The Klein-Gordon Equation

The KG equation

$$(\Box + m^2)\phi = 0 \tag{1}$$

was derived in the very early days of quantum mechanics (see [16], bottom of p. 25). It can be regarded as a quantize form of the relativistic relation $E^2 - \mathbf{p}^2 = m^2$, where $i\partial/\partial t,\ -i\nabla$ replace E and $\mathbf{p}$, respectively. Hence, there is no doubt concerning its correctness *as a formula.* Indeed, as is well known, components of a solution of the Dirac equation satisfy the KG equation.

The problem discussed here is the status of the KG equation (1) *as a fundamental quantum mechanical equation derived from a Lagrangian density.* Here the Lagrangian density of an electrically charged KG particle is

$$\mathcal{L} = (\phi^*_{,0} - ieV\phi^*)(\phi_{,0} + ieV\phi) - \sum_{k=1}^{3}(\phi^*_{,k} + ieA_k\phi^*)(\phi_{,k} - ieA_k\phi) - m^2\phi^*\phi \tag{2}$$

(See [17,18], eq. (37). Note that here units where $\hbar = c = 1$ are introduced.) This aspect of the KG equation had a controversial status for a very long time. Dirac's negative opinion on this equation (see [19] and [20], pp. 3-8) directed him to construct his famous equation which is now regarded as the relativistic quantum mechanical Hamiltonian of spin-1/2 particles.

Other researchers disagree with Dirac (see [18], pp. 70-72, 105, 188-205; [16], second column of p. 24). In particular, Pauli and Weisskopf constructed the second order Lagrangian density (2). Unlike the case of the Dirac equation, this Lagrangian density does not yield an expression for the particle density but for its charge density.

Before examining the experimental side, let us state a fundamental property of particles described by a wave function $\psi(x^\mu)$. Due to the fact that $\psi(x^\mu)$ depends on a *single* set of space-time coordinates x^μ, one concludes that a particle *truly described* by $\psi(x^\mu)$ must be elementary, namely a pointlike structureless particle.

The experimental data of elementary massive spin 1/2 (Dirac) particles, like the electron, the muon and the u, d quarks is consistent with the pointlike requirement. This is not true for the old candidates for the KG particles, namely the three 0^- π mesons. Indeed, it is now known that a π meson contains a quark and an antiquark. The charge radius of the $\pi^\pm$ is 0.672 ± 0.008 fm (see [4], p.499). Hence, π mesons are definitely not pointlike particles. For this reason, they cannot be regarded as Klein-Gordon particles.

A recent analysis of the KG Lagrangian density proves that it is also not free of theoretical difficulties [21]. Thus, it is proved that the theory derived from the KG Lagrangian density (2) has the following difficulties:

1. There is no expression for the particle's density. The expression for the charge density depends on coordinates of *external particles.*

2. The Hamiltonian density depends on time derivative of ϕ. Hence, if a Hamiltonian of the KG particle exists then the Hamiltonian density depends on the Hamiltonian.

3. There is no *covariant differential operator* that serves as a Hamiltonian [21]. Furthermore, the Hamiltonian matrix of a charged KG particle destroys the inner product of the Hilbert space [21]. There is no Hilbert space for an uncharged KG particle because in this case density is undefined [17].

4. The *second order* KG equation (1), which is derived from the KG Lagrangian density (2), is not identical to the *first order* fundamental quantum mechanical equation $i\partial\phi/\partial t = H\phi$.

5. One cannot construct a self-consistent electromagnetic interaction of a charged KG particle. The linear interaction $eA_\mu j^\mu$ entails an equation imbalance [22] and the quadratic term $(p^\mu - eA^\mu)(p_\mu - eA_\mu)$ destroys the inner product of the Hilbert space [21].

6. There is no explanation why the energy-momentum operators $(i\partial/\partial t, -i\nabla)$ are used for the *different* task of representing charge density and current.

7. The nonrelativistic limit of the KG equation disagrees with the Schroedinger equation. Indeed, in the case of the Schroedinger equation, $\Psi^*\Psi$ represents probability density [23] whereas the KG equation has no expression for probability density. Hence, the KG equation is inconsistent with a restriction imposed by a lower rank theory.

8. Another aspect of the previous point is the dimension of the corresponding wave functions. Examining the Schroedinger equation, one finds that $\Psi^*\Psi$ represents probability density. It follows that the dimension of Ψ is $[L^{-3/2}]$. On the other hand $m^2\phi^*\phi$ is a term of the Lagrangian density of the KG field [17]. Hence, in units where $\hbar = c = 1$, the dimension of $m^2\phi^*\phi$ is $[L^{-4}]$ and that of ϕ is $[L^{-1}]$. Therefore, due to the difference in dimension, the nonrelativistic limit of the KG equation disagrees with the Schroedinger equation.

(By contrast, it is proved in [21] that an analogous analysis of the Dirac equation yields completely acceptable relations.)

These theoretical difficulties, together with the lack of support from the experimental side (there is no candidate for a *pointlike* KG particle) indicate that, unlike the case of the Dirac equation, the existence of a genuine KG particle is not very likely.

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