system, it is necessary to generalize the law of evolution to that of a nontrivial integral operator over the time.

The most general linear evolution law has the form

$$(U(\tau)\psi)_{t+\tau} = \int_{-\infty}^{+\infty} W_{t,t'}(\tau)\psi_{t'} dt' . \tag{3.15}$$

We shall show that this type of evolution, which goes beyond the formulation of Flesia and Piron [19] and Floquet theory [24], can correspond to unitary evolution in $\bar{\mathcal{H}}$ with a nontrivial S-matrix for which the singularities of its Fourier transform are associated with the spectrum of the generator of the Lax-Phillips semigroup. As we shall show in Section 5, the form of the evolution law (3.15) has a natural realization in Liouville space. For mathematical and conceptual purposes, and possibly for actual physical applications, we wish, however, to study first the condition for which the full structure of the Lax-Phillips theory is applicable in the framework of ordinary Hilbert space theory as well. In this framework, one can understand the structure of Eq. (3.15) by examining the class of physical systems actually studied by Lax-Phillips , i.e., that of hyperbolic systems, as for electromagnetic scattering theory. Their foliation of the Hilbert space of solutions along the time axis, leads to the existence of a nontrivial semigroup behavior. According to our remark following Eq. (3.5), it follows that the evolution law must be of the type of Eq.(3.15). The action of the Green's function for a hyperbolic system requires an integral over t, introducing the correlations necessary for the construction of a non-trivial Lax-Phillips theory.

In the framework of the ordinary quantum theory, one generally constructs Hamiltonian dynamical models with Hamiltonian functions which are either time-independent, or depend on time in a pointwise manner. One can understand the time parameter entering this construction as given in terms of a particular representation for the time operator which can be defined in the larger Hilbert space $\bar{\mathcal{H}}$. In this representation, the subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$, in general, may not have definite support properties. The transformation to a representation in which they have definite support properties (as in the "free " representation in the Flesia-Piron theory) may result in a representation for the Hamiltonian which is not decomposable, inducing a non-pointwise evolution as in (3.15). Such transformations were discussed, for example, by Friedrichs [9]. We shall use such a representation in the following.

Assuming that the evolution is represented by (2.10), i.e., pointwise on the t-axis, Flesia and Piron proved that the generator K of the evolution on $\bar{\mathcal{H}}$ is of the form $-i\partial_t$ plus a decomposable operator. This follows from the application of the Trotter's formula [30]

$$e^{-iH\tau} = s - \lim_{n\to\infty} (e^{-\partial_t \tau/n} e^{-iK\tau/n})^n \,. \tag{3.16}$$

and the assumption that the evolution is generated pointwise as in Eq. (1.4). One examines

$$\begin{aligned}(e^{-\partial_t \tau/n} e^{-iK\tau/n} f)_t &= (e^{-iK\tau/n} f)_{t+\tau/n} \\ &= (U(\tau/n) f)_{t+\tau/n} = W_t(\tau/n) f_t,\end{aligned} \tag{3.17}$$

where the first follows from the action of translation, and the last from the assumption that the evolution is represented by an operator acting pointwise on the t-axis. Taking the limit of the sequence, one obtains

$$(e^{-iH\tau} f)_t = s - \lim_{n\to\infty} (W_t(\tau/n))^n f_t \,, \tag{3.18}$$

which is clearly decomposable.

On the other hand, if the evolution acts as a non-trivial kernel on the time variable, as in (3.15), one obtains

$$\begin{aligned}(e^{-\partial_t \tau/n} e^{-iK\tau/n} f)_t &= (e^{-iK\tau/n} f)_{t+\tau/n} = \\ &= \int W_{t,t'}(\tau/n) f_{t'} dt'\end{aligned} \tag{3.19}$$

Applying this operator again n times, one obtains

$$\begin{aligned}&((e^{-\partial_t \tau/n} e^{-iK\tau/n})^n f)_t = \\ &= \int W_{t,t_1}(\tau/n) W_{t_1,t_2}(\tau/n) \ldots W_{t_{n-1},t_n}(\tau/n) f_{t_n} dt_1 dt_2 \ldots dt_n \,;\end{aligned} \tag{3.20}$$

this n-fold convolution converges, since the product (3.16) converges if $-i\partial_t, K, i\partial_t + K$ have a common dense domain. It is clear that the right hand side is, in general, not decomposable.

The operators $W_{tt'}(\tau)$ must satisfy some conditions if the $U(\tau)$ are to form a one-parameter unitary group. Since

$$\begin{aligned}(U(\tau_1+\tau_2)\psi)_{t+\tau_1+\tau_2} &= \int W_{t,t'}(\tau_1+\tau_2)\psi_{t'}dt' \\ &= (U(\tau_1)U(\tau_2)\psi)_{t+\tau_1+\tau_2} \\ &= \int W_{t+\tau_2,t'}(\tau_1)(U(\tau_2)\psi)_{t'}dt' \\ &= \int\int W_{t+\tau_2,t'}(\tau_1)W_{t'-\tau_2,t''}(\tau_2)\psi_{t''}dt'dt''\end{aligned}$$

must be true for arbitrary ψ, we require the relation

$$W_{t,t'}(\tau_1+\tau_2) = \int W_{t+\tau_2,t''+\tau_2}(\tau_1)W_{t'',t'}(\tau_2)dt'' \tag{3.21}$$

¿From the property $U^{-1}(\tau) = U(-\tau) = U^\dagger(\tau)$, i.e.,

$$(\psi, U(\tau)\phi)_{\bar{\mathcal{H}}} = (U(-\tau)\psi, \phi)_{\bar{\mathcal{H}}}$$

one finds

$$\int(\psi_t, \int W_{t-\tau,t'}\phi_{t'}dt')_{\mathcal{H}}dt = \int(\int W_{t+\tau,t'}(-\tau)\psi_{t'}dt', \phi_t)_{\mathcal{H}}dt\,.$$

Since this is true for arbitrary ψ and ϕ, we obtain

$$W_{t_1-\tau,t_2}(\tau) = W_{t_2+\tau,t_1}(-\tau)^\dagger$$

or

$$W_{t,t'}(-\tau) = W_{t'-\tau,t-\tau}(\tau)^\dagger\,. \tag{3.22}$$

The two conditions (3.21),(3.22) ensure that the evolution is unitary. If we use the chain property, putting $\tau_1 = -\tau_2 = \tau$ in (3.21), we have

$$\int dt'' W_{t-\tau,t''-\tau}(\tau)W_{t'',t'}(-\tau) = W_{t,t'}(0) = \delta(t-t')\,. \tag{3.23}$$

With (3.22), (3.23) becomes

$$\int W_{t,t''}(\tau)W_{t',t''}(\tau)^\dagger dt'' = \delta(t-t')\,.$$

Let us now study for this general evolution, some properties of the S-matrix

$$(S\psi)_t = \int S_{t,t'}\psi_{t'}dt',$$

and show that in this general case the S-matrix must have the form $S_{t,t'} = S(t-t')$. Using the definition $S = W_+^{-1} W_-$, where

$$W_{\pm} = s - \lim_{\tau \to \pm\infty} U(-\tau) U_0(\tau)\,,$$

we find

$$(S\psi)_t = s - \lim_{\tau_1, \tau_2 \to \infty} (U_0(-\tau_1) U(\tau_1) U(\tau_2) U_0(-\tau_2)\psi)_t$$

But,

$$(U_0(-\tau_1) U(\tau_1+\tau_2) U_0(-\tau_2)\psi)_t = (U(\tau_1+\tau_2) U_0(-\tau_2)\psi)_{t+\tau_1} =$$
$$= \int W_{t+\tau_1,t'}(\tau_1+\tau_2)(U_0(-\tau_2)\psi)_{t'} dt' =$$
$$= \int W_{t+\tau_1,t'}(\tau_1+\tau_2)\psi_{t'+\tau_2} dt' = \int W_{t+\tau_1,t'-\tau_2}(\tau_1+\tau_2)\psi_{t'} dt',$$

and therefore the matrix elements of S are

$$S_{t,t'} = s - \lim_{\tau_1,\tau_2 \to \infty} W_{t+\tau_1,t'-\tau_2}(\tau_1+\tau_2) =$$
$$= s - \lim_{\tau_1',\tau_2' \to \infty} W_{t-t'+\tau_1',-\tau_2'}(\tau_1'+\tau_2') = S(t-t') \tag{3.24}$$

(where $\tau_1' = \tau_1 + t'$ $\tau_2' = \tau_2 - t'$) . This is a very important property of the S-matrix, according to which, when one goes to the spectral representation $\hat{\psi}_\sigma = \int e^{-i\sigma t}\psi_t dt$, the S-matrix takes the simple form

$$\hat{S}_{\sigma,\sigma'} = \frac{1}{2\pi}\int e^{-i\sigma t} S_{t,t'} e^{i\sigma' t'} dt dt' = \delta(\sigma-\sigma')\hat{S}(\sigma)$$

where

$$\hat{S}(\sigma) = \int e^{-i\sigma t} S(t) dt \tag{3.25}$$

i.e. in this basis the S-matrix is diagonal, and the S-operator is multiplication on the subspaces, labeled by σ, of $\{\mathcal{H}_\sigma\}$, the set of (isomorphic) Hilbert spaces which are the Fourier dual to the set $\{\mathcal{H}_t\}$. This result can be obtained also by looking at the definition of the S-matrix,

$$S = s - \lim_{\tau_1,\tau_2 \to \infty} U_0(-\tau_1) U(\tau_1+\tau_2) U_0(-\tau_2)$$

from which it follows that

$$S U_0(\tau) = U_0(\tau) S\,.$$

Since $U_0(\tau)$ is the translation operator one obtains the result $[\,S\,,\,i\partial_t\,] = 0$ (which correspond to the usual result of scattering theory $[\,S\,,\,H_0\,] = 0$). It follows from this commutation relation that $S_{t,t'} = S(t-t')$.

Since $U(\tau)$ is a continuous group of unitary operators, one can write it in the form $U(\tau) = e^{-iK\tau}$. We shall now find the form of the S-matrix in terms of the generator K.

The generator may be calculated from the equation $i\partial_\tau U(\tau) = KU(\tau)$. Looking at the components of this equation one has

$$i\partial_\tau (U(\tau)\psi)_t = (KU(\tau)\psi)_t \,. \tag{3.26}$$

On the other hand, by the definition of $U(\tau)$,

$$i\partial_\tau (U(\tau))_t = i\partial_\tau \int W_{t-\tau,t'}(\tau)\psi_{t'}dt' =$$

$$= -i\partial_t \int W_{t-\tau,t'}(\tau)\psi_{t'}dt' + i\partial_\tau \int W_{\tilde{t},t'}(\tau)\psi_{t'}dt' \tag{3.27}$$

where we define $\tilde{t} = t - \tau$ with the implication that ∂_τ operates only on the argument τ of $W_{\tilde{t},t'}(\tau)$ which is explicitly displayed. Let us write the generator in the (general) form, writing the t-derivative explicitly to take into account the form of (3.27),

$$(K\psi)_t = -i\partial_t \psi_t + \int \kappa_{t,t'}\psi_{t'}dt' \,,$$

so that

$$(KU(\tau)\psi)_t = -i\partial_t \int W_{t-\tau,t'}(\tau)\psi_{t'}dt' + \int\int \kappa_{t,t'}W_{t'-\tau,t''}(\tau)\psi_{t''}dt'dt'' \,. \tag{3.28}$$

Comparing this result to the equation (3.27) one obtains

$$i\partial_\tau \int W_{\tilde{t},t'}(\tau)\psi_{t'}dt' = \int\int \kappa_{t,t'}W_{t'-\tau,t''}(\tau)\psi_{t''}dt'dt'' \,,$$

and since this equation holds for arbitrary ψ

$$i\partial_\tau W_{\tilde{t},t''}(\tau) = \int \kappa_{t,t'}W_{t'-\tau,t''}(\tau)dt' \,,$$

i.e.,

$$i\partial_\tau W_{t,t'}(\tau) = \int \kappa_{t+\tau,t''+\tau}W_{t'',t'}(\tau)dt'' \,. \tag{3.29}$$

This differential equation determines the evolution operators W in terms of the generator K, and may be formally expanded in a series (convergent for sufficiently small κ). The first terms in the series are:

$$W_{t,t'}(\tau) = \delta(t-t') - i\int_0^\tau \kappa_{t+\tau',t'+\tau'}d\tau' -$$

$$-\frac{1}{2}\int_0^\tau d\tau'' \int_0^\tau d\tau' \int dt''\, T[\kappa_{t+\tau'',t''+\tau''}\kappa_{t+\tau',t''+\tau'}] + ...$$

where the T implies the τ-ordered product . Using the formula (3.6), we obtain a perturbative formula for S in the form

$$S_{t,t'} = s - \lim_{\tau_1,\tau_2\to\infty} \big(\delta(t-t') - i\int_0^{\tau_1+\tau_2} \kappa_{t-t'+\tau-\tau_2,\tau-\tau_2} d\tau -$$

$$- \frac{1}{2}\int_0^{\tau_1+\tau_2} d\tau \int_0^{\tau_1+\tau_2} d\tau' \int dt''\, T[\kappa_{t-t'-\tau_2+\tau,t''+\tau}\kappa_{t-t'-\tau_2+\tau',t''+\tau'}] \big) + ... =$$

$$= s - \lim_{\tau_1,\tau_2\to\infty} \big(\delta(t-t') - i\int_{-\tau_2}^{\tau_1} \kappa_{t-t'+\tau,\tau} d\tau -$$

$$- \frac{1}{2}\int_{-\tau_2}^{\tau_1} d\tau \int_{-\tau_2}^{\tau_1} d\tau' \int dt'' T[\kappa_{t-t'+\tau,t''+\tau}\kappa_{t''+\tau',\tau'}] \big) + ... =$$

$$= \delta(t-t') - i\int_{-\infty}^{\infty} \kappa_{t-t'+\tau,\tau} d\tau -$$

$$- \frac{1}{2}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \int dt'' T[\kappa_{t-t'+\tau,t''+\tau}\kappa_{t''+\tau',\tau'}] + \ldots \tag{3.30}$$

It is interesting to examine one particular case for which Eq. (3.29) can be solved exactly. Consider the case in which the perturbation κ is of the form $\kappa_{t,t'} = \kappa(t-t')$. Let us take the Fourier transform of (3.29) with respect to t and t'.

$$i\partial_\tau W_{\sigma,\sigma'}(\tau) = \int e^{i\sigma\tau}\kappa_{\sigma,\sigma''}e^{-i\sigma''\tau}W_{\sigma'',\sigma'}(\tau)d\sigma'' . \tag{3.31}$$

Since $\kappa_{t,t'} = \kappa_{t-t'}$, one obtains

$$\kappa_{\sigma,\sigma'} = \tilde{\kappa}(\sigma)\delta(\sigma-\sigma') . \tag{3.32}$$

Using (3.32) in (3.31), one obtains

$$i\partial_\tau W_{\sigma,\sigma'}(\tau) = \tilde{\kappa}(\sigma)W_{\sigma,\sigma'}(\tau) ,$$

from which follows

$$W_{\sigma,\sigma'}(\tau) = e^{-i\tilde{\kappa}(\sigma)\tau}\delta(\sigma-\sigma') . \tag{3.33}$$

Taking the inverse transform of (3.33) we get

$$W_{t,t'}(\tau) = \frac{1}{2\pi}\int d\sigma d\sigma' e^{i\sigma t}e^{-i\tilde{\kappa}(\sigma)\tau}\delta(\sigma-\sigma')e^{-i\sigma' t'} =$$

$$= \frac{1}{2\pi}\int d\sigma e^{i\sigma(t-t')}e^{-i\tilde{\kappa}(\sigma)\tau} . \tag{3.34}$$

Using (3.24) and (3.34) one obtains

$$S_{t,t'} \;=\; s - \lim_{\tau\to\infty} \int_{-\infty}^{\infty} d\sigma e^{i\sigma(t-t')} e^{i\sigma\tau} e^{-i\tilde{\kappa}(\sigma)\tau} \,, \tag{3.35}$$

where $\tau = \tau_1 + \tau_2$, and it follows that

$$\tilde{S}(\sigma) \;=\; s - \lim_{\tau\to\infty} e^{i(\sigma - \tilde{\kappa}(\sigma))\tau} \,. \tag{3.36}$$

If there is a subset in $\bar{\mathcal{H}}$ for which this limit exists, one may replace it by the limit in the sense of distributions (that is, pointwise in σ). We will show now that the limit in the sense of distributions is either trivial, $\tilde{S}(\sigma) = 0$, or it is not meromorphic as a function of σ. Therefore, either the strong limit does not exist, or it is not meromorphic (since the strong limit has unity norm when it exists, it can not vanish identically). Assuming that the limit exists as a distribution,

$$\begin{aligned}\tilde{S}(\sigma) \;&=\; s - \lim_{\tau\to\infty} e^{i(\sigma-\tilde{\kappa}(\sigma))\tau} \;= \\ &=\; \lim_{\epsilon\to 0} \epsilon \int_0^{\epsilon} e^{i(\sigma-\tilde{\kappa}(\sigma))\tau} e^{-\epsilon\tau} d\tau \;= \\ &=\; \lim_{\epsilon\to 0} \frac{i\epsilon}{\sigma - \tilde{\kappa}(\sigma) + i\epsilon} \,.\end{aligned} \tag{3.37}$$

This expression obviously vanishes for $\sigma \neq \tilde{\kappa}(\sigma)$, and is 1 where they are equal, and therefore it is clear that either it vanishes everywhere, or it is not a meromorphic function.

We now consider the structure of the Lax-Phillips semigroup

$$\mathcal{Z}(\tau) \;=\; P_+ U(\tau) P_- \,. \tag{3.38}$$

It is shown by Lax-Phillips that since $\{U(\tau)\mathcal{D}_+\}$ is dense in $\bar{\mathcal{H}}$, $\mathcal{Z}(\tau)$ is strongly contractive [17]. We show now that under the general evolution (3.15), the semigroup is still contractive, as for the Flesia-Piron case [19][20]. Let us calculate the generator of the semigroup B. We use the free translation representation in which both $\mathcal{D}_\pm$ have definite support properties. In this representation ,

$$\mathcal{Z}(\tau) \;=\; P_+ U(\tau) P_- \;=\; E(\rho) U(\tau)(I - E(0)) \,, \tag{3.39}$$

where $E(t)$ is the spectral resolution corresponding to T_0, the free-time-operator (the conjugate of K_0 which is, in the free translation representation , $-i\partial_t$). Then, the generator (in the subspace $\mathcal{K}$) of $\mathcal{Z}(\tau)$ is

$$\begin{aligned}B \;&=\; i \lim_{\tau\to 0} \frac{\mathcal{Z}(\tau) - I_{\mathcal{K}}}{\tau} \;=\; i \lim_{\tau\to 0} \frac{E(\rho)(I - iK\tau)(I - E(0)) - I_{\mathcal{K}}}{\tau} \;= \\ &=\; E(\rho) K (I - E(0)) \;=\; P_+ K P_- \,.\end{aligned} \tag{3.40}$$

Note that one can not apply the imprimitivity relations directly here, since as remarked above, K is not conjugate to T_0. According to the requirements on $\mathcal{D}_\pm$ the matrix elements of κ between states from $\mathcal{D}_-$ to $\mathcal{D}_+$, or $\mathcal{D}_\pm$ to $\mathcal{K}$ vanish, and therefore

$$B = P_+ K_0 P_- + \kappa_{\mathcal{K}} . \tag{3.41}$$

An operator B is called dissipative [31][32] if

$$-i\big((\phi, B\phi) - (B\phi, \phi)\big) \leq 0 , \tag{3.42}$$

for all ϕ in the domain of B. Since $\kappa_{\mathcal{K}}$ is self-adjoint only the first term determines whether the operator is dissipative, i.e., this property does not depend on the perturbation. As shown by Horwitz and Piron[20], the operator $P_+ K_0 P_-$ is, in fact, dissipative. It is known [32] that $\mathcal{Z}(\tau)$ is a contractive semigroup if and only if its generator is dissipative. It therefore follows, independently of (self-adjoint) interaction, that the semigroup $\mathcal{Z}(\tau)$ is contractive. We see from this [20] the *essential mechanism of Lax-Phillips theory.* The non-self-adjointness of $P_+ K_0 P_-$ corresponds to the restriction of $-i\partial_t$ to a finite interval, so that, in fact the operator has imaginary eigenvalues. In the presence of interaction (non-trivial κ), these eigenvalues emerge as the actual eigenvalues of B, corresponding to the singularities of $S(\sigma)$.

We remark that the direct integral space provides a framework as a functional space for quantum mechanics in which the Nagy-Foias construction can be realized, i.e., for which unitary evolution can be restricted to a contractive semigroup. We shall now introduce an extension of the conceptual framework which considers the set $\{\psi_t\}$, corresponding to the Lax-Phillips vector ψ, as an ensemble of the same type, for example, as $\{\psi(x)\} \in \mathcal{H}$, where x is a point of the spectrum of the position observable, in the usual form of the quantum theory. In concluding this section, we investigate some consequences of this interpretation.

In particular, we discuss some properties of the time operator and the realization of the superselection rule in time. In the next section, we discuss the possibility of decoherence in $\mathcal{H}$ induced by the unitary evolution in $\bar{\mathcal{H}}$.

In fact, there are three distinct types of time operator. One, which we call the incoming time operator T^{in}, provides a spectral family in terms of which the incoming representation can be constructed, and in which functions in $\mathcal{D}_-$ have definite support and functions in $\bar{\mathcal{H}}$ evolve by translation. In this representation, the norm of the evolving states in $L^2(-\infty, 0; \mathcal{H})$ must decrease. After sufficient laboratory time τ passes, the states evolve to $\mathcal{D}_+$, and in the outgoing representation, provided by the spectral family of the outgoing time operator T^{out}, they have definite support in $L^2(\rho, \infty; \mathcal{H})$. The mapping of functions in the incoming representation to the outgoing representation is provided by the Lax-Phillips S-matrix, and the time operators are related by

$$T^{out} = S T^{in} S^{\dagger} . \tag{3.43}$$

The third type of time operator correspond to the "free" representation and is related to T^{in},T^{out} by the Lax-Phillips wave operators. The spectral family for this operator provides the "standard" representation (analogous to Dirac's choice of "standard" spectral families), which we have used above.

There is an interval, in general, when the system is in interaction, and its state is neither in $\mathcal{D}_-$ nor $\mathcal{D}_+$. The expectation value of the operator T^{in} in the state ψ^τ projected into $\mathcal{K} \oplus \mathcal{D}_+$ (corresponding to the projection P_-) can be interpreted as the interaction interval. If the system in interaction is considered as an unstable particle (a resonance), this interval is its *age* after creation at $t = 0$. This interpretation follows from that of T^{in}, i.e.,

$$\frac{dT^{in}}{d\tau} = i[K, T^{in}] = 1 . \tag{3.44}$$

Hence, the expectation value of T^{in}, corresponding to the support properties of the states in $\mathcal{D}_-$ in the incoming representation, moves with the laboratory time τ. The expectation value of T^{in} then moves out of $(-\infty, 0)$. The expectation value of T^{in} in the state $P_-\psi^\tau$ is

$$< T^{in} >_\tau = \int t|_{in}\langle t|P_-\psi^\tau)|^2 dt ; \tag{3.45}$$

here, $|_{in}\langle t|P_-\psi^\tau)|^2$ is the probability density for the age t at time τ, an intrinsic dynamical property of the system. The positive value that the expectation value develops corresponds to the average age.

Similarly, when an unstable system decays, it moves to the subspace $\mathcal{D}_+$, the subspace of outgoing states. In the outgoing representation, these states have support in $L^2(\rho, \infty; \mathcal{H})$. The operator T^{out} satisfies

$$\frac{dT^{out}}{d\tau} = i[K, T^{out}] = 1 . \tag{3.46}$$

and its expectation value goes with the laboratory time. Hence for an unstable system which decays, the expectation value of $T^{out} - \rho$ in states in $\mathcal{D}_+$ is the time after decay, i.e.,

$$\text{time after decay} = \int (t - \rho)\ |_{out}\langle t|(1 - P_+)\psi^\tau)|^2 dt , \tag{3.47}$$

where $|_{out}\langle t|(1 - P_+)\psi^\tau)|^2$ is the probability density at each τ that the system has decayed at laboratory time $t - \rho$ (for τ sufficiently early, this quantity vanishes). Similarly, the expectation value of $\rho - T^{out}$ in states in $\mathcal{K} \oplus \mathcal{D}_-$ is the average time interval for the system to decay, i.e.,

$$\text{time interval to decay} = \int (\rho - t)\ |_{out}\langle t|P_+\psi^\tau)|^2 dt ; \tag{3.48}$$

where, $|_{out}\langle t|P_+\psi^\tau)|^2$ is the probability density at each τ, that the system will decay at laboratory time $\rho - t$. We then understand the subspace $\mathcal{K}$ as corresponding to the unstable system.

An unstable system must be characterized by (at least) *two* time operators; if there were only one, in terms of which both the age and the time of decay are described, incoming and outgoing representations would coincide (up to isomorphisms of $\bar{\mathcal{H}}$), as discussed above, and the Lax-Phillips S-matrix would be unity. In this representation, both $\mathcal{D}_-$ and $\mathcal{D}_+$

have definite support properties and the evolution is represented by translation. Hence, the sum of the age and time of decay would be necessarily constant. This is not consistent with observation of known unstable systems, for which the time of decay, given the time of creation of the system, is not definite. Therefore, our treatment which introduces two time operators is necessary for the construction of a physically consistent theory.

The Lax-Phillips description of an unstable system developed here has the following important characteristics:

1. A state in $\mathcal{K}$ is indistinguishable from any other in $\mathcal{K}$ by its support property in t, if detected according to operators that are independent of t, i.e., the time of decay associated with these states cannot be determined by measurements of time-independent observables. This characteristic is consistent with our experience, in which one cannot predict the time of decay, or distinguish different stages of development of the undecayed system.
2. The structure of the theory is somewhat similar to the Wigner-Weisskopf idea, in that a subspace is associated with the decaying system. The decay of the system is also associated with the probability flow out of the subspace. However, in Wigner-Weisskopf theory, the process of decay is represented as a continuous evolution from the original unstable state to the final state through a changing linear superposition. In the Lax-Phillips theory the expectation value of an observable which is decomposable in the free or outgoing representations, where $\mathcal{D}_+$ has definite support properties, necessarily reduces to the sum of the expectation values in the subspaces $\mathcal{K} \oplus \mathcal{D}_-$ and in the subspace $\mathcal{D}_+$ (the decay products).

There is, therefore, an exact superselection rule for measurements of the system by means of such decomposable operators.

4. Mixing of States under the Lax-Phillips Evolution

Recently, Machida and Namiki [33] have proposed a measurement theory based on a direct integral space of continuously many Hilbert spaces and a continuous superselection rule. As pointed out by Tasaki *et al* [34], although they had some success, their theory has a conceptual difficulty. Indeed, in their theory, while the apparatus is described by many Hilbert spaces, the system corresponds to a single Hilbert space as in the conventional theory. Thus, one needs to specify the boundary between the system and the apparatus. As discussed by von Neumann, this is impossible.

In this section we will investigate the possibility of using the quantum version of Lax-Phillips theory as discussed above to solve this problem by describing both the system and the apparatus by a direct integral space of the form of Eq. (2.1).

In the direct integral space $\bar{\mathcal{H}}$, the most general operator $\hat{A}$ takes the form

$$(\hat{A}\psi)_t \;=\; \int dt' A_{t,t'} \psi_{t'} \,, \tag{4.1}$$

where $A_{t,t'}$ is an operator from $\mathcal{H}_{t'}$ to $\mathcal{H}_t$. However, if the operator is self-adjoint in $\bar{\mathcal{H}}$, the foliation may be changed such that the operator is decomposable, i.e.,

$$(\hat{A}\psi)_t \;=\; A_t \psi_t \,. \tag{4.2}$$

Moreover, any (time-dependent) observable $A(t)$ defined in the usual quantum Hilbert space $\mathcal{H}$ can be naturally lifted to the direct integral space $\bar{\mathcal{H}}$ as follows

$$(\hat{A}\psi)_t \;=\; A(t)\psi_t \,. \tag{4.3}$$

For any such decomposable self-adjoint operator in the direct integral space, we define an "expectation value" (consistent with our discussion in Section 3) as

$$\langle \hat{A} \rangle_\psi \;=\; \frac{(\psi, \hat{A}\psi)_{\bar{\mathcal{H}}}}{(\psi,\psi)_{\bar{\mathcal{H}}}} \;=\; \frac{\int dt (\psi_t, A_t \psi_t)_{\mathcal{H}}}{\int dt (\psi_t, \psi_t)_{\mathcal{H}}} \,. \tag{4.4}$$

This definition is a natural generalization of the expectation value in the conventional quantum mechanics. Indeed, for a state

$$(\psi^\epsilon)_t \;=\; \sqrt{\frac{1}{\pi}\frac{\epsilon}{(t-t_0)^2+\epsilon^2}}\,\psi_0 \,, \tag{4.5}$$

(where ψ_0 is in $\mathcal{H}$), the average value of the operator $\hat{A}$ of (4.3) is given by

$$\begin{aligned} \langle A \rangle_{\psi^\epsilon} = & \int dt (\psi_0, A(t)\psi_0)_{\mathcal{H}} \Big(\frac{1}{\pi}\frac{\epsilon}{(t-t_0)^2+\epsilon^2} \Big) \\ & \times \Big[\int dt (\psi_0, \psi_0)_{\mathcal{H}} \Big(\frac{1}{\pi}\frac{\epsilon}{(t-t_0)^2+\epsilon^2} \Big) \Big]^{-1} \\ \rightarrow & \frac{\int dt (\psi_0, A(t)\psi_0)_{\mathcal{H}} \delta(t-t_0)}{\int dt (\psi_0, \psi_0)_{\mathcal{H}} \delta(t-t_0)} \;=\; \frac{(\psi_0, A(t_0)\psi_0)_{\mathcal{H}}}{(\psi_0,\psi_0)_{\mathcal{H}}} \,, \end{aligned} \tag{4.6}$$

for $\epsilon \rightarrow 0$, clearly the usual quantum mechanical expectation value.

We wish to show now that a vector in the direct integral space (which we will refer to as a Lax-Phillips state), can represent both pure and mixed states in the usual sense.

Most measurement processes are concerned with measurements of observables which are time-independent in the Schrödinger picture. Therefore, if two different Lax-Phillips states give the same expectation value for all time-independent observables, these two states are essentially indistinguishable. In this sense, we define the following:

1. A Lax-Phillips vector $\psi \in \bar{\mathcal{H}}$ is called " pure-like" if there exists a pure state

   $$\rho_0 = \phi_0 \phi_0^*, \qquad \phi_0 \in \mathcal{H},$$

   such that

   $$\langle \hat{A} \rangle_\psi = Tr \rho_0 A = (\phi_0, A\phi_0) \tag{4.7}$$

   for every element of the algebra of bounded linear operators associated with the spectral families of the time-independent observables* on the original space $\mathcal{H}$.

* We wish to emphasize that what is meant is *explicit* time-dependence in the Schrödinger picture; we do not refer here to the dynamical time-dependence that may arise in the Heisenberg picture if A is not a constant of the motion.

2. A Lax-Phillips vector is called "mixed-like" if no such (pure) ρ_0 exists.
   We now show that $\psi = \{\psi_t\} \in \mathcal{H}$ is pure-like if an only if it has the form

$$\psi_t = f(t)\phi_0. \tag{4.8}$$

The proof is as follows (we take $\int dt\ \|\psi_t\|^2_{\mathcal{H}} = 1$ henceforth).

$$\langle \hat{A} \rangle_\psi = \int dt\ (\psi_t, A\psi_t) = w(A) \tag{4.9}$$

is a convex linear functional of A. Consider a sequence of projection operators P_n which converge to some projection P in operator norm. Then,

$$w(P_n) = \int dt\ (\psi_t, P_n\psi_t) = \int dt\ \|P_n\psi_t\|^2 \leq 1 \tag{4.10}$$

is a positive sequence converging to $w(P) = \int dt\ \|P\psi_t\|^2$. Hence $w(P)$ is a continuous linear functional on the projection operators on $\mathcal{H}$, and Gleason's theorem (see, e.g., ref. 20) assures, for a Hilbert space of ≥ 3 real dimensions, that there exists a density operator ρ_ψ (for which Tr $\rho_\psi = 1$, Tr $\rho_\psi^2 \leq 1$, $\rho_\psi \geq 0$) such that

$$w(A) = \mathrm{Tr}(\rho_\psi A). \tag{4.11}$$

If $\rho_\psi = \rho_0 = \phi_0\phi_0^*$, a pure state in $\mathcal{H}$, then the condition that must be satisfied is (ρ_0 is a time-independent bounded self-adjoint operator, in fact, a projection), by (4.10),

$$w(\rho_0) = \mathrm{Tr}\rho_0^2 = 1, \tag{4.12}$$

and therefore, from (4.9), we must have

$$\int dt\ |(\psi_t, \phi_0)|^2 = 1. \tag{4.13}$$

By the Schwartz inequality in $\mathcal{H}$ ($\|\phi_0\|_{\mathcal{H}} = 1$),

$$|(\psi_t, \phi_0)|^2 \leq \|\psi_t\|^2_{\mathcal{H}}, \tag{4.14}$$

and $\int dt\ \|\psi_t\|^2_{\mathcal{H}} = 1$, the right hand side of (4.13) is an upper bound on the integral. To achieve this upper bound, ψ_t must be proportional to ϕ_0, i.e., (4.8) must hold, and $\int dt\ |f(t)|^2 = 1$.

We remark that the lift $\hat{\rho}_\psi$ of the time independent operator ρ_ψ defined in (4.11) (or of ρ_0), is, according to (4.3), defined by

$$(\hat{\rho}_\psi\psi)_t = \rho_\psi\psi_t$$

for $\psi_t \in \mathcal{H}$. The operator valued (on $\mathcal{H}$) kernel of $\hat{\rho}_\psi$ in $\bar{\mathcal{H}}$ in the t representation is therefore formally of the form

$$\langle t|\hat{\rho}_\psi|t'\rangle = \delta(t - t')\rho_\psi,$$

so that clearly the trace of $\hat{\rho}_\psi$ in $\bar{\mathcal{H}}$ does not exist. Our discussion has been primarily with the definition of *states on* $\mathcal{H}$ *induced by the vectors of* $\bar{\mathcal{H}}$.

We now discuss the possibility of decoherence, or the evolution from pure-like to mixed-like states. First, we consider the Schrödinger evolution for a time-dependent Hamiltonian. We shall then study a more general evolution, for which we obtain stronger results. The solution of the time-dependent Schrödinger equation can always be written formally as $\psi_t = U(t,t')\psi_{t'}$, where $U(t,t')$ satisfies the chain property $U(t,t')U(t',t'') = U(t,t'')$, and can be expressed in terms of the integral of a time-ordered product. We define $W_t(\tau) = U(t+\tau,t)$, and lift the evolution to $\bar{\mathcal{H}}$ as follows

$$\psi^\tau_{t+\tau} = W_t(\tau)\psi_t\,, \tag{4.15}$$

where $W_t(\tau)$ is given by (T implies the time-ordered product)

$$W_t(\tau) = T\left(e^{-i\int_t^{t+\tau} H(t')dt'}\right)\,. \tag{4.16}$$

For this kind of time-evolution we obtain

$$\langle\hat{A}\rangle_\psi = \int dt\ (W_t(\tau)\psi_t, AW_t(\tau)\psi_t)_{\mathcal{H}}, \tag{4.17}$$

where we have taken the normalization as unity. For the pure-like state introduced in (4.8), we then have

$$\langle\hat{A}\rangle_\psi = \int dt\ |f(t)|^2(W_t(\tau)\phi_0, AW_t(\tau)\phi_0)_{\mathcal{H}}. \tag{4.18}$$

It follows from our previous argument that the effective state corresponding to (4.18) (in the sense of (4.9) and (4.11)) is mixed-like if $W_t(\tau)\phi_0 \neq W_{t'}(\tau)\phi_0$ (i.e., the state ρ_ψ induced from $\psi^\tau_{t+\tau} = W_t(\tau)\psi_t = f(t)W_t(\tau)\phi_0$ is not pure in $\mathcal{H}$).

The evolution operator $W_t(\tau)$, in a full evolution model, does not depend on t if the Hamiltonian is time-independent. In this case, $W_t(\tau) = W(\tau) = e^{-iH\tau}$, and

$$\begin{aligned}\langle\hat{A}\rangle_\psi &= \Big(\int dt\ |f(t)|^2\Big)(W(\tau)\phi_0, AW(\tau)\phi_0)_{\mathcal{H}}\\ &= (W(\tau)\phi_0, AW(\tau)\phi_0)_{\mathcal{H}}\end{aligned} \tag{4.19}$$

so that the corresponding state is pure-like. If the Hamiltonian does not depend on time explicitly, a pure-like state remains pure-like, and no apparent decoherence (in the state induced in $\mathcal{H}$) arises. On the other hand, if the Hamiltonian depends on time explicitly, the states induced in $\mathcal{H}$ do not, in general, maintain their purity and decoherence may take place. For the interaction picture model of Flesia and Piron [19], decoherence may occur. If, however, consistently with the interaction picture model, one takes for the time-independent observables their corresponding interaction picture forms, no decoherence takes place. The result is, in this case, of course independent of the choice of the picture.

As we shall see in a concrete example, the degree of decoherence depends not only on the time-dependence of the Hamiltonian, but also on the initial states.

Example

Tasaki *et al* [34] consider a simple example described by the following Hamiltonian

$$H(t) \;=\; -\frac{\Omega_0}{2}\sigma_z + \frac{\Omega}{2}\left[\sigma_+ e^{i\Omega_0 t} + \sigma_- e^{-i\Omega_0 t}\right], \tag{4.20}$$

where σ_i are the Pauli matrices:

$$\sigma_z \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_+ \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Hilbert space for this model is the two dimensional complex space $\mathcal{H} = \mathbf{C}^2$. It is easy to derive the evolution operator $W_t(\tau)$ corresponding to the Hamiltonian $H(t)$. One obtains

$$W_t(\tau) \;=\; u(\tau)\left\{\cos\frac{\Omega}{2}\tau - i\sin\frac{\Omega}{2}\tau(\sigma_+ e^{i\Omega_0 t} + \sigma_- e^{-i\Omega_0 t})\right\}, \tag{4.21}$$

where the operator $u(\tau)$ is given by

$$u(\tau) \;=\; \begin{pmatrix} \exp(i\frac{\Omega_0}{2}\tau) & 0 \\ 0 & \exp(-i\frac{\Omega_0}{2}\tau) \end{pmatrix}. \tag{4.22}$$

The direct integral space for this model is given by $L^2(-\infty,\infty;\mathbf{C}^2)$. We wish to study now the time evolution of the pure-like state given by Eq. (4.8). ¿From Eqs. (4.15) and (4.21) we have

$$\begin{aligned}
\rho_{\psi_P} \;=\; & u(\tau)\int dt|f(t)|^2\left\{\cos(\frac{\Omega}{2}\tau) - i\sin(\frac{\Omega}{2}\tau)(\sigma_+ e^{i\Omega_0 t} + \sigma_- e^{-i\Omega_0 t})\right\}|\psi_0\rangle \\
& \times \langle\psi_0|\left\{\cos(\frac{\Omega}{2}\tau) + i\sin(\frac{\Omega}{2}\tau)(\sigma_+ e^{i\Omega_0 t} + \sigma_- e^{-i\Omega_0 t})\right\}u^\dagger(\tau) \\
& = \; u(\tau)\Bigg[\cos^2(\frac{\Omega}{2}\tau)|\psi_0\rangle\langle\psi_0| + \sin^2(\frac{\Omega}{2}\tau)\sigma_+|\psi_0\rangle\langle\psi_0|\sigma_- \\
& + \sin^2(\frac{\Omega}{2}\tau)\sigma_-|\psi_0\rangle\langle\psi_0|\sigma_+ \\
& + \left\{iF(\Omega_0)\cos(\frac{\Omega}{2}\tau)\sin(\frac{\Omega}{2}\tau)(|\psi_0\rangle\langle\psi_0|\sigma_+ - \sigma_+|\psi_0\rangle\langle\psi_0|)\right. \\
& \left. + \sin^2(\frac{\Omega}{2}\tau)F(2\Omega_0)\sigma_+|\psi_0\rangle\langle\psi_0|\sigma_+ + h.c.\right\}\Bigg]u^\dagger(\tau) \\
& = u(\tau)W(\tau,\alpha)|\psi_0\rangle\langle\psi_0|W^\dagger(\tau,\alpha)u^\dagger(\tau) \\
& + u(\tau)\Bigg\{i(F(\Omega_0) - e^{i\alpha})\cos(\frac{\Omega}{2}\tau)\sin(\frac{\Omega}{2}\tau)(|\psi_0\rangle\langle\psi_0|\sigma_+ - \sigma_+|\psi_0\rangle\langle\psi_0|) \\
& + \sin^2(\frac{\Omega}{2}\tau)(F(2\Omega_0) - e^{2i\alpha})\sigma_+|\psi_0\rangle\langle\psi_0|\sigma_+ + h.c.\Bigg\}u^\dagger(\tau),
\end{aligned} \tag{4.23}$$

where $F(\omega)$ is the Fourier transform of $|f(t)|^2$

$$F(\omega) \equiv \int dt |f(t)|^2 e^{i\omega t} , \tag{4.24}$$

α is an arbitrary real number (to be chosen for convenience) and the operator $W(\tau, \alpha)$ is given by

$$W(\tau, \alpha) = \cos(\frac{\Omega}{2}\tau) - i\sin(\frac{\Omega}{2}\tau)\big(e^{i\alpha}\sigma_+ + e^{-i\alpha}\sigma_-\big) . \tag{4.25}$$

The first term clearly keeps the purity. Therefore, if we can choose α such that the remaining terms vanish, the time evolution does not destroy the purity of the state. The necessary and sufficient condition is the existence of a real α satisfying

$$F(\Omega_0) = e^{i\alpha} , \qquad F(2\Omega_0) = e^{2i\alpha} . \tag{4.26}$$

As an example, suppose we take $|f(t)|^2$ to be the Gaussian form

$$|f(t)|^2 = \frac{1}{2\sqrt{\pi}\Delta} \exp\left[\frac{(t-t_0)^2}{4\Delta^2}\right] , \tag{4.27}$$

then we obtain

$$F(\omega) = e^{i\omega t_0} e^{-(\omega\Delta)^2} . \tag{4.28}$$

This function clearly cannot satisfy the conditions (4.26), but in order to minimize the terms in (4.23) which destroy the purity, we may take α to be $\alpha = \Omega_0 t_0$. Then we have

$$\begin{aligned} \rho_{\psi_P}(\tau) &= u(\tau)W(\tau, \Omega_0 t_0)|\psi_0\rangle\langle\psi_0|W^\dagger(\tau, \Omega_0 t_0)u^\dagger(\tau) \\ &+ u(\tau)\Big\{ig(\Omega_0)e^{i\Omega_0 t_0}\cos(\frac{\Omega}{2}\tau)\sin(\frac{\Omega}{2}\tau)(|\psi_0\rangle\langle\psi_0|\sigma_+ - \sigma_+|\psi_0\rangle\langle\psi_0|) \\ &+ \sin^2(\frac{\Omega}{2}\tau)g(2\Omega_0)e^{2i\Omega_0 t_0}\sigma_+|\psi_0\rangle\langle\psi_0|\sigma_+ + h.c.\Big\}u^\dagger(\tau), \end{aligned} \tag{4.29}$$

where the function $g(\omega) = |F(\omega)| - 1 = \exp(-(\omega\Delta)^2) - 1$ describes the initial state dependence of the degree of decoherence. Strictly speaking, as $g(\Omega_0)$ and $g(2\Omega_0)$ are different from zero, decoherence takes place irrespective to the value of $\Delta(\neq 0)$. However, if the g-terms are very small, the first term dominates, and the state ρ_{ψ_P} corresponds to an almost pure state. In short, we find for the Gaussian example, that when the initial state is well localized on the t-axis compared with the time scale of the change of the Hamiltonian, i.e. $\Omega_0\Delta << 1$, the state ρ_{ψ_P} remains practically pure. Otherwise, decoherence takes place.

We now generalize the above discussion to the generalized evolution introduced in the previous section. In particular we find that even for *closed* systems mixing of pure states is possible, when the interaction is not local on the time axis.

As we have seen, the generator K of the evolution $U(\tau)$ described by (3.15), may, in general, be written in the form

$$(K\psi)_t \;=\; -i\partial_t\psi_t \;+\; \int \kappa_{t,t'}\psi_{t'}dt' \,. \tag{4.30}$$

When the kernel κ is in the form

$$\kappa_{t,t'} \;=\; \kappa_{t-t'} \,, \tag{4.31}$$

it follows that (here , $-i\partial_t$ stands for the operator on $\bar{\mathcal{H}}$ which is represented as a derivative in the t-representation)

$$[\kappa\,,\, -i\partial_t] \;=\; 0\,. \tag{4.32}$$

Therefore, the system described by a generator of this form is closed in the sense that it is invariant to translations on the time-axis, i.e.,

$$[K\,,\, -i\partial_t] \;=\; 0\,. \tag{4.33}$$

As we have seen in the previous section (Eq. (3.34)), this kind of interaction leads to an evolution operator of the form

$$W_{t,t'}(\tau) \;=\; \frac{1}{2\pi}\int e^{i(t-t')\sigma}e^{-i\kappa(\sigma)\tau}d\sigma \;=\; W_{t-t'}(\tau) \tag{4.34}$$

where $\kappa(\sigma)$ is the Fourier transform of $\kappa_{t-t'}$ with respect to $t-t'$.

Now, consider the most general form of pure state, $\psi_t \;=\; f(t)\psi_0$. The time evolution of such a state is

$$(\psi^\tau)_{t+\tau} \;=\; \int W_{t,t'}(\tau)\psi_0 f(t')dt' \,. \tag{4.35}$$

For an evolution operator of the form (4.34), it follows that

$$\begin{aligned}(\psi^\tau)_{t+\tau} \;&=\; \int W_{t-t'}(\tau)\psi_0 f(t')dt' \;= \\ &=\; \int W_{t'}(\tau)\psi_0 f(t-t')dt' \,.\end{aligned} \tag{4.36}$$

This corresponds, for every t, to a superposition of the states $W_{t'}(\tau)\psi_0$, but, in general, for each t, the weights are different, and we conclude that the state may be mixed by the evolution. The purity of the state will be conserved if and only if all the states $W_{t'}(\tau)\psi_0$ are the same up to a factor which is a function of t' (and τ; the discussion which follows is, however, for each τ). We shall now prove that this occurs for *any* ψ_0 if and only if $\kappa_{t-t'} \;=\; \kappa\delta(t-t')$, where κ is some constant operator.

Let us assume that the state remains pure under evolution, i.e.,

$$W_t(\tau)\psi_0 \;=\; \alpha_t\psi_1 \tag{4.37}$$

for any arbitrary ψ_0 and corresponding ψ_1. Let $\{\phi_n\}$ be a complete orthonormal set in $\mathcal{H}$; then for each τ,

$$W_t(\tau)\phi_n \;=\; \alpha_t \psi_n \;=\; \alpha_t \sum_n \beta_{mn}\phi_m \,, \tag{4.38}$$

and therefore,

$$(\phi_m, W_t(\tau)\phi_n) \;=\; \beta_{mn}\alpha_t \,. \tag{4.39}$$

Hence,

$$W_t(\tau) \;=\; \alpha_t W(\tau) \,, \tag{4.40}$$

where

$$(\phi_m, W(\tau)\phi_n) \;=\; \beta_{mn} \,. \tag{4.41}$$

Taking the Fourier transform of (4.40) one obtains

$$\tilde{W}_\sigma(\tau) \;=\; \tilde{\alpha}(\sigma) W(\tau) \,. \tag{4.42}$$

On the other hand, from (4.34) it follows that

$$\tilde{W}_\sigma(\tau) \;=\; e^{-i\kappa(\sigma)\tau} \,. \tag{4.43}$$

We now show that $W(\tau)$ has an inverse. As we have seen, the evolution operators satisfy the relation

$$\int W_{t,t''}(\tau) W_{t',t''}(\tau)^\dagger dt'' \;=\; \delta(t-t') \,. \tag{4.44}$$

It follows from (4.44) and (4.40) that

$$\int \alpha_{t-t''}\alpha^*_{t'-t''} dt''\, W(\tau) W(\tau)^\dagger \;=\; \delta(t-t') \,, \tag{4.45}$$

and therefore

$$\int \alpha_{t-t''}\alpha^*_{t'-t''} dt'' \;=\; \lambda\delta(t-t') \quad \lambda \neq 0 \,, \tag{4.46}$$

so that

$$\lambda W(\tau) W(\tau)^\dagger \;=\; 1 \,. \tag{4.47}$$

It follows from (3.23) and (3.22), by shifting t, t' and taking $\tau \to -\tau$, that the conjugate can appear on the first instead of the second factor in (4.44) ; it then follows that (λ must be real), $\lambda W(\tau)^\dagger W(\tau) = 1$ as well, i.e.,

$$W^{-1}(\tau) \;=\; \lambda W(\tau)^\dagger \,. \tag{4.48}$$

Hence, W^{-1} exists. Then, from (4.42) and (4.43) it follows that

$$\tilde{W}_\sigma(\tau_1)\tilde{W}_\sigma(\tau_2)^{-1} \;=\; e^{-i\kappa(\sigma)(\tau_1-\tau_2)} \;=\; W(\tau_1)W(\tau_2)^{-1} \,, \tag{4.49}$$

independently of σ; hence,

$$\kappa(\sigma) = \text{const.} \Rightarrow \kappa_{t-t'} = \kappa\delta(t-t'). \tag{4.50}$$

Thus, we realize that pure states remain pure if and only if condition (4.50) is satisfied, which is exactly the case of a time-independent, pointwise Hamiltonian.

We therefore see that a generalized evolution of the form (3.15) may lead to mixing of pure states without assuming that the system is open or non-conservative, in the sense that $W_{t,t'}(\tau)$ may be of the form $W_{t-t'}(\tau)$, as discussed after Eq. (4.30). This result is in agreement with the result of the previous section that in the framework of Lax-Phillips theory, the relation between the singularities of the S-matrix and the spectrum of the generator of the semigroup can be obtained only from the more general evolution, which indicates that the origin of irreversibility may be found in such structures.

In conclusion, we have seen that the Lax-Phillips theory provides a description of the quantum states which admits the possibility of decoherence for time-dependent Hamiltonian systems, and even for closed (but not Hamiltonian in the original Hilbert space) systems. Therefore, Machida-Namiki theory can be formulated naturally in this framework, and it is not necessary to specify the limit between the system and the measuring apparatus.

5. Intrinsic Decoherence in Classical and Quantum Evolution

It has long been emphasized by Prigogine and his co-workers [27] that the natural description for the evolution of a system with many degrees of freedom is that of the evolution of the density matrix ρ, through the Liouville equation,

$$i\frac{d\rho}{dt} = [H, \rho]. \tag{5.1}$$

The density matrix ρ ($\rho \geq 0, Tr\rho = 1$) has the property that $Tr\rho^2 \leq 1$, where the equality is attained only for a pure state. In general, one considers the space of Hilbert-Schmidt operators A for which

$$Tr\, A^* A < \infty; \tag{5.2}$$

the positive (normalized) elements of such a space correspond to the physical states, the density matrices. On this space, the commutator with the Hamiltonian H defines a linear operator $\mathcal{L}$, called the Liouvillian, for which

$$i\frac{d\rho}{d\tau} = \mathcal{L}\rho, \tag{5.3}$$

where one assumes that $\mathcal{L}$ is self-adjoint in the Liouville space. The spectrum of the Liouvillian is, in general, continuous in $(-\infty, \infty)$, and hence there may exist an operator T conjugate to $\mathcal{L}$, such that

$$[T, \mathcal{L}] = i. \tag{5.4}$$

Suppose T is self-adjoint and has the spectral representation

$$T = \int t'\, dE(t'). \tag{5.5}$$

Then it follows from the commutation relation (5.4) that

$$e^{i\mathcal{L}\tau}Te^{-i\mathcal{L}\tau} = T + \tau\,,$$

or

$$e^{i\mathcal{L}\tau}dE(t')e^{-i\mathcal{L}\tau} = dE(t'-\tau)\,, \tag{5.6}$$

i.e., $\mathcal{L}$,T, and $dE(t')$ form an imprimitivity system [35]. With this, we see that the spectral family of the operator T shifts with $\mathcal{L}$ in the same way as the time evolution of the state ρ (in (5.3)), and we may therefore identify T as the "time operator".

In particular, for a Hamiltonian of the form of the sum of an unperturbed operator H_0 and a perturbation V, i.e., $H = H_0 + V$, the corresponding Liouvillian is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I\,. \tag{5.7}$$

Now suppose we consider the "time operator" T_0, conjugate to $\mathcal{L}_0$; it satisfies

$$[\,T_0\,,\,\mathcal{L}_0\,] = i\,.$$

Then, in the spectral representation of T_0,

$${}_0\langle t|[T_0,\mathcal{L}_0]|t'\rangle_0 = i\delta(t-t')\,,$$

or

$$(t-t')\;{}_0\langle t|\mathcal{L}_0|t'\rangle_0 = i\delta(t-t')\,. \tag{5.8}$$

It follows that

$${}_0\langle t|\mathcal{L}_0|t'\rangle_0 = -i\partial_t\delta(t-t')\,. \tag{5.9}$$

Hence,

$${}_0\langle t|\mathcal{L}|t'\rangle_0 = -i\partial_t\delta(t-t') + {}_0\langle t|\mathcal{L}_I|t'\rangle_0\,, \tag{5.10}$$

where the last term is, in general, not diagonal.

We therefore see that the Liouville space formulation of dynamics provides a physical example of a structure in which the evolution law (for which the evolution parameter τ corresponds to the laboratory time) is a nontrivial kernel (non-decomposable) on the time axis, and hence, a Lax-Phillips system which may have an S-matrix with non-trivial analytic properties.

We shall show in this section that the existence of a time operator in the Liouville space provides a natural and consistent mechanism for the decoherence of physical states, i.e., that pure states become mixed during the evolution, both for quantum and classical systems. The Hamiltonian evolution of states in classical mechanics is known by the Liouville theorem to be non-mixing, i.e., to preserve the entropy of the system [36]. The same property holds for the quantum evolution as well, and follows from the unitarity of the evolution operator. This has been an obstacle to the consistent description of irreversible processes from first principles [37]. The usual use of techniques of coarse-graining or truncation to achieve a realization of the second law does not follow from basic dynamical laws, and is fundamentally not consistent with the underlying Hamiltonian dynamical structure [38].

To show that the Liouville space provides a natural framework for such mixing to develop, we first make some definitions. The notion of a pure state is defined by means of expectation values of observables, i.e., a state is called "pure" if the expectation value of each observable in this state is equal to the corresponding expectation value computed with respect to some well-defined wave-function. We wish to weaken this condition, and require such an equality only for a t-independent subset of observables (to be defined precisely below). One obtains all the physical information concerning this subset of observables from an *effective* state resulting from the reduction of the full state by integration over the degree of freedom which is not relevant for this subset, i.e., the spectrum of the time operator. We call this reduced state the *physical state*. We show that there exist mixed states for which the effective physical state is pure and denote them as "effectively pure". These states may become effectively mixed during the evolution of the system. We will formulate these ideas in the framework of the quantum Liouville space, and will consider later their application to classical mechanics. We also consider a simple example to illustrate this mechanism.

The kernel representing a Hilbert-Schmidt operator A on the original Hilbert space of n degrees of freedom, $\langle \mathbf{k}|A|\mathbf{k}'\rangle$, where $\mathbf{k}$ consists of n parameters, corresponds to the function $A(\mathbf{k},\mathbf{k}') \equiv \langle \mathbf{k},\mathbf{k}'|A\rangle$ representing the vector A of the Liouville space. We then change variables from $\mathbf{k},\mathbf{k}'$ to t, the spectrum of T, and $(2n-1)$ other independent parameters β. This transformation is defined by a kernel $K(t,\beta|\mathbf{k},\mathbf{k}')$ such that

$$A(t,\beta) \equiv \langle t,\beta|A\rangle = \int K(t,\beta|\mathbf{k},\mathbf{k}')\langle \mathbf{k},\mathbf{k}'|A\rangle d\mathbf{k}d\mathbf{k}' , \tag{5.11}$$

and, in particular, for the density operator ρ (positive A),

$$\rho_t(\beta) \equiv \langle t,\beta|\rho\rangle = \int K(t,\beta|\mathbf{k},\mathbf{k}')\langle \mathbf{k},\mathbf{k}'|\rho\rangle d\mathbf{k}d\mathbf{k}' . \tag{5.12}$$

In what follows, we shall use the time operator $T \equiv T_0$ conjugate to the unperturbed Liouville operator, which is defined according to the decomposition [39]

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I , \tag{5.13}$$

i.e., on a suitable domain

$$[T,\mathcal{L}_0] = i . \tag{5.14}$$

It is clear from (5.14) that

$$\big(e^{-i\mathcal{L}_0\tau}A\big)(t,\beta) = A(t+\tau,\beta) . \tag{5.15}$$

Under the free evolution, the representation of A on the Liouville space undergoes translation $t \to t+\tau$, so that t acquires the meaning of a label for the free translation in time.

Using this new basis, the expectation value of an observable is written as

$$\langle A \rangle_\rho \;=\; \mathrm{Tr}(A\rho) \;=\; \int \rho_t(\beta) A(t,\beta) dt d\beta \,, \tag{5.16}$$

where

$$A(t,\beta) \;=\; \int K(t,\beta|\mathbf{k},\mathbf{k}')\langle \mathbf{k},\mathbf{k}'|A\rangle d\mathbf{k} d\mathbf{k}' \,. \tag{5.17}$$

If A belongs to the subset of t-independent operators, i.e., $A(t,\beta) \equiv A(\beta)$, then from Eq. (5.16) it follows that

$$\langle A \rangle \;=\; \mathrm{Tr}(A\rho) \;=\; \int \hat{\rho}(\beta) A(\beta) d\beta \,, \tag{5.18}$$

where $\hat{\rho}$ is defined as

$$\hat{\rho}(\beta) \equiv \int dt \rho_t(\beta) \,. \tag{5.8}$$

We therefore see that with respect to the set of t-independent observables, all of the information available in the state is contained in $\hat{\rho}$. It follows from Eq. (5.15) that t-independent observables commute with the free Hamiltonian H_0. In this case, clearly the asymptotic form of the observable A (in Heisenberg picture) exists if the wave operator for the scattering theory exists, i.e.,

$$\begin{aligned} \lim_{\tau \to \pm\infty} e^{-i\mathcal{L}\tau} A &= \lim_{\tau \to \pm\infty} U(\tau)^{-1} A U(\tau) \\ &= \lim_{\tau \to \pm\infty} U(\tau)^{-1} U_0(\tau) A U_0(\tau)^{-1} U(\tau) \,, \\ &= \Omega_\pm A \Omega_\pm^{-1} = A_\pm \end{aligned} \tag{5.20}$$

where $U(\tau)$ is the full evolution operator, and $U_0(\tau)$ is that of the unperturbed evolution. The t-independent observables therefore correspond to the asymptotic variables in a scattering theory [40].

Ludwig [41] has emphasized that measurements on a quantum system are made by means of the detection of signals corresponding to observables which are operationally on a semi-classical or classical level. These measurable signals, which characterize the state are the properties propagating to the detectors, and are therefore asymptotic variables, i.e., ξ-independent. We do not argue that observables which are time dependent in Heisenberg picture (such as the electro-magnetic field) play no role. These operators may be even useful for calculations of measurable quantities, and their expectation values can be evaluated using, for example, the Schwinger-Keldysh technique [42]. However, from a physical point of view, based on the above mentioned theoretical arguments on the nature of measurement, only functions of these observables which have asymptotic limits (in the case of electro-magnetic field, the free number density and the momentum, for example) provide for experimental measurement. Measurements carried out upon an evolving system involve, in fact, interactions with apparatus which are essentially asymptotic (e.g. magnetic fields far from an electron beam, or the e-ν or photon signal from the pions in the final state of K-meson decay). These asymptotic observables determine the structure of the state, and hence (with a sufficient number of such measurements) can be used to define the nature of the evolution, i.e., whether a pure state tends to a mixed state. We thus conclude that the subset of ξ-independent observables corresponds to all the experimentally accessible measurements, and is therefore the subset of observables which can be used to characterize experimentally the structure of a physical state.

Note that $\hat{\rho}$ is not simply related to the density matrix of the system, but is given by the integral of (2.12) over the variable t. In fact, the unit operator on the original Hilbert space is represented by

$$\begin{aligned} 1(t,\beta) &= \int K(t,\beta|\mathbf{k},\mathbf{k}')\delta(\mathbf{k}-\mathbf{k}')d\mathbf{k}d\mathbf{k}' \\ &= \int K(t,\beta|\mathbf{k},\mathbf{k})d\mathbf{k} \end{aligned} \quad . \tag{5.21}$$

Now, $\mathcal{L}_0$ annihilates the unit operator, i.e.,

$$\left(e^{-i\mathcal{L}_0 t}1\right)(\mathbf{k},\mathbf{k}') = 1(\mathbf{k},\mathbf{k}'), \tag{5.22}$$

so that, according to (5.15), $1(t,\beta)$ is independent of t, i.e., $1(t,\beta) \equiv 1(\beta)$. The function $1(\beta)$ is, moreover, invariant under all automorphisms of the algebra of observables which leave $t(\mathbf{k},\mathbf{k}')$ invariant. We shall discuss the properties of the representations provided by t,β in more detail elsewhere [43]. For our present purpose, we note that

$$\begin{aligned} Tr\rho &= \int \rho(t,\beta)1(t,\beta)dt\ d\beta \\ &= \int \hat{\rho}(\beta)\ 1(\beta)\ d\beta = 1, \end{aligned} \tag{5.23}$$

so that $1(\beta)$ provides the appropriate measure for what we have called the physical state.

A state $\hat{\rho}$ is called *effectively pure* if there exists a wave function ψ such that for every t-independent observable A

$$\langle\psi|A|\psi\rangle \ = \ \langle A\rangle_{\hat{\rho}} \ = \ \int \hat{\rho}(\beta)A(\beta)d\beta\,. \tag{5.24}$$

The form of $\hat{\rho}(\beta)$ can be determined by the measurement of all the t-independent observables. As will be shown below, in the basis of (generalized) eigenstates of the free Hamiltonian, $\hat{\rho}(\beta)$ must be represented as a sum of bilinear functions over equal energy subspaces. The state $\hat{\rho}(\beta)$ is effective pure if and only if the coefficients of these bilinear functions are factorizable.

If ρ is pure in the usual sense, i.e., $\mathrm{Tr}\rho^2 = 1$, then the condition (5.24) holds for any observable, and therefore the resulting $\hat{\rho}$ is effectively pure. On the other hand, it is clear that the reduction Eq. (5.19) is not one to one and therefore each $\hat{\rho}$ corresponds to an *equivalence class* of states in Liouville space. Even if only one of these states is pure, $\hat{\rho}$ would be effectively pure, since it does not distinguish between elements of the equivalence class. We thus see that strict purity implies effective purity but not the opposite, i.e., *even mixed states may appear as physically pure.*

We wish to show now that while unitarity excludes the possibility of mixing of pure states, mixing of effectively pure states is still possible. Generally, in the presence of interaction, the full Liouvillian takes the form (from (5.10)

$$\langle t|\mathcal{L}|t'\rangle \;=\; -i\partial_t\delta(t-t') \;+\; \langle t|\mathcal{L}_I|t'\rangle\,, \tag{5.25}$$

where the second term is, in general, not diagonal, but rather acts as an integral operator on t. The resulting evolution is also of an integral operator structure and takes the form (we call the evolution parameter τ to distinguish it from the spectrum of the T-operator)

$$\rho^\tau_t \;=\; \int W_{t,t'}(\tau)\rho^0_{t'}dt'\,, \tag{5.26}$$

where the operator $W_{t,t'}(\tau)$ acts only on the β dependence.

For simplicity we use the Fourier transform representation

$$\rho(\alpha,\beta) \;=\; \int e^{-it\alpha}\rho_t(\beta)dt;$$

$$\overline{W}_{\alpha,\alpha'}(\tau) \;=\; \int e^{-it\alpha}e^{it'\alpha'}W_{t,t'}(\tau)dtdt'\,. \tag{5.27}$$

Note that $\hat\rho(\beta)=\rho(\alpha,\beta)|_{\alpha=0}$, and therefore

$$\hat\rho^\tau(\beta)=\rho^\tau(\alpha,\beta)|_{\alpha=0}=\int \overline{W}_{0,\alpha'}(\tau;\beta,\beta')\rho(\alpha',\beta')d\alpha'd\beta'\,. \tag{5.28}$$

The initial effective purity of ρ provides information only on its $\alpha=0$ component while the other components may be effectively mixed, but, as we see from Eq. (5.28), during the evolution the $\alpha=0$ component develops contributions from the other components, and therefore it may become mixed. The states keep their effective purity, in general, only if $\overline{W}_{0,\alpha'}\sim\delta(\alpha')$.

We are now in a position to characterize an effectively pure state more explicitly. Since α is the Fourier dual of the variable t, it follows from (5.3) that α is the spectrum of the unperturbed Liouvillian $\mathcal{L}_0$. Hence, the $\alpha=0$ component of a state $\sum c(\mathbf{k},\mathbf{k}')\psi_\mathbf{k}\psi^*_{\mathbf{k}'}$, for a basis $\{\psi_\mathbf{k}\}$ which are (generalized) eigenfunctions of H_0, is the partial sum over the terms for which the (unperturbed) energy eigenvalues associated with $\psi_\mathbf{k}$ and $\psi_{\mathbf{k}'}$ are equal. For a pure state corresponding to $\psi=\sum a(\mathbf{k})\psi_\mathbf{k}$, $c(\mathbf{k},\mathbf{k}')=a(\mathbf{k})a(\mathbf{k}')^*$ is factorizable. An effectively pure state coincides with the $\alpha=0$ component of a pure state, and therefore satisfies this factorizability condition in the equal energy subspaces. On the other hand, this condition in the equal energy subspaces does not imply its general validity (for $\alpha\neq 0$). Hence, an effectively pure state is associated with an equivalence class which includes mixed states as well.

Note that if H_0 is nondegenerate, the effective purity condition holds trivially for every state (diagonal elements of the density matrix in H_0 representation are positive definite), and the evolution cannot induce mixing.

Example

We wish to consider now a simple concrete example to illustrate the above ideas. Consider the evolution of a particle in three dimensions in the presence of a screened Coulomb (Yukawa) potential. The matrix elements of the free Liouvillian are given by (we take $2m = 1$)

$$\langle \mathbf{k}_1, \mathbf{k}_2 | \mathcal{L}_0 | \mathbf{k}_3, \mathbf{k}_4 \rangle = \delta^3(\mathbf{k}_1 - \mathbf{k}_3)\delta^3(\mathbf{k}_2 - \mathbf{k}_4)(\mathbf{k}_2^2 - \mathbf{k}_1^2) \,. \tag{5.29}$$

We change the variables in Liouville space from $(\mathbf{k}_1, \mathbf{k}_2)$ to $(\alpha, \beta, \Omega_1, \Omega_2)$ by the transformation

$$\alpha = \mathbf{k}_2^2 - \mathbf{k}_1^2, \qquad \beta = \mathbf{k}_2^2 + \mathbf{k}_1^2, \tag{5.30}$$

and Ω_1, Ω_2 are the angle variables of the momenta $\mathbf{k}_1, \mathbf{k}_2$, respectively. We denote the set of variables $\beta, \Omega_1, \Omega_2$ by $\bar{\beta}$. In this new basis the matrix elements of the free Liouvillian are given by

$$\langle \alpha, \bar{\beta} | \mathcal{L}_0 | \alpha', \bar{\beta}' \rangle = \alpha \delta(\alpha - \alpha')\delta(\bar{\beta} - \bar{\beta}') \,. \tag{5.31}$$

The variable α defined by this change of basis coincides with the α of our general discussion above.

As mentioned before, effectively pure states are mixed during the evolution unless $\overline{W}_{0,\alpha'} \sim \delta(\alpha')$. We therefore look at the evolution operators induced by the perturbation to see whether this is the case. The matrix elements of the interaction Liouvillian are given by

$$\langle \mathbf{k}_1, \mathbf{k}_2 | \mathcal{L}_I | \mathbf{k}_3, \mathbf{k}_4 \rangle = \delta^3(\mathbf{k}_1 - \mathbf{k}_3)\tilde{V}_{\mathbf{k}_2 - \mathbf{k}_4} - \delta^3(\mathbf{k}_2 - \mathbf{k}_4)\tilde{V}_{\mathbf{k}_1 - \mathbf{k}_3} \,, \tag{5.32}$$

where $\tilde{V}_{\mathbf{k}}$ is the Fourier transform of the potential V, taken at the point $\mathbf{k}$.

For the screened Coulomb potential

$$V(r) = \frac{Ae^{-\mu r}}{\mu r} \,, \tag{5.33}$$

$\tilde{V}_{\mathbf{k}}$ is given by

$$\tilde{V}_{\mathbf{k}} = \frac{4\pi A}{\mu(\mathbf{k}^2 + \mu^2)} \,, \tag{5.34}$$

and the matrix element takes the form

$$\langle \mathbf{k}_1, \mathbf{k}_2 | \mathcal{L}_I | \mathbf{k}_3, \mathbf{k}_4 \rangle = \frac{4\pi A}{\mu} \left(\frac{\delta^3(\mathbf{k}_1 - \mathbf{k}_3)}{(\mathbf{k}_2 - \mathbf{k}_4)^2 + \mu^2} - \frac{\delta^3(\mathbf{k}_2 - \mathbf{k}_4)}{(\mathbf{k}_1 - \mathbf{k}_3)^2 + \mu^2} \right) . \tag{5.35}$$

Changing the variables to $(\alpha, \bar{\beta})$, one obtains

$$
\begin{aligned}
\angle\alpha, \bar{\beta}|\mathcal{L}_I|\alpha', \bar{\beta}'\rangle \equiv \mathcal{L}_I(\alpha, \alpha', \bar{\beta}, \bar{\beta}') = \\
= \frac{64\pi A}{\sqrt{2}\mu}\Bigg(& \frac{\left[\delta(\beta - \alpha - \beta' + \alpha')\frac{1}{\sqrt{\beta-\alpha}}\right]\delta(\Omega_1, \Omega_3)}{\beta + \beta' + \alpha + \alpha' - 2\sqrt{(\beta+\alpha)(\beta'+\alpha')}B(\Omega_2, \Omega_4) + \mu^2} \\
& - \frac{\left[\delta(\beta + \alpha - \beta' - \alpha')\frac{1}{\sqrt{\beta+\alpha}}\right]\delta(\Omega_2, \Omega_4)}{(\beta + \beta' - (\alpha + \alpha') - 2\sqrt{(\beta-\alpha)(\beta'-\alpha')}B(\Omega_1, \Omega_3) + \mu^2}\Bigg),
\end{aligned}
\tag{5.36}
$$

where $B(\Omega_1, \Omega_2)$ is defined by

$$
B(\Omega_1, \Omega_2) = \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) + \cos\theta_1 \cos\theta_2 \,. \tag{5.37}
$$

It is therefore clear that the kernel $\mathcal{L}_I(\alpha, \alpha', \bar{\beta}, \bar{\beta}')$ is *not* of the form $\delta(\alpha - \alpha')\hat{A}(\bar{\beta}, \bar{\beta}')$ and therefore the evolution operators also do not have this form. In particular, for weak interactions, first order perturbation theory gives

$$
\overline{W}_{0,\alpha}(\tau) = \delta(\alpha) - i\tau\mathcal{L}_I(0, \alpha, \bar{\beta}, \bar{\beta}') + O(\tau^2 A^2)\,, \tag{5.38}
$$

where the second term induces mixing.

Application to Classical Mechanics

The method that we have described above applies as well to the formulation of classical mechanics on a Hilbert space defined on the manifold of phase space which was introduced by Koopman [44] and used extensively in statistical mechanics [38]. Misra [26] has shown that dynamical systems which admits a Lyapunov operator necessarily have absolutely continuous spectrum; therefore one can construct a time operator on the classical Liouville space for such systems. We identify the variables $\mathbf{k}, \mathbf{k}'$ with the variables of the classical phase space, and consider the trace as an integral over this space. The expectation value of a t-independent operator defines a reduced density function in the form (5.19). Since a pure state is defined by a density function concentrated at a point of the phase space, a state which is effectively pure must have the form $\hat{\rho}(\beta) = \delta(\beta - \beta_0)$. The equivalence class associated with this reduced density contains mixed states as well, such as $\rho(t, \beta) = \delta(\beta - \beta_0) f(t)$ corresponding to a non-localized function on the phase space $(\mathbf{k}, \mathbf{k}')$. The structure of the theory, and the conclusions we have reached, are therefore identical to those of the quantum case.

6. Conclusions

The Lax-Phillips theory assumes the existence of a unitary evolution $U(\tau)$ on a Hilbert space $\bar{\mathcal{H}}$, and incoming and outgoing subspaces satisfying the conditions (2.19). Under these conditions, incoming (outgoing) representations of $\bar{\mathcal{H}}$ which are of the form $\bar{\mathcal{H}} = L^2(\mathcal{R}; \mathcal{H})$ are defined, where $\mathcal{H}$ is an auxiliary Hilbert space, for which the evolution is represented by translation and the incoming (outgoing) subspace has definite support properties. The unitary transformation between these representations is called the S-matrix, and is of a multiplicative form in the spectral representation of the generator of the evolution (which is the Fourier transform of the outgoing representation). They define a (strongly contractive) semigroup in the form $\mathcal{Z}(\tau) = P_+ U(\tau) P_-$, where $P_\pm$ is the projection on the orthogonal complement of $\mathcal{D}_\pm$. The main result is that the eigenvalues of the generator of the semigroup, which determine the life-time for the contraction, correspond to the singularities (of the analytic continuation) of the S-matrix in the spectral representation of the generator of the evolution.

The physical system Lax-Phillips had primarily in mind is scattering of waves (electromagnetic or acoustic) from an obstacle. For this system they identified the incoming and outgoing subspaces with the subspaces of incoming and outgoing waves, respectively. Thus, the abstract Lax-Phillips semigroup corresponds in this case to the contracted evolution of waves in the region of the obstacle. It originally appeared that the ideas of the Lax-Phillips theory could not be extended to the quantum theory [14], since the generator of translations, which are the representation of the evolution in the incoming and outgoing representation, has a spectrum on the whole real line, and the generator of the evolution for the quantum case is semi-bounded for most cases.

Flesia and Piron [19] have shown that the use of the larger Hilbert space formulation of quantum mechanics solves this problem since the spectrum of the generator of the evolution in the larger space is indeed the whole real line. In this formulation the evolution takes place in a direct integral Hilbert space, over the usual quantum mechanical Hilbert spaces, indexed by t, and the evolution is according to a new evolution parameter τ. In this framework, they have calculated the Lax-Phillips S-matrix, and shown that it has the form $(S\psi)_t = S_t\psi_t$, where S_t is a map from $\mathcal{H}$ to $\mathcal{H}$, independent of t, and corresponds to the usual scattering theory S-matrix in the limit of the gap between the subspaces $\mathcal{D}_\pm$ going to ∞.

However, this theory has a fundamental difficulty. As we have mentioned above, the relation between the S-matrix and the semigroup is established in terms of the Fourier transform of the S-matrix. On the other hand, since S_t is independent of t, $S_t \equiv S_0$, the Fourier transform of S is trivial, i.e., $S(\sigma) = S_0\delta(\sigma)$, and is not a meromorphic function of σ. Therefore this theory cannot use the full structure of the Lax-Phillips theory, and in particular, is not applicable to the description of unstable systems.

The Flesia-Piron construction can, however, be extended to *non-decomposable* evolution. As discussed in Section 3, this kind of evolution occurs naturally in the framework of the Liouville space; it may also correspond to action at a distance as a consequence of the Green's function, or to a pointwise Hamiltonian in another representation, which becomes non-decomposable in the representation in which $\mathcal{D}_\pm$ have definite support properties.

We have shown for this general evolution law, that the S-matrix is indeed multiplicative in the Fourier transform basis, but, in general, is not trivial. Furthermore, the semigroup is contractive, independently of the form of the non-decomposable (self-adjoint) perturbation. One finds, in this construction, an exact superselection rule separating the states of the unstable system and the decay products.

We have reviewed the application of Lax-Phillips theory to the idea of Machida and Namiki[33] for the transition to a mixed state during the measurement[34]. Such a transition can be consistently realized in the framework of the Lax-Phillips theory, with no necessity for distinguishing the system from the measurement apparatus. The measurement is characterized by expectation values in the Lax-Phillips pure state of time independent observables; if the Lax-Phillips state contains non-trivial time-dependence (i.e., not simply factorizable), then the resulting measurement appears as that of a mixed state, resulting in one or another state with some probability, but not a linear superposition.

Extending this idea to the generalized evolution, we have found that decoherence takes place even for closed systems, i.e., systems which are invariant for translations along the t-axis, (excluding generators of the form $-i\partial_t + H$, where H is t-independent). Therefore Namiki's measurement theory can be applied for closed systems also. This may be considered a first step towards a theory of the measurement process.

It has also been recently shown, as discussed in Section 5, following similar ideas, that the Liouville space which provides a natural framework for the realization of Lax-Phillips theory in its most general form, also admits the construction of a mechanism for the decoherence of physical states. This mechanism is contained in the formation of equivalence classes of states with respect to the measurement of operators which are independent of the Liouville time, for example, corresponding to asymptotic observables. It is shown that an effective pure state may actually be mixed, and its evolution will lead, in general, to an effective mixed state. The evolution of such equivalence classes rather than the evolution of a particular state, therefore become a proper subject of study.

Lockhart and Misra [45] have discussed the notion of dynamical evolution (following the unitary or measure preserving standard mathematical models of evolution) and *physical* *evolution*, for which the evolution is realized as a dissipative semigroup consistent with the second law of thermodynamics. In fact, they state that the central problem of quantum measurement theory is the reconciliation of the Schrödinger evolution with the statistical evolution caused by the measurement process generally referred to as the "collapse of the wave packet". Criteria are given by Lockhart and Misra [45], where a rather detailed model of irreversible processes is developed, for the type of systems which admit irreversible behavior. It appears that it may be possible to imbed these ideas into the framework of the Lax-Phillips theory. We conjecture that the dissipative semi-group behavior we have discussed in the framework of the Lax-Phillips theory is representative of the evolution of non-integrable systems, and that there are no complex singularities of the spectral representation of the S-matrix (corresponding to the spectrum of the semigroup generator) for an integrable system.

In this review, we have discussed states in a space which includes the time as a variable, with an additional parameter that corresponds to the laboratory clock. The interpretation of such a time variable (subject, as well, to transformations from one representation to another) is not at all trivial; the *a priori* distribution of its values varies systematically under the Lax-Phillips or Liouville evolution, with the evolution of the state. The realization of a non-trivial Lax-Phillips theory, as well as the structure of the Liouville evolution, impose a dynamical role for this variable, as would be expected to be necessary for the description of a physical *process*, reversible or irreversible. Some progress, as we have pointed out, has been made in understanding this essential aspect of the theory, and it is hoped that future work will bring further insight.

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39. The Liouville operator is defined by

$$e^{-i\mathcal{L}\tau}\rho = e^{-iH\tau}\rho e^{iH\tau} ,$$

and $\mathcal{L}_0$ by

$$e^{-i\mathcal{L}_0\tau}\rho = e^{-iH_0\tau}\rho e^{iH_0\tau} .$$

Then, $\mathcal{L}_I \equiv \mathcal{L} - \mathcal{L}_0$. We thank I. Antoniou for a communication on this point, as well as for a discussion of the interpretation of Eq. (5.15).
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