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Limited Sensitivity to Analyticity: a Manifestation of Quantum Chaos

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Classical Hamiltonian systems generally exhibit an intricate mixture of regular and chaotic motions on all scales of phase space. As a nonintegrability parameter K (the "strength of chaos") is gradually increased, the analyticity domains of functions describing regular-motion components [e.g., Kolmogorov–Arnol'd–Moser (KAM) tori] usually shrink and vanish at the onset of global chaos (breakup of all KAM tori). It is shown that these phenomena have quantum-dynamical analogs in simple but representative classes of model systems, the kicked rotors and the two-sided kicked rotors. Namely, as K is gradually increased, the analyticity domain $\mathcal{R}_{QE}$ of the quantum-dynamical eigenstates decreases monotonically, and the width of $\mathcal{R}_{QE}$ in the global-chaos regime vanishes in the semiclassical limit. These phenomena are presented as particular aspects of a more general scenario: As K is increased, $\mathcal{R}_{QE}$ gradually becomes less sensitive to an increase in the analyticity domain of the system.

INTRODUCTION

By the term "chaos" we generally mean a complex, highly irregular motion exhibited by a relatively simple nonlinear dynamical system. This motion is deterministic, namely it corresponds to an exact solution of the equation (or the system of equations) describing the system. This solution is associated, however, with an extremely nonanalytic structure in the space of the motion,[1] and cannot therefore be described exactly by well-behaved functions or convergent series expansions.

Of particular interest are Hamiltonian systems[2–5] for which the Newton equations can be derived from a Hamiltonian H, generally a function of both

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phase-space variables and time. Hamiltonian systems are interesting because of two main reasons: (a) They exhibit, generically, a very rich dynamical structure, consisting of regular-motion components (either stable or unstable) intricately mixed with chaotic motion on all scales of phase space.[2–6] These two kinds of motion then affect significantly each other. Chaotic orbits usually look like a random sequence of "quasiregular" segments[6–8] each resembling some ordered orbit in its immediate vicinity; this quasiregularity leads to slow decays of correlations[6] and related phenomena in Hamiltonian chaos. At the same time, the basic features of regular-motion components depend strongly on the level of chaos in the system (see below). (b) The existence of a Hamiltonian allows for a well-defined first-principles quantization of the classical system.[9, 10] Since the basic equations of quantum mechanics, e.g., the Schrödinger equation, are linear in the wave-function, they cannot exhibit a strictly chaotic behavior. On the other hand, in the semiclassical limit ($\hbar$ much smaller than typical classical actions), quantum dynamics is expected to resemble classical dynamics to a significant extent. The fundamental question is then how precisely classical chaos manifests itself in the behavior of the corresponding quantum-dynamical system in the semiclassical limit. This is the problem of "quantum chaos".[9, 10].

In this paper, we show a vivid manifestation of quantum chaos in simple but representative classes of model systems, the kicked rotors (KRs) and the two-sided kicked rotors (TKRs). The general Hamiltonians for these systems are

$$H_{KR}=\frac{L^2}{2I}+\hat{k}V(\theta)\,\Delta_T(t) \tag{1}$$

$$H_{TKR}=\frac{L^2}{2I}+\hat{k}V(\theta)[\Delta_T(t)-\Delta_T(t-T/2)] \tag{2}$$

where L is the angular momentum, I is the moment of inertia, $\hat{k}$ is a parameter, $V(\theta)$ is a periodic and analytic function of the angle θ, and $\Delta_T(t)=\sum_{s=-\infty}^{\infty}\delta(t-sT)$ is the periodic delta function with time period T. In order to state our results, it is necessary to give first some background concerning the classical dynamics of these systems.

The classical dynamics can be visualized in a most clear way by the Poincaré map, connecting the phase-space variables at times t and $t+T$. For simplicity, we shall restrict ourselves here to the KR case (1), for which the Poincaré map reads as follows:

$$\begin{aligned} L_{s+1}&=L_s-\hat{k}V'(\theta_s)\\ \theta_{s+1}&=\theta_s+(T/I)\,L_{s+1} \end{aligned} \tag{3}$$

where $L_s = L(t = sT - 0)$ and $\theta_s = \theta(t = sT = 0)$, for all integers s. After defining the variable $p = TL/I$, it becomes clear that the map (3) depends only on the parameter $K = T\hat{k}/I$. Different initial conditions (θ_0, p_0) lead to different orbits (θ_s, p_s) of (3). Some typical orbits are shown in Fig. 1 for the "standard" potential $V(\theta) = \cos(\theta)$ and for different values of K. For large K (Fig. 1d), chaos prevails, with chaotic orbits filling almost

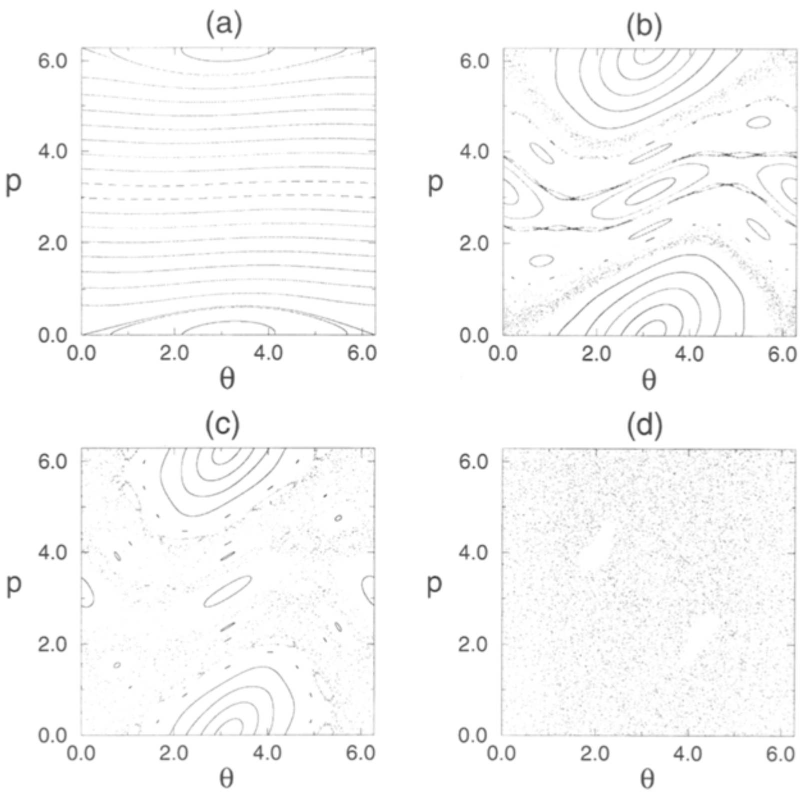

Fig. 1. Typical orbits of the "standard map" (3) with $V(\theta) = \cos(\theta)$, for different values of $K = T\hat{k}/I$: (a) $K = 0.1$. Here most of the orbits are "horizontal," including many KAM tori, and regular motion prevails. Notice the "separatrix" orbit, connecting the points (0, 0) and $(2\pi, 0)$ as well as the points $(0, 2\pi)$ and $(2\pi, 2\pi)$. This orbit is actually a narrow chaotic "layer." (b) $K = 0.9716 \approx K_c$. The chaotic separatrix layer is now much thicker than in (a), and smaller chaotic layers, associated with stable island chains, are also visible. Here the chaos is still "local," since the golden-mean KAM tori still exist (see text). (c) $K = 1.2 > K_c$. Here no KAM tori exist, so that all the localized chaotic layers can merge into a single connected chaotic region (global chaos). (d) $K = 5$, a strong-chaos case. The chaotic region covers a large fraction of the phase space.

randomly a finite area[11] of the (θ_s, p_s) phase space. For $K \ll 1$, on the other hand, almost all the orbits are "horizontal" (Fig. 1a), resembling those of the integrable case $K=0$ of the free rotor for which $p_{s+1}=p_s$ is a constant of the motion. Such an orbit generally fills densely a "rotational torus," i.e., a curve connecting the points $(0, p)$ and $(2\pi, p)$ for some p. The existence of these tori for $K \neq 0$ is predicted by the celebrated Kolmogorov–Arnol'd–Moser (KAM) theorem,[12] but they are actually observed for $K \leqslant K_c$ where K_c is much larger than typical bounds from KAM theory. For example, $K_c \approx 0.9716$ for $V(\theta)=\cos(\theta)$ (see Fig. 1b). For $K \ll K_c$, the phase space is foliated to a large extent by the KAM tori, as in Fig. 1a, and chaotic orbits are *locally* confined to a very small fraction of the area between two neighboring tori, which are "barriers" to chaotic motion.[13] As K is gradually increased, KAM tori successively "break" into "cantori,"[14] i.e., orbits leaving an infinite family of gaps in θ. Chaotic orbits can cross the cantori through the gaps,[13] leading to a gradual increase of the local chaotic regions. At $K=K_c$ (see Fig. 1b), there remains only one discrete set of KAM tori, the so-called "golden-mean tori," characterized by $\langle p_s \rangle/(2\pi) = \pm(\sqrt{5}-1)/2+r$ for all integers r.[15] For $K>K_c$, also these tori break into cantori, leading to a *transition from local to global chaos* (Fig. 1c). Global chaos features an unbounded, almost random diffusion in the p direction.[3, 13, 16]

The breakup of the golden-mean tori, at the onset of the local-to-global-chaos transition, is preceded (for $K<K_c$) by an interesting phenomenon. In general, a KAM curve is described by a *single-valued* 2π-periodic function $p=p(\theta)$:[17]

$$p(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \tag{4}$$

where the Fourier coefficients a_n be determined by perturbation theory in K starting from functional equations satisfied by $p(\theta)$,[18–20] e.g., $p[\theta + p(\theta) - KV'(\theta)] = p(\theta) - KV'(\theta)$.[20] Now, both numerical[18] and analytical[21] results for $V(\theta)=\cos(\theta)$ indicate that, asymptotically, the Fourier coefficients a_n for the golden-mean curve $p(\theta)$ in (4) decay exponentially for $K<K_c$:

$$a_n \sim \exp[-\beta(K)\,|n|] \tag{5}$$

where $\beta(K)$ is a positive function satisfying $\beta(K) \to 0$ as $K \to K_c$ from below. For K sufficiently close to K_c, $\beta(K)$ is essentially linear in $K_c - K$. At the critical point $K=K_c$, a_n decays only algebraically with n. This fact is closely related to the self-similar structure of the critical golden-mean tori,[15, 18] represented by continuous but nondifferentiable functions $p(\theta)$. Because of (5), the Fourier-series representation (4) of $p(\theta)$ is analytic precisely in the infinite strip $|\mathrm{Im}(\theta)| < \beta(K)$ in the complex θ-plane. One may then say that as K is increased the domain of analyticity of a golden-mean torus gradually decreases. This phenomenon reflects in a natural way the increase in the region of nonanalytic dynamical structure (i.e., the chaotic region) as K is increased. For $K \geqslant K_c$, the torus or its remnants (the cantorus) is a nonanalytic object.

In this paper, we show that this phenomenon has an analogue in the quantum dynamics. Classical orbits or invariant sets, in particular KAM tori or chaotic regions, correspond to quantum eigenstates. For time-periodic systems such as (1) and (2), these are the eigenstates of the unitary evolution operator U in one period, the so-called "quasi-energy" (QE) states. We consider the systems (1) and (2) for naturally defined potentials $V_N(\theta)$, $N = 1, 2,...,$ converging in the derivative to $V(\theta) = \cos(\theta)$. The domain of analyticity $\mathscr{R}_N$ of $V_N(\theta)$ generally increases without bounds with N. For a given potential $V_N(\theta)$, we show that the domain of analyticity $\mathscr{R}_{QE, N}$ of the QE states is not larger than $\mathscr{R}_N$. As K is increased at fixed $\hat{k}$ (by increasing T/I), the width of $\mathscr{R}_{QE, N}$ decreases monotonically. In the global-chaos regime ($K > K_c$) and as $N \to \infty$, the width of $\mathscr{R}_{QE, N}$ "saturates" to a value which, under certain assumptions, is proportional to τ^2, where τ is a scaled (dimensionless) $\hbar$ for the problem. Thus, for K arbitrarily close to K_c from above, $\mathscr{R}_{QE, N}$ shrinks to zero in the semiclassical limit ($\tau \to 0$), in conformity with the classical situation. These phenomena are particular aspects of a more general scenario: As K is increased, $\mathscr{R}_{QE, N}$ gradually becomes less sensitive to an increase in the domain of analyticity $\mathscr{R}_N$ of the system.

This paper is mostly a review and reformulation of results in previous works,[22–24] so as to emphasize and extend the idea of limited sensitivity to analyticity as a manifestation of quantum chaos. In Sec. 2, we consider basic phenomena in the systems (1) and (2), the quantum resonance and the quantum antiresonance (QAR). In Sec. 3, we show that asymptotic exponential "QAR-localization" in angular momentum occurs for τ in the immediate vicinity of QAR. This localization is associated with a classically integrable limit and is totally determined by the analytical properties of the potential (full sensitivity to analyticity). The main results are in Sec. 4, where we study the transition to dynamical localization[10] as the level of chaos is increased. We show, on the basis of extensive numerical data, that this transition is accompanied by a gradual decrease in the sensitivity to analyticity.

2. QUANTUM RESONANCE AND QUANTUM ANTIRESONANCE

The evolution operators in one period (from $t=-0$ to $t=T-0$), for (1) and (2), are given by

$$U_{KR}(\tau)=e^{-i\tau\hat{n}^2}e^{-ikV(\theta)} \tag{6}$$

$$U_{TKR}(\tau)=e^{-i\tau\hat{n}^2/2}e^{ikV(\theta)}e^{-i\tau\hat{n}^2/2}e^{-ikV(\theta)} \tag{7}$$

where $\hat{n}\equiv L/\hbar=-id/d\theta$, $\tau\equiv\hbar T/2I$, and $k\equiv\hat{k}/\hbar$. Here τ is the scaled (dimensionless) $\hbar$ for the problem.

Since the spectrum of the angular-momentum operator $L=\hat{n}\hbar$ consists of all the integer multiples of $\hbar$, one has, identically,

$$U_{KR}(\tau=2\pi m)=e^{-ikV(\theta)} \tag{8}$$

$$U_{TKR}(\tau=4\pi m)=1 \tag{9}$$

for all integers m. Relation (8) implies that the spectrum $e^{-i\omega}$ of (6) for $\tau=2\pi m$ is absolutely continuous, with the "quasienergy" (QE) ω ranging in the interval $[kV_{\min}, kV_{\max}]$, where $V_{\min}(V_{\max})$ is the minimum (maximum) value of $V(\theta)$. This continuity of the QE spectrum implies that the expectation value $\langle L^2\rangle$, in an arbitrary quantum state, increases quadratically with time, $\langle L^2\rangle \propto t^2$.[25] This quantum "ballistic" motion, which is in sharp contrast with the classical diffusive motion $\langle L^2\rangle \propto t$ expected for $K>K_c$ (see Secs. 1 and 4), is called "quantum resonance" (QR).[25] One can show[25] that QR occurs for general rational values of τ/π, but the rate of quadratic growth Q in $\langle L^2\rangle=Qt^2$ is generally much slower than that for $\tau=2\pi m$. One then refers to the simple case of $\tau=2\pi m$ in (8) as to the "fundamental" QR.

The fundamental QR for $m=0$, i.e., $\tau\to 0$, can be easily understood "classically." The limit $\tau\to 0$ is equivalent to $T/I\to 0$. In the latter limit, θ in the map (3) becomes a constant of the motion, leading to the ballistic motion $L_s=L_0-s\hat{k}V'(\theta)$, completely analogous to the quantum one. QRs associated with general, rational values of $\tau/\pi\neq 0$ cannot be explained classically.

Relation (9) for the TKR implies a phenomenon diametrically opposite to QR: *exactly* periodic recurrences of an arbitrary wave packet with the basic period T. Since this phenomenon occurs at values of τ where QR takes place in the KR, we call it "quantum antiresonance" (QAR).[23, 24] At QAR, the QE spectrum consists just of the single, infinitely degenerate level $\omega=0$. In the next section we show that the QAR for $m=0$ (i.e., $\tau\to 0$) has a classical explanation, as in the QR case.

QAR occurs also in the KR provided the potential satisfies $V(\theta+\pi) = -V(\theta)$. This can be seen most clearly using the following relation between the operators (6) and (7), valid only for such a potential:

$$U_{KR}^2(\pi + \tau/2) = U_{TKR}(\tau) \tag{10}$$

Together with (9), relation (10) implies that *period*-2 QAR occurs in the KR for $\tau = (2m+1)\pi$, m integer. This QAR was first noticed in Ref. 25 for $V(\theta) = \cos(\theta)$. Unlike the general QAR (9) in the TKR, this QAR has no classical explanation since $\tau = (2m+1)\pi \neq 0$. In fact, since the two sides of relation (10) are evaluated at different values of τ (the scaled $\hbar$), this relation has no classical counterpart.

3. QAR-LOCALIZATION AND FULL SENSITIVITY TO ANALYTICITY

As we have seen in the previous section, the QE spectrum at QAR is infinitely degenerate. The natural question is then precisely how this degeneracy is removed by slightly perturbing τ near $\tau = 4\pi m$. This question was investigated in our previous works,[23, 24] and the following statement was proven in the framework of a self-consistent framework: The QE spectrum of the TKR for $\tau = 4\pi m + \varepsilon$ and infinitesimal ε is a pure point, and the QE states exhibit asymptotic exponential localization ("QAR-localization") in L space. The inverse localization length, which gives the width of the analyticity domain of the QE states, *is completely determined by the analyticity properties of the system.* Thus, infinitesimally close to QAR, one has *full sensitivity to analyticity.*

We give here the main steps of the proof. Using the operator identity

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots \tag{11}$$

and expanding the operator $\exp(-i\tau\hat{n}^2/2) = \exp(-i\varepsilon\hat{n}^2)$ in powers of ε, it is easy to show that the evolution operators (7) can be written, to first order in ε, as $\exp(-i2\varepsilon G_1)$, where

$$G_1 = \left[\hat{n} - \frac{k}{2}V'(\theta)\right]^2 + \frac{k^2}{4}V'^2(\theta) \tag{12}$$

If the eigenvalue problem for G_1 is $G_1\psi = \zeta\psi$, we perform the gauge transformation

$$\varphi = \exp[-ikV(\theta)/2]\,\psi \tag{13}$$

and obtain for φ, using (12), the eigenvalue equation

$$-\frac{d^2\varphi}{d\theta^2}+\frac{k^2}{4}V'^2(\theta)\,\varphi=\zeta\varphi \tag{14}$$

The problem has thus been reduced to that of a Schrödinger equation with a periodic potential. The spectrum ζ then has a band structure, but because of the periodicity condition $\varphi(2\pi)=\varphi(0)$, only the level with zero quasimomentum is picked out from each band. This gives, in general, a point spectrum. Now, being the solution of the linear differential equation (14), $\varphi(\theta)$ is analytic at least in the domain of analyticity of $V'(\theta)$.[26] Let γ be the smallest distance of a singularity of $V'(\theta)$ from the real θ-axis. Then the Fourier-series expansion of the QE states φ or ψ in (13) will converge in an infinite horizontal strip symmetrically positioned around the real θ-axis and having width $2\gamma^{(0)}$, where $\gamma^{(0)}\geqslant\gamma$.[26] This strip will be naturally identified as the domain of analyticity of the QE states. Its half width $\gamma^{(0)}$ *is totally determined by the analyticity properties of the potential* in (14), as is well known from the theory of linear differential equations in the complex plane.[26] Clearly, the Fourier coefficients of the QE states decay asymptotically as $\exp(-\gamma^{(0)}|n|)$. This is the QAR-localization in the angular momentum $L=nh$, with *localization length* $\xi^{(0)}=1/\gamma^{(0)}\leqslant 1/\gamma$.

As an example, the standard potential $V(\theta)=\cos(\theta)$ is analytic in the entire complex plane, so that $\gamma^{(0)}=\gamma=\infty$ and $\xi^{(0)}=0$. In fact, Eq. (14) reduces in this case to the Mathieu equation,[26, 27] whose solutions in angular-momentum space are known to decay, asymptotically, faster than exponentially (i.e., like n^{-n}).

Another interesting example is the potential

$$V(\theta)=A\arctan[\kappa\cos(\theta)-\kappa_0] \tag{15}$$

where A,κ and κ_0 are some constants. One can show[24] that in the case of (15) $\gamma^{(0)}$ is exactly equal to γ (not just $\gamma^{(0)}\geqslant\gamma$), and γ can be exactly related to κ and κ_0.

As in the case of the fundamental QR for $m=0$ (see the previous section), also the $m=0$ QAR-localization can be understood classically by considering the limit $T\to 0$ (i.e., $\tau\to 0$) of the TKR. One can show[24] that the Hamiltonian (2) reduces in this limit to

$$H_{\text{eff}}=\frac{\hbar^2}{2I}G_1=\frac{1}{2I}\left[L-\frac{\hat{k}}{2}V'(\theta)\right]^2+\frac{\hat{k}^2}{8I}V'^2(\theta) \tag{16}$$

The effective Hamiltonian (16) is, obviously, integrable (no chaos!), and can be considered as the classical counterpart of the quantum operator

(12). After the canonical transformation $L' = L - \hat{k}V'(\theta)/2$ [analogous to the gauge transformation (13)], H_{eff} becomes essentially the Schrödinger Hamiltonian in (14). Thus, QAR-localization is associated with a classically integrable limit. The full sensitivity to analyticity in the infinitesimal neighborhood of QAR is consistent with this fact.

4. TRANSITION TO DYNAMICAL LOCALIZATION AND LIMITED SENSITIVITY TO ANALYTICITY

In the previous sections we have seen that the limit $\tau \to 0$ of both the KR and the TKR Hamiltonians *at fixed k* is classically integrable. In the TKR case, the QE eigenvalue problem reduces in this limit to Eq. (14), from which it follows that the QE states are exponentially localized in L space, with the localization length $\xi^{(0)}$ completely determined by the analytical properties of the potential (full sensitivity to analyticity). In this section, we consider the case of τ small ($\tau < 2\pi$) at fixed k, such that, for the maximal values of τ considered, $K = 2\tau k > K_c$. Then, as τ is gradually increased within the *semiclassical regime* (τ small), the transition from local to global chaos takes place. It is well known[10, 28, 29] that the QE states for the KR are exponentially localized in L space also in the global-chaos regime for generic values of τ. This localization, called "dynamical localization" (DL),[10] is, however, basically different from QAR-localization. The DL length ξ in the semiclassical regime appears to be almost proportional to the chaotic diffusion coefficient D [see relation (27) below],[29] in contrast with the QAR-localization length $\xi^{(0)}$ which is totally determined by the analyticity properties of the potential. The transition from QAR-localization to DL as τ is increased is then accompanied by a gradual decrease in the sensitivity to analyticity.

Our analysis makes use of an important and well-known result in quantum chaos. The QE problem for time-periodic systems such as (1) and (2) is exactly equivalent, in the angular-momentum representation and for generic (irrational) values of τ/π, to the equation describing a *pseudo-random tight-binding model.*[23, 24, 28, 31] It was originally[28] assumed, and later[31] shown in detail, that the localization properties of pseudorandom tight-binding models are similar to those of truly random ones, exhibiting Anderson exponential localization.[32] It then follows that the QE states are exponentially localized in L space for generic values of τ.

We shall use the general tight-binding model for the KR introduced by Shepelyansky.[29] Briefly, this model is derived as follows. Let $u^{\pm}(\theta)$ denote a QE state with quasienergy ω at time $t = \pm 0$. From the simple time evolution with (6) one gets:

$$u^+(\theta) = \exp[-ikV(\theta)]\, u^-(\theta), \qquad u_n^- = \exp[i(\omega - \tau n^2)]\, u_n^+ \tag{17}$$

where $u_n^\pm$ are the Fourier coefficients (or L representation) of $u^\pm(\theta)$. Because of the first relation in (17), there exist functions $\bar{u}$ and g, where g can be assumed to be real, such that $u^\pm(\theta) = g(\theta) \exp[\mp ikV(\theta)/2]\, \bar{u}(\theta)$. Together with the second relation in (17), this gives

$$\sum_{r=-\infty}^{\infty} W_r \sin[(\tau n^2 - \omega)/2 + \alpha_r]\, \bar{u}_{n+r} = 0 \tag{18}$$

where the real numbers W_r and α_r are defined by

$$W(\theta) = g(\theta) \exp[-ikV(\theta)/2] = \sum_{r=-\infty}^{\infty} W_r \exp[i(r\theta + \alpha_r)] \tag{19}$$

and it is assumed that $W(-\theta) = W(\theta)$. Equation (18) describes a general tight-binding model for the KR, with hopping constants W_r. For generic, irrational values of τ/π, the term τn^2 in (18) gives rise to pseudorandom disorder,[28, 31] which may lead to Anderson-like exponential localization of $\bar{u}_n$ in n. We shall restrict our attention from now on to the standard potential $V(\theta) = \cos(\theta)$ with the simple choice $g = 1$. From (19) one finds in this case that $W_r = J_r(k/2)$, a Bessel function, and $\alpha_r = -\pi r/2$. Using relation (10), one then obtains the corresponding tight-binding model for the TKR:

$$\sum_{r=-\infty}^{\infty} J_r(k/2) \sin\{[(2\pi + \tau)\, n^2 - \omega + 2\pi r]/4\}\, \bar{u}_{n+r} = 0 \tag{20}$$

where the quantities ω and $\bar{u}_n$ now refer to the TKR. Let us recall how the asymptotic localization length ξ can be calculated from such tight-binding models.[29, 30] Since the Bessel function $J_r(k/2)$ decays faster than exponentially for $|r| > k/2$,[27] it is reasonable to approximate (20) by restricting r to the finite range $|r| \leqslant N$, for sufficiently large N. The truncated form of Eq. (20) can be easily written as a transfer-matrix problem[29, 30]

$$\sigma_{s+1} = \Gamma_s \sigma_s \tag{21}$$

where σ_s is the $2N$-dimensional vector with components $\sigma_s^{(r)} = \bar{u}_{s-r}$, $r = -N+1, \ldots, N$, and Γ_s is a $2N \times 2N$ symplectic matrix. One may interpret (21) as a map describing a Hamiltonian dynamical system with N degrees of freedom.[4] The vector σ_0 is mapped into σ_n, for arbitrary $n > 0$, by the product matrix

$$\Lambda_n = \Gamma_{n-1}\Gamma_{n-2}\cdots\Gamma_0 \tag{22}$$

Since the matrix (22) is, obviously, symplectic, its eigenvalues $\lambda(n)$ come in N reciprocal pairs $[\lambda_r(n), \lambda_r^{-1}(n)]$, $r = 1,..., N$, and we can always assume the ordering $1 \leqslant |\lambda_1(n)| \leqslant |\lambda_2(n)| \leqslant \cdots |\lambda_N(n)|$. The *minimal Lyapunov exponent* for the map (21),

$$\gamma_N = \lim_{n\to\infty} \frac{1}{n} \ln |\lambda_1(n)| \tag{23}$$

determines then an "Nth-order approximation" $\xi_N = 1/\gamma_N$ to ξ.[29, 30] A key observation is, however, that ξ_N has an *exact* quantum-dynamical meaning *per se.*

This can be easily shown by observing that the truncated version of the tight-binding model (20) is exactly equivalent to the dynamical problem (KR or TKR) with a potential $V_N(\theta)$ replacing $V(\theta) = \cos(\theta)$. The potential $V_N(\theta)$ can be easily determined from the truncated version of Eq. (19) in our case:

$$W_N(\theta) = g_N(\theta) \exp[-ikV_N(\theta)/2] = \sum_{r=-N}^{N} J_r(k/2)\, e^{ir(\theta-\pi/2)} \tag{24}$$

Solving Eq. (24) for $V_N(\theta)$, we obtain

$$V_N(\theta) = -\frac{2}{k} \arctan\left[\frac{\sum_{r=-N}^{N} J_r(k/2)\sin(r\theta - r\pi/2)}{\sum_{r=-N}^{N} J_r(k/2)\cos(r\theta - r\pi/2)}\right] \tag{25}$$

It is then clear that γ_N in (23) is the half-width of the strip of analyticity of the QE states (Fourier-series representation) for the potential (25), and $\xi_N = 1/\gamma_N$ is the localization length.

In the limit $\tau \to 0$, $\xi_N \to \xi_{N,0}$, where $\xi_{N,0}$ is the QAR-localization length for (25). As in the case of the potential (15), it can be shown[24] that $\xi_{N,0} = 1/\gamma_{N,0}$, where $\gamma_{N,0}$ is the half-width of the strip of analyticity of $V_N(\theta)$ (Fourier-series representation) or the smallest distance of a pole θ_0 of $V'_N(\theta)$ from the real θ axis. Here θ_0 is a solution of the equation[24]

$$\sum_{r=-N}^{N} J_r(k/2)\, e^{ir(\theta_0-\pi/2)} = 0 \tag{26}$$

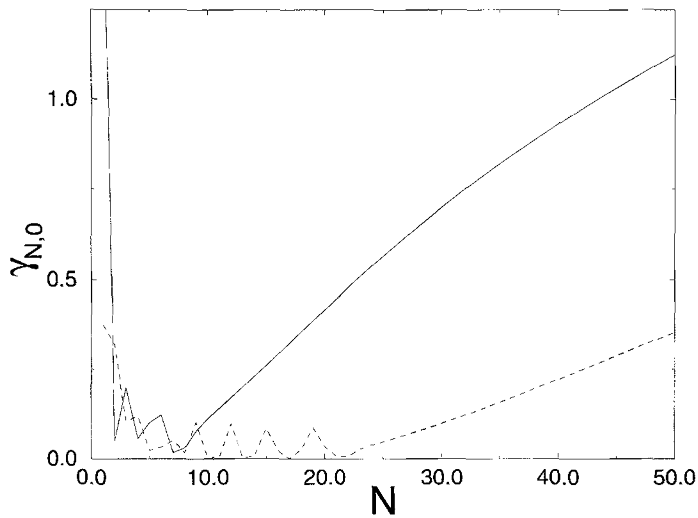

Fig. 2. Half-width $\gamma_{N,0}$ of the strip of analyticity of the potential (25) for $k=20$ (solid line) and $k=50$ (dashed line), obtained by solving Eq. (26).

Figure 2 shows $\gamma_{N,0}$ as a function of N for $k=20$ and $k=50$, obtained by solving the exact equation (26) up to $N=50$. After an "oscillation" transient in the interval $N<k/2$, $\gamma_{N,0}$ appears to increase monotonically without bounds. In fact, one must have $\gamma_{N,0} \to \infty$ as $N \to \infty$ since in this limit one recovers the original, untruncated model (20) for $V(\theta)=\cos(\theta)$, which is analytic over the entire complex plane ($\gamma=\infty$). It is actually not hard to show that $\lim_{N\to\infty} V'_N(\theta)=V'(\theta)=-\sin(\theta)$. We thus see that the truncated models define in a natural way a sequence of potentials $V_N(\theta)$ whose analyticity domains increase without bounds with N.

Given this, we have performed an accurate numerical study of γ_N as a function of N for several values of $\tau \neq 0$. We have calculated γ_N from (23) using the well-known method[4, 29, 30] for determining the Lyapunov spectra of products of matrices such as (22). The method is based on direct application of the map (21) a large number $n=n_{\max}$ of times, such that for $n>n_{\max}$ the matrices Γ_n can be considered as random. This randomness should be realized to some extent by the pseudorandom term $\tau n^2/4$ in (20) if $\tau[(n+1)^2-n^2]/4 \sim 2\pi$, or $n=n_{\max} \sim 4\pi/\tau$. In practice, it was sufficient to use $n_{\max} \leqslant 10^6$ even for values of τ as small as 5×10^{-7}. This is demonstrated in Fig. 3, where we plot the quantity of interest, $\gamma_N(n)=n^{-1}\ln|\lambda_1(n)|$ (solid line), for $N=6$, $k=10$, and $\tau=5\times 10^{-7}$. Also plotted are the same quantity for the KR tight-binding model [i.e., the original model (18) with $|r|\leqslant 6$-truncation] (dot-dashed line) and the value of

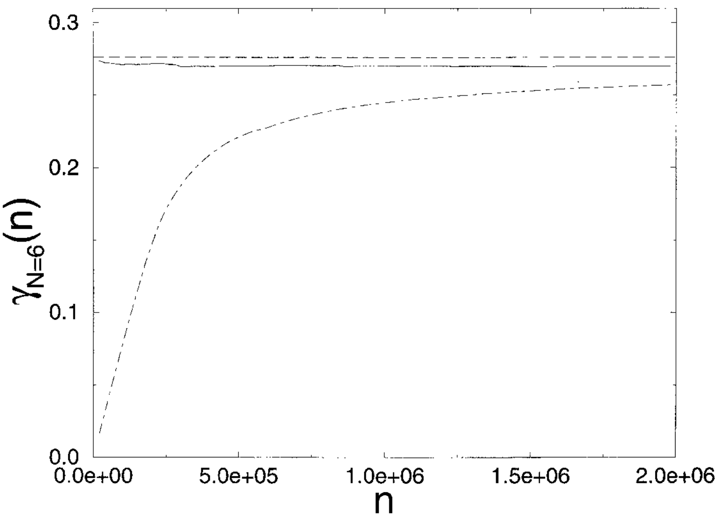

Fig. 3. $\gamma_N(n) = n^{-1} \ln |\lambda_1(n)|$ [see (23)] for $N=6, k=10$, and $\tau = 5 \times 10^{-7}$ in the case of the TKR model (20) (solid line) and in the case of the KR model (18) (dot-dashed line). The dashed straight line corresponds to the value of $\gamma_{N=6,0} = 0.2759$ from Eq. (26).

$\gamma_{N=6,0}$ from Eq. (26) (dashed line). It is clear from Fig. 3 that $\gamma_N(n)$ converges very fast to its limit value γ_N, which is close from below to $\gamma_{N,0}$. The fact that $\gamma_N(n)$ for the KR (dot-dashed line) appears to converge to the same limit value (but much more slowly) suggests that also in the KR case the asymptotic localization length $\gamma_N \to \gamma_{N,0}$ as $\tau \to 0$. At the moment, however, we are unable to prove this as in the TKR case. We notice that the regime of small τ in the KR corresponds to the neighborhood of the fundamental QR (see Sec. 2), where the ballistic motion manifests itself in the fact that the QE states are extended over a large interval of n, $|n - n_0| < \bar{n} \gg 1$, and the asymptotic exponential localization is observed only for $|n - n_0| > \bar{n}$.[28] This is reflected in the small value of the "local" inverse localization length $\gamma_N(n)$ for small n in Fig. 3. On the other hand, the much faster convergence of $\gamma_N(n)$ in the TKR case indicates that the initial localization profile of the QE states is well approximated by the asymptotic QAR-localization.

The final results for γ_N are shown in Figs. 4–6, corresponding to $k = 5, 10, 20$. In each figure, the curve of γ_N is plotted for 15 values of τ. For $k = 5$ (Fig. 4), $2 \times 10^{-6} \leqslant \tau \leqslant 4$, and the values of τ for $k = 10$ and $k = 20$ (Figs. 5 and 6) are, respectively, 1/2 and 1/4 of those for $k = 5$. Thus, the different curves are associated with the same values of $K = 2\tau k$ in all the three figures, and the most "classical" case (smallest τ's) is that of $k = 20$

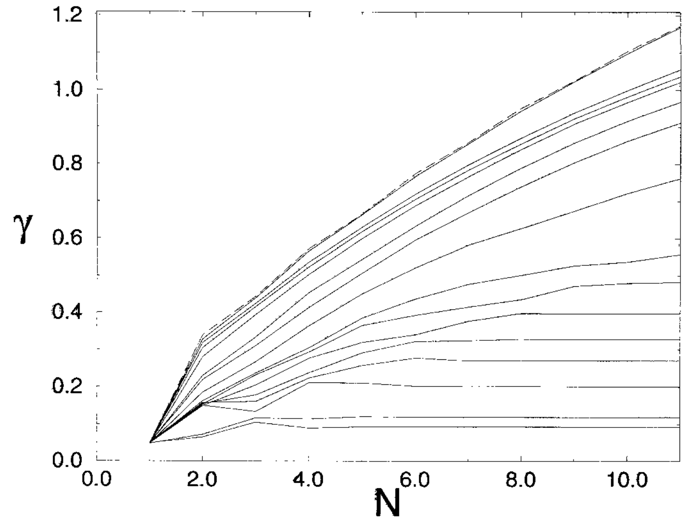

Fig. 4. Solid lines: minimal Lyapunov exponent γ_N [see (23)], associated with truncations of the TKR model (20) for $k=5$ and 15 values of $\tau=2\times10^{-6}$, 2×10^{-5}, 2×10^{-4}, 2×10^{-3}, 0.02, 0.04, 0.1, 0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 2.0, 4.0, in order of the descending curves. The dashed line is the curve of $\gamma_{N,0}$ for $k=5$ (see Fig. 2).

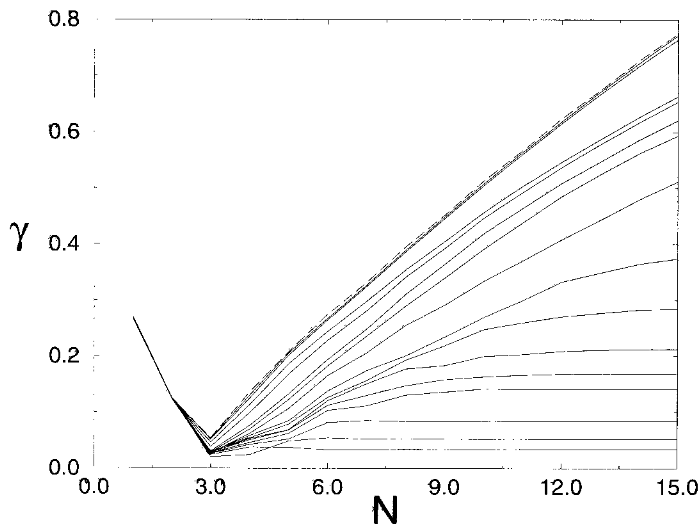

Fig. 5. Similar to Fig. 4, but for $k=10$ and $\tau=10^{-6}$, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 0.02, 0.05, 0.1, 0.125, 0.15, 0.175, 0.2, 0.25, 1.0, 2.0, in order of the descending curves at $N=15$.

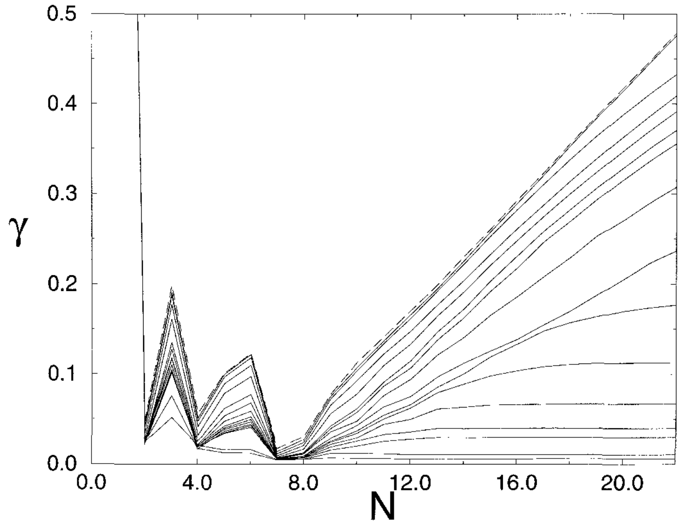

Fig. 6. Similar to Fig. 4, but for $k = 20$ and $\tau = 0.5 \times 10^{-6}$, 0.5×10^{-5}, 0.5×10^{-4}, 0.5×10^{-3}, 0.005, 0.01, 0.025, 0.05, 0.0625, 0.075, 0.0875, 0.1, 0.125, 0.5, 1.0, in order of the descending curves.

(Fig. 6). We also plot in each figure the curve for $\gamma_{N,0}$ (dashed line), obtained by solving the exact equation (26). It is important to remark here that for τ not too small the results in Figs. 4–6 were found to be almost identical to the corresponding ones in the KR, obtained by using the truncations of the original model (18), instead of (20). Thus, Figs. 4–6 describe, essentially, also the KR case.

As expected, for τ sufficiently small γ_N is well approximated by $\gamma_{N,0}$. As τ is increased at fixed N (thus increasing the level of chaos, $K = 2\tau k$), γ_N generally decreases monotonically. This phenomenon is completely analogous to the decrease of the domain of analyticity of the golden-mean torus in the KR as K approaches K_c from below (see Introduction). In a future work,[33] we plan to investigate the precise relation between γ_N and the half-width of the strip of analyticity of KAM tori for (25). These two quantities are expected to coincide in the semiclassical limit ($\tau \to 0$ at fixed K).

We notice that for values of τ corresponding to $K \geqslant 2.5$ in all the three figures, γ_N appears to "saturate," i.e., to be almost N-independent for N sufficiently large. This saturation effect is most pronounced in the strong-chaos regime ($K \gg K_c$), where γ_N becomes essentially independent of N. The saturation value can be identified, of course, with the value of γ for $V(\theta) = \cos(\theta)$, γ_∞. Shepelyansky[29] found that in a semiclassical regime (small τ) of global chaos ($K > K_c$) in the KR the localization length $\xi_\infty = 1/\gamma_\infty$

satisfies the following approximate relation (in our notation and in units such that $\hbar = 1$):

$$\xi_\infty \approx \frac{D(K)}{8\tau^2} \tag{27}$$

where $D(K)$ is the classical chaotic diffusion coefficient for the KR:

$$D(K) = \lim_{s \to \infty} \frac{\langle (p_s - p_0)^2 \rangle}{s} \tag{28}$$

Here $p_s = TL_s/I$ (see Introduction) and $\langle\ \rangle$ denotes average over an ensemble of initial conditions $\{(\theta_0, p_0)\}$. The main condition for the validity of relation (27) is $\xi_\infty > k$.[29] Using $\tau = K/(2k)$ in (27), we see that at fixed $K > K_c$ this condition is satisfied provided

$$k > k_0 \equiv \frac{2K^2}{D(K)} \tag{29}$$

Notice that k_0 increases without bounds as $K \to K_c[D(K) \to 0]$. Assuming the general and exact validity of relation (27) with (29) in the semiclassical limit ($\tau \to 0$ or $k \to \infty$ at fixed K), it follows immediately that γ_∞ is simply proportional to τ^2 in this limit. Thus, for K arbitrarily close to K_c from above, the domain of analyticity of the QE states shrinks to zero in the semiclassical limit. This fully agrees with the fact that the domain of analyticity of a golden-mean torus or cantorus for $K \geqslant K_c$ is zero (see Introduction).

The two phenomena above, i.e., the decrease of γ_N as τ (or K) is increased and the saturation of γ_N as N is increased (for $K > K_c$) may be viewed as particular aspects of a more general scenario, which is well illustrated by Figs. 4–6. We already know that the region of analyticity $\mathscr{R}_N$ of $V_N(\theta)$ increases with N for N sufficiently large, $N > k/2$ (see Fig. 2), and that $2\gamma_N$ is the width of the strip of analyticity $\mathscr{R}_{QE,N}$ of the QE states for $V_N(\theta)$. For infinitesimal τ, $\mathscr{R}_{QE,N} = \mathscr{R}_N$, so that an increase in the analyticity of the potential results in a corresponding increase of the analyticity of the QE states. For finite τ, however, the increase of $\mathscr{R}_{QE,N}$ with N is slower than that of $\mathscr{R}_N$. In the global-chaos regime ($K > K_c$), the increase of $\mathscr{R}_N$ leads to an increase of $\mathscr{R}_{QE,N}$ only up to $N \sim N^*$, where γ_N saturates to a value $\approx \gamma_{N^*}$. Both N^* and γ_{N^*} decrease as K is increased, so that the influence of the analyticity of $V_N(\theta)$ on the analyticity of the QE states is gradually reduced. The extreme case corresponds to the strong-chaos regime ($K \gg K_c$). Here the increase of $\mathscr{R}_N$ does not lead to any increase in $\mathscr{R}_{QE,N}$, and $\gamma_N \approx \gamma_\infty$ is totally determined by the diffusion coefficient D, as in relation (27). This case of dynamical localization is thus completely different in nature from QAR-localization, which is associated with a classically integrable limit.

We may therefore conclude by saying that the transition between these two kinds of quantum localization takes place via *a gradual reduction of the influence of the analyticity of the system on that of the eigenstates, as the level of chaos is increased.* This is a vivid manifestation of quantum chaos.

Dedication. We would like to dedicate this work to our teacher, colleague, and friend, Prof. Lawrence P. Horwitz, on the occasion of his 65th birthday.

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