Chapter V

THE GOLDEN AGE

The fifth century, the age of Pericles, the era of the widest unfurlment of power of the Athenian empire, and of its decline, is the golden age of Hellenic culture. It is the century during which the glorious temples of the Acropolis arose, when sculpture produced its ever admired, never equalled masterworks, the age also of the great tragic poets Aeschylus, Sophocles and Euripides, of the great historians Herodotus and Thucydides.

The fifth century begins with the unsuccessful revolt of the Ionian cities in Asia Minor against the Persian domination, which led to the great Persian wars, from which Athens and Sparta emerged as victors. Athens, which had been completely destroyed, was now splendidly rebuilt and Athenian naval power was established. And, democracy was introduced over the entire range of this power. Thus the aristocratic Pythagoreans were divested of their power and driven out of Italy (about 430). The contrasts between democratic Athens and the military aristocracy of Sparta became more and more pronounced, until at last the frightful Peloponnesian war brought about the end of the power of Athens and thus of the greatness of Hellas.

In philosophy, this century was the period of the Enlightenment (Aufklärung), of rationalism. The ancient religion and ethics were ridiculed by the intellectuals. The sophists taught that all truth and all values are only relative. Materialistic systems, such as that of the atomist Democritus, were at a premium in wide circles. Anaxagoras and Democritus held that the sun and the moon were not living beings, but inert rocks, tossed to and fro in the whirl of the atoms.

In contrast to this view stood the religious-mystical metaphysics of the Pythagoreans, which adhered for instance to the belief that the planets were animated, divine beings, carrying out perfect circular motions as a result of their perfection and their divine insight.

It was not until the fourth century, the century of Plato, that philosophy and mathematics reached their zenith. We can follow the growth which took place in the fifth century and which was preparatory to the period of flowering. For mathematics, this growth occurs chiefly in the Pythagorean school. At the beginning of the fifth century, the most interesting figure in this school is

Hippasus.

The Pythagorean doctrine had two aspects, a scientific one and a mystical-religious one. Pythagoras called himself a philosopher, not a sage therefore, but a seeker after wisdom, a lover of wisdom; but he was also a prophet whose teachings were looked upon as infallible, divine revelations.

This twofold aspect was bound to lead to a serious dilemma after his death.

Should the divine revelations of Pythagoras, the "ipse dixit" be looked upon as the source of all knowledge, concerning numbers, harmony and the stars? In this case there was little sense in attempting to perfect this absolute wisdom by one's own studies. Or should one search for the truth, relying upon one's own thoughts, in the manner of a genuine philosopher? In that case, reason should be ranged above revelation, as the supreme judge in all scientific questions.

And, there is something else. Any one who wants to advance in science can not look upon this discipline as a secret accessible only to the initiates. He has to take into account the results of others and should not conceal his own views. It is thus inevitable that he should come into conflict with the pledge of silence.

This conflict did indeed break out soon after the death of Pythagoras. Hippasus made bold to add several novelties to the doctrine of Pythagoras and to communicate his views to others. He constructed the circumscribed sphere of a dodecahedron; he developed the theory of the harmonic mean; to the three consonant intervals, octave, fifth and fourth, he added two more, the double octave and the fifth beyond the octave; he discussed the theory of musical ratios with the musician Lasos, who was led thereby to experiment with vases, empty and half-filled. These and other similar indiscretions caused a split: Hippasus was expelled. Later on he lost his life in a shipwreck, as a punishment for his sacrilege, according to his opponents. His followers called themselves "Mathematikoi", in opposition to the "Akousmatikoi", who strictly observed the sacred tenets and who faithfully transmitted the "Akousmata", the sacred sayings. The two sects entered into vicious disputes, as is customary among sects of one religion.

A century after Hippasus, there were various groups of Pythagoreans; their mutal relations and their relations to the two old sects are not entirely clear. One of these groups was driven out of Italy around 430 by the advancing democracy. Aristoxenes (a pupil of Aristotle, about 320) says that he had known the last remnant of this group, disciples of Philolaus and Eurytus. We are not certain of the truth of this; it would not be the only lie of which Aristoxenes is guilty.[1] Another group must have remained in Italy; for late in the 4th century, Pythagorean orgies there are still mentioned. A third group, which appears to carry on the tradition of the "Mathematikoi", is formed by the "so-called Pythagoreans in Italy", repeatedly referred to by Aristotle. One of the most important and most interesting figures among these is Archytas of Taras, friend and contemporary of Plato, whose work will be discussed in the next chapter. Around 350, mention is still made of the astronomers Ecphantus and Hicetas of Syracuse who taught that the earth revolves on its axis.

No names of Mathematikoi are known between Hippasus and Archytas. But, more important than such names is the development of mathematics itself in the school of the Pythagoreans; about this we do know something.

[1] "Who will believe Aristoxenes?", exclaims Aristotle in Eusebius, Praep. Evang. XV 2. See E. Frank in the *American Journal of Philology 64*, p. 222.

The Mathemata of the Pythagoreans.

Four "mathemata", i.e. subjects of study, were known to the Pythagoreans: Theory of Numbers (Arithmetica), Theory of Music (Harmonica), Geometry (Geometria) and Astronomy (Astrologia). Astronomy will be discussed fully elsewhere. For the theory of music, I refer to my memoir in Hermes 78 (1943). We begin therefore with

The theory of numbers.

Here and in the sequel, the word "numbers" always refers, according to Greek usage, to *integral* positive numbers (quantities).[1] The theory of numbers is therefore the theory of natural numbers.

The oldest bit of the theory of numbers that we can reconstruct, is:

The theory of the even and the odd.

If we turn to the ninth book of Euclid's Elements, we find a sequence of propositions (21—34), which have no connection whatever with what precedes, and which constitutes, according to O. Becker[2], together with IX 36, a piece of archaic, and indeed typically Pythagorean mathematics. In abbreviated form, they are:

21. A sum of even numbers is even.
22. A sum of an even number of odd numbers is even.
23. A sum of an odd number of odd numbers is odd.
24. Even less even is even.
25. Even less odd is odd.
26. Odd less odd is even.
27. Odd less even is odd.
28. Odd times even is even.
29. Odd times odd is odd.
30. If an odd number measure an even number, it will also measure the half of it.
31. If an odd number be prime to any number, it will also be prime to the double of it.
32. A number that results from (repeated) duplication of 2, is exclusively even times even.
33. If a number has its half odd, it is even times odd only.
34. Every even number that does not belong among those mentioned in 32 and 33, is even times even and also even times odd.

[1] The Greeks exclude even unity from the numbers because unity is not a quantity. This compels clumsy formulations, such as "if a is a number or 1 . . .". We shall take no notice of these quibbles and we shall simply count 1 among the numbers.

[2] O Becker, *Quellen und Studien*, B 4, p. 533.

The climax of the theory is proposition 36, which asserts that numbers of the form

$$2^n(1 + 2 + 2^2 + \ldots + 2^n)$$

are perfect, provided

$$p = 1 + 2 + \ldots + 2^n$$

is a prime number.

Becker has shown that this proposition can be derived from 21—34 alone. Euclid gives a somewhat different proof, probably to establish the connection with the Theory of Numbers developed in Books VII—IX, but there is no doubt that the original proof depended on the theory of the even and the odd. In particular, proposition 30 seems to have been introduced especially to make possible the proof that $2^n p$ has no other divisors than $1, 2, 2^2 \ldots 2^n$ and $p, 2p, \ldots, 2^n p$.

Propositions 32—34 are related to the classification of even numbers into even times even and even times odd, which is found, in a different form in the work of Nicomachus of Gerasa, and at which Plato hints here and there.

Plato always defines Arithmetica as "the theory of the even and the odd".

For the Pythagoreans, even and odd are not only the fundamental concepts of arithmetic, but indeed the basic principles of all nature. Their point of view is expressed by Aristotle (Metaphysics A5) in the following words:

> The elements of number are the even and the odd . . . Unity consists of both . . . Number develops from Unity . . . Numbers constitute the entire heaven.
>
> Others of the same school say that there are 10 principles, which they group in pairs:

bounded	–	unbounded
odd	–	even
unity	–	plurality
male	–	female, etc.

It becomes clear that the antithesis even-odd plays a very fundamental role in Pythagorean metaphysics.

In a fragment of the old comic poet Epicharmus (around 500 B.C. or perhaps even earlier), we find an amusing allusion to the philosophy of the Pythagoreans. The following dialogue takes place:

> "When there is an even number present or, for all I care, an odd number, and some one wants to add a pebble or to take one away, do you think that the number remains unchanged?"
>
> "God forbid!"
>
> "And when some one wants to add some length to a cubit or wants to cut off a piece, will the measure continue as before?"
>
> "Of course not".
>
> "Well then, look at people. One grows, another one perhaps gets shorter, they are constantly subject to change. But whatever is changeable in character and does not remain the same, that is certainly different from what has changed. You and I are also different people from what we were yesterday, and we will be still different in future, so that by the same argument we are never the same."
>
> (Diels, Fragmente der Vorsokratiker, Epicharmus A 2).

To appreciate this properly, we must remember that this conversation takes place in a comedy, not in a philosophical discussion. Quite likely the first speaker owes the other some money and wants to prove by a philosophical argument, that it is not he who has borrowed the money, but a totally different person!

Epicharmus is poking fun therefore at the disputes of the philosophers of his day. But who are the philosophers he is thinking of? Obviously the Pythagoreans.[1] For, why should he speak in his first sentence of the antithesis even-odd? Epicharmus lived in Sicily, and the Pythagoreans played a big role throughout Southern Italy in this period; so the audience probably understood the allusion.

But it is time to return to the mathematics of the Pythagoreans!

The only place at which the theory of the even and the odd is applied in the Elements themselves, is the proof of the incommensurability of the side and the diagonal of a square, at the end of Book X; we will reproduce this proof in abbreviated form.

If the diagonal AC and the side AB of a square $ABCD$ are commensurable, let their ratio, reduced to lowest form be $m : n$. From $AC : AB = m : n$ follows $AC^2 : AB^2 = m^2 : n^2$; but, since $AC^2 = 2AB^2$, this leads to $m^2 = 2n^2$, so that m^2 is even. It follows that m is even; for, if m were odd, we would conclude from Proposition 29 that m^2 is also odd. Hence m is even; let one half of m be equal to h. Since m and n have no common factor, and m is even, it follows that n is odd. But, from $m = 2h$, we obtain $m^2 = 4h^2$ and hence $n^2 = 2h^2$. Therefore n^2 is even, so that a repetition of the earlier reasoning shows that n is also even. The conclusion would then be that the number n is both even and odd; this is impossible.

Aristotle repeatedly alludes to this proof. According to Plato, Theodorus of Cyrene (around 430) proved the irrationality of $\sqrt{3}$, $\sqrt{5}$, etc. to $\sqrt{17}$, (stated more accurately: of the sides of squares of areas 3, 5, . . . 17), but the tacitly omitted $\sqrt{2}$. From this Zeuthen draws the conclusion, that the irrationality of $\sqrt{2}$, i.e. of the diagonal of the unit square, was known before his time. On the other hand, Pappus states that the theory of irrationals started in the school of Pythagoras, and their theory of the even and the odd gave the Pythagoreans the means to prove the irrationality of $\sqrt{2}$; it is therefore highly probable that the proof given in the preceding paragraph, came from them.

Proportions in numbers.

The systematic organization of the theory of the ratios of numbers and of divisibility, found in Book VII of the Elements, came later than the theory of the even and the odd. It is my judgment that this entire book should be attributed to the Pythagoreans before Archytas. To justify this conclusion, it is necessary to take first a closer look at the Theory of Numbers of Archytas himself.

[1] Compare A. Rostagni, *Il Verbo di Pitagora* (1924)

In the theory of music of Archytas, the following two propositions from the theory of numbers, which we shall call A and B, play a leading role:

A. *Between two numbers in epimoric ratio*[1] *there exists no mean proportional.*

B. *When the "combination" of a ratio with itself*[2] *is a "multiple ratio"* [3], *then the ratio itself is a multiple ratio.*

These two propositions are found, e.g., at the beginning of Euclid's "Sectio canonis". Boethius says that Archytas' proof of proposition A ran as follows: Let $A : B$ be the given epimoric ratio, and let C and $D + E$ be the smallest numbers in this ratio.[4] Then $D + E$ exceeds the number C by an amount D, which is a divisor of $D + E$ itself and also of C. I assert now that D must be unity. For, suppose that D is a number greater than 1 and a divisor of $D + E$; then D is also a divisor of E and hence of C. It follows that D is a common divisor of C and $D + E$; but this is impossible, because the smallest of the numbers whose ratio equals their ratio, are relatively prime (cf. Euclid's Elements VII 22). Therefore D is unity, i.e. $D + E$ exceeds C by unity, and hence a mean proportional between C and $D + E$ can not be found. Consequently, a mean proportional between the two original numbers A and B can not exist, since their ratio equals that of C and $D + E$.

This proof quotes almost word for word a proposition from Book VII, viz.: *VII 22. The least numbers of those which have the same ratio with them are prime to one another.*

Furthermore, at the end of the proof a proposition from Book VIII is applied, viz.:

VIII 8. If between two numbers there fall numbers in continued proportion with them (i.e. if one has a geometric progression of which the given numbers are the extreme terms), *then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers.*

The proof of Proposition A in Euclid's Sectio Canonis is essentially the same as that of Archytas. Proposition B is proved in the Sectio Canonis by appeal to another proposition from the Elements, viz.

VIII 7. If there be as many numbers as we please in continued proportion, and the first measure the last, it will measure the second also.

It becomes evident that Archytas' theory of music presupposes quite a bit of the theory of numbers. Tannery and Heath rightly conclude from this, that at the time of Archytas, i.e. around 400, there must have existed in the school of Pytha-

[1] See footnote 2 on p. 97 for the meaning of this term.

[2] The "combination" of the ratios $a{:}b$ and $b : c$ is the ratio $a : c$. The combination of two ratios can $a : b$ and $c : d$ can always be obtained by reducing them to the form $ac : bc$ and $bc : bd$. This operation corresponds to our multiplication of fractions.

[3] Two numbers are said to be in "multiple ratio" if one of them is a multiple of the other.

[4] For the sake of clarity I have written $D + E$. The Latin text simply writes DE. The sequel indicates that the equality of C and E is assumed.

goras some kind of "traité d'arithmétique", a "treatise of some kind on the Elements of Arithmetics", something similar to Books VII—IX of Euclid's Elements. I am now going to inquire more closely what must certainly have been contained in these Pythagorean Elements of Arithmetic.

Propositions VIII 7 and 8 both depend necessarily upon

VIII 3. *If as many numbers as we please in continued proportion*[1] *be the least of those which have the same ratio with them*[2], *the extremes of them are prime to one another.*

This proposition in turn depends on VII 27 and on

VIII 2. *To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.*

The solution of this problem proceeds as follows: Let the given ratio, in its lowest terms, be $A : B$. Multiply A and B, each by themselves and by each other; we obtain then a geometric progression of 3 terms A^2, AB, B^2. Multiply these by A and the last also by B, thus getting a progression of 4 terms: A^3, A^2B, AB^2, B^3. Continue in this manner. The same pattern

$$\begin{array}{ccccccc} & & A & & B & & \\ & A^2 & & AB & & B^2 & \\ A^3 & & A^2B & & AB^2 & & B^3 \end{array}$$

is also found in Nicomachus with an application to the theory of music.

This construction of geometric progressions must therefore certainly belong to the arithmetic of the Pythagoreans. The same is true of the rest of Book VIII. Indeed, it is exactly those Platonic dialogues, in which the largest number of Pythagorean ideas have been used (Timaeus and Epinomis), that contain numerous allusions to propositions and concepts of Book VIII. For example, in the Timaeus it is stated that between two squares there is one mean proportional, between two cubes, two mean proportionals; and the Epinomis speaks of similar plane and space numbers. (Two "plane numbers" $A \cdot B$ and $C \cdot D$ are called similar if their "sides" are proportional; an analogous definition is given for "space numbers" $A \cdot B \cdot C$ and $D \cdot E \cdot F$). The contents of Book VIII must therefore be credited to the Pythagoreans, in particular to Archytas and his circle.

But Book VIII is based on Book VII. We have already seen that Archytas quotes VII 22 almost word for word. The logical genealogy of Archytas' propositions A and B is the following[3]

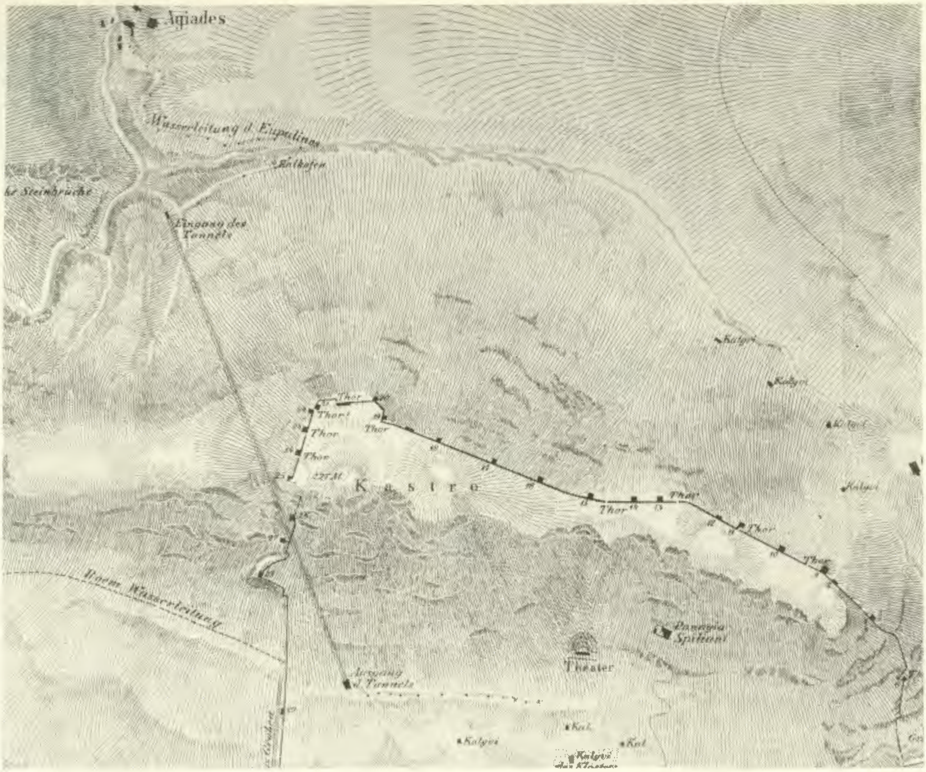
[1] i.e. if $a : b = b : c = c : d \ldots$

[2] i.e. that there are no smaller numbers $a', b', c', \ldots$ which have the same ratios as $a, b, c, \ldots$

[3] For a more detailed elaboration, see my article "*Arithmetik der Pythagoreer*", Math. Ann. *130* (1948), p. 127.

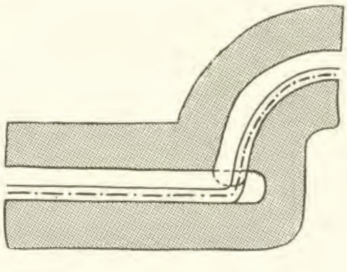
PL. 13. Bronze head of Pythagoras (?). Copy of a Greek original, probably from the end of the 4th century B.C., from the Villa dei Papiri in Herculaneum, Museo Nazionale, Naples. Aelianus relates that Pythagoras wore oriental dress; this would explain the turban. As Schefold points out, it is also possible that the head represents Archytas of Taras, the most important Pythagorean mathematician and musicologist (see p. 149). If this be the case we have probably a real, though somewhat idealized, portrait.

(Photo Alinari)

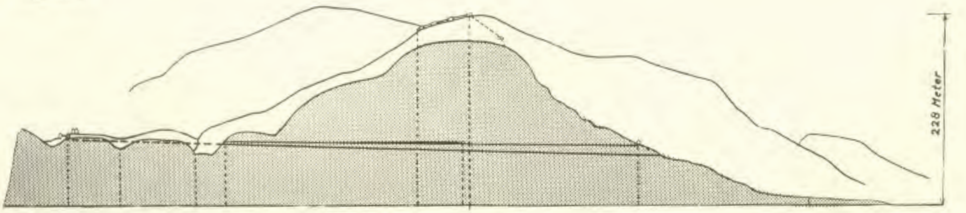


Pl. 14*a*. Mt. Castro on the island of Samos with the conduit of Eupalinos. From E. Fabricius, Mitt. Deutsches Archaol. Institut, Athens, 9 (1884).

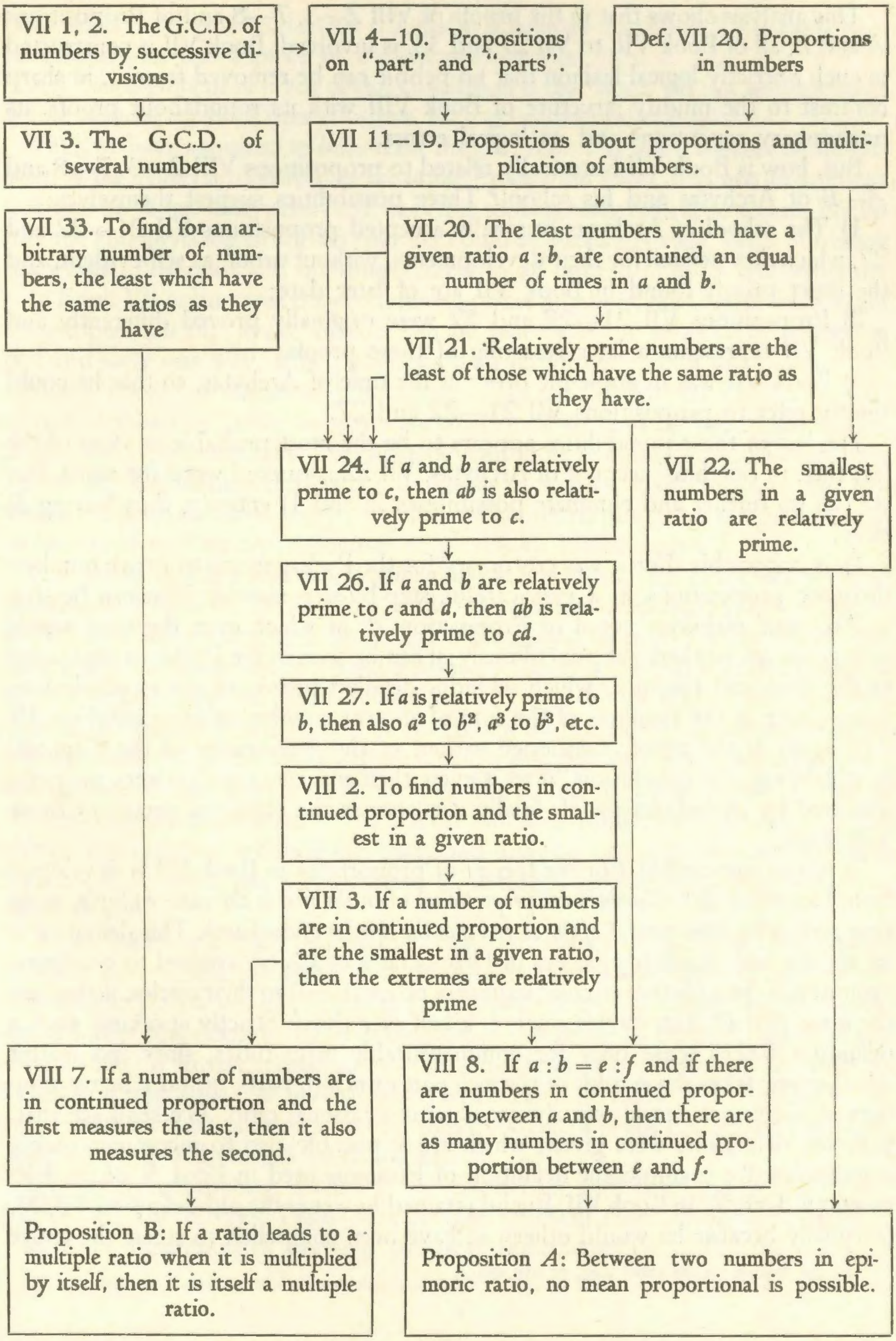
Pl. 14*c*. Cross-section of the tunnel and ditch



Pl. 14*d*. How the northern tunnel was made to join the southern tunnel; ground-plan.



Pl. 14*b*. Cross-section of Mt. Castro, the altitudes enlarged twofold. The upper line across the mountain represents the original tunnel, the lower one the ditch with greater fall.


VII 1, 2. The G.C.D. of numbers by successive divisions.
VII 4–10. Propositions on "part" and "parts".
Def. VII 20. Proportions in numbers
VII 3. The G.C.D. of several numbers
VII 11–19. Propositions about proportions and multiplication of numbers.
VII 33. To find for an arbitrary number of numbers, the least which have the same ratios as they have
VII 20. The least numbers which have a given ratio a : b, are contained an equal number of times in a and b.
VII 21. Relatively prime numbers are the least of those which have the same ratio as they have.
VII 24. If a and b are relatively prime to c, then ab is also relatively prime to c.
VII 22. The smallest numbers in a given ratio are relatively prime.
VII 26. If a and b are relatively prime to c and d, then ab is relatively prime to cd.
VII 27. If a is relatively prime to b, then also a² to b², a³ to b³, etc.
VIII 2. To find numbers in continued proportion and the smallest in a given ratio.
VIII 3. If a number of numbers are in continued proportion and are the smallest in a given ratio, then the extremes are relatively prime
VIII 7. If a number of numbers are in continued proportion and the first measures the last, then it also measures the second.
VIII 8. If a : b = e : f and if there are numbers in continued proportion between a and b, then there are as many numbers in continued proportion between e and f.
Proposition B: If a ratio leads to a multiple ratio when it is multiplied by itself, then it is itself a multiple ratio.
Proposition A: Between two numbers in epimoric ratio, no mean proportional is possible.

This analysis shows that in the proofs of VIII 2—3, 7—8 and of Propositions *A* and *B*, all of Book VII, to VII 27 and 33, is involved. Book VII is constructed in such a strictly logical fashion that no pebble can be removed from it, in sharp contrast to the muddy structure of Book VIII with its roundabout proofs, its unnecessary repetitions and its logical errors.

But, how is Book VII historically related to propositions VIII 2—3, 7—8 and *A*—*B* of Archytas and his school? Three possibilities suggest themselves:

1) The school of Archytas originally accepted propositions VII 21—22 and 27, which they needed for their developments, without proof, as self-evident, and the strict proofs found in Book VII are of later date;

2) Propositions VII 21—22 and 27 were originally proved differently and Book VII represents a later recasting of these proofs;

3) Book VII was in apple-pie order at the time of Archytas, so that he could simply refer to propositions VII 21—22 and 27.

The last of these possibilities appears to be the most probable in view of the fact that, in Boethius' account of Archytas, VII 22 is quoted word for word. But we can go further and eliminate possibilities 2) and 1) entirely, thus leaving 3) only!

1) is impossible. For it was customary for the Pythagoreans to prove number-theoretic propositions in a very careful, step-by-step manner. This can be seen in Archytas' elaborate proof of Proposition *A*, in which even the most trivial syllogisms are worked out punctiliously; it can be seen in the Pythagorean theory of the even and the odd, which includes detailed proofs of the most obvious things, such as the evenness of the sum of an even number of even numbers (IX 21); again in the proof, mentioned earlier, of the irrationality of the diagonal, in which, e.g., the conclusion "if m^2 is even, then m is also even" is very properly obtained by an indirect proof; finally it is seen in the elaborate proofs of Book VIII itself.

2) is also impossible. For the theory of proportions in Book VII is developed from Definition 20: *Numbers are proportional when the first is the same multiple, or the same part, or the same parts,* [1] *of the second that the third is of the fourth.* This definition is an ancient one. It already occurs in Hippocrates of Chios, applied to geometric magnitudes; he calls two circular segments proportional to their circles, if they are the same part of their circles, such as a half or a third. Strictly speaking, such a definition makes sense only for commensurable magnitudes, since, no matter whether one takes the m-fold, or the n-th part or m n-th parts, m and n being numbers in the Greek sense, one always obtains a rational ratio. We shall see that, later on, definitions were given, which are applicable also to incommensurable magnitudes; for example, the definition of Eudoxus used in Book V of the Elements (V, Def. 5). In Book VII, Euclid retained however the old definition VII 20, obviously because he would otherwise have been compelled to recast the entire

[1] e.g., the double, or the third part, or two thirds.

book so as to adapt it to the new definition. It follows that Book VII is not a later reconstruction, but a piece of ancient mathematics.

Another argument in favor of this conclusion is that Book VII is a well-rounded whole without traces of later revisions, such as occur in other books.

Thus we have acquired an important insight: *Book VII was a textbook on the elements of the Theory of Numbers, in use in the Pythagorean school.* It is not an accident that this book has been preserved; it is due to its strictly logical structure. Euclid did not find anything to correct; neither could he make a change, short of tearing the whole thing apart and building it anew — an indication of its firm structure.

In Book VII occur some propositions on prime numbers, but it does not contain the proposition that every number can be written as the product of primes in one and only one way, which plays such an important role in our modern elementary Theory of Numbers. The theory of the G.C.D. and the L.C.M., as well as that of relatively prime numbers are developed without the use of the factorization into prime numbers. The entire development rests on the "Euclidean algorithm" for determining the G.C.D. by means of successive subtractions. The emphasis is not on the divisibility properties of numbers, but rather on the theory of proportions and on the reduction of a ratio to lowest terms.

It is probable that it was the *calculation with fractions* which led to the setting up of this theory. Fractions do not occur in the official Greek mathematics before Archimedes, but, in practice, commercial calculations had of course to use them. The reason why fractions were eliminated from the theory is the theoretical *indivisibility of unity*. In the Republic (525E), Plato says: "For you are doubtless aware that experts in this study, if any one attempts to cut up the 'one' in argument, laugh at him and refuse to allow it; but if *you* mince it up, *they* multiply, always on guard lest the one should appear to be not one but a multiplicity of parts". Theon of Smyrna explains this further [1]: "When the unit is divided in the domain of visible things, it is certainly reduced as a body and divided into parts which are smaller than the body itself, but it is increased in numbers, because many things take the place of one".

When fractions are thus thrown out of the pure theory of numbers, the question arises whether it is not possible to create a mathematical equivalent of the concept fraction and thus to establish a theoretical foundation for computing with fractions. Indeed, this equivalent is found in the ratio of numbers. In place of the reduction of fractions to lowest terms, comes the simplification of a ratio of numbers to smallest terms which is investigated theoretically in Book VII. Reduction of fractions to a common denominator brings the concept of the L.C.M. on the stage; and this is also discussed in Book VII.

The art of computing, including the calculation with fractions, is called "logistics" by the Greeks. Plato distinguishes practical from theoretical logistics, just as he discriminates between practical and theoretical arithmetic (Gorgias

[1] *Theonis Smyrnaei Expositio rerum mathematicarum*, ed. Hiller (1878), p. 18.

451 *A—C*). Theoretical logistics deals especially with the study of numbers in their mutual ratios, exactly the sort of thing treated in Book VII, while theoretical arithmetic is concerned with "the even and the odd, how much each amounts to in every individual case". Apparently therefore, Plato considers the old Pythagorean theory of the even and the odd (Book IX) as belonging to theoretical arithmetic, and the theory of the ratios of numbers (Books VII, VIII) as a part of theoretical logistics. He contrasts these theoretical sciences with practical arithmetic, i.e. counting, and practical logistics, i.e. calculation, in particular with fractions.[1]

Ancient arithmetic also includes something which is thought of at present as a topic in algebra, namely

The solution of systems of equations of the first degree.

The solution of the special system of equations

$$\begin{aligned} x + x_1 + x_2 + \ldots + x_{n-1} &= s \\ x + x_1 &= a_1 \\ \ldots\ldots\ldots\ldots & \\ x + x_{n-1} &= a_{n-1} \end{aligned} \tag{1}$$

is known by the name "*flower of Thymaridas*". The solution is of course

$$x = \frac{(a_1 + \ldots + a_{n-1}) - s}{n - 2}.$$

The Neo-Pythagorean Iamblichus, who atrributes this solution to the old Pythagorean Thymaridas, shows furthermore, how systems of the form

$$x + y = \alpha(z + u), \quad x + z = \beta(u + y), \quad x + u = \gamma(y + z)$$

can be reduced to the form (1).

We see from this that the Pythagoreans, like the Babylonians, occupied themselves with the solution of systems of equations with more than one unknown.

Geometry.

Tannery has called attention to the following remarkable passage in Iamblichus' Pythagorica Vita:

> In the following manner the Pythagoreans explain how it came about that geometry was made known publicly: Through the fault of one of their number, the Pythagoreans lost their money. After this misfortune, it was decided to allow him to earn money with Geometry — thus Geometry came to be designated as "The Tradition of Pythagoras".

In further explanation, Tannery recalls (La géométrie grecque, p. 81) that the Pythagoreans had common ownership of goods. Tannery does not believe

[1] For a further discussion of the Greek concept of number, see J. Klein, *Die griechische Logistik und die Entstehung der Algebra, Quellen und Studien*, B 3, p. 18.

in the original secrecy of the mathematical sciences (I do), but he thinks (and very rightly) that this legend must nevertheless contain a core of truth. Indeed, after the middle of the 5th century, it became quite feasible to earn money with science; there existed a great thirst for knowledge. The sophists, the "teachers of wisdom" received excellent pay; wealthy people took pride in having their sons taught by the best and the most famous sophists. It is therefore quite possible that, in their distress, the Pythagoreans tried to increase their revenues in this manner.

The last sentence of the quoted fragment indicates that a textbook on Geometry must have existed with the title "The Tradition of Pythagoras", some kind of written course of lectures from which the Pythagoreans made money. This also explains how it happened that later writers attributed to Pythagoras all kinds of geometric discoveries, although some of these (e.g. the irrational) were certainly found only much later. It is only necessary to assume that these things occurred in "The Tradition of Pythagoras".

Tannery supposes that the statements of Eudemus concerning the geometry of the Pythagoreans must also have been taken from this book. According to Eudemus, the Pythagoreans discovered the proposition, that in every triangle the angle sum is equal to two right angles, and proved it as follows:

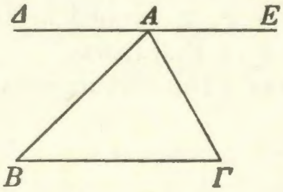


Fig. 31.

"Let $AB\Gamma$ be a triangle. Draw through A a line $\Delta E \,//\, B\Gamma$. Since $B\Gamma$ and ΔE are parallel, the alternate interior angles are equal, so that $\angle \Delta AB = \angle AB\Gamma$ and $\angle EA\Gamma = \angle A\Gamma B$. Add $\angle BA\Gamma$ to both sides. Then $\angle \Delta AB + \angle BA\Gamma + \angle \Gamma AE$, i.e. two right angles will be equal to the sum of the three angles of the triangle. Therefore the sum of the three angles of a triangle is equal to two right angles. This is the proof of the Pythagoreans" (Proclus, Commentary on Euclid, I 32).

This shows that the geometry of the Pythagoreans was constructed logically and that they knew not only the proposition concerning the angle sum of a triangle, but also the one on alternate interior angles formed by parallel lines.

It is probable that the Pythagoreans had a theory of regular polygons. They certainly knew the star-pentagon, and the dodecahedron; and they also knew that there are only three regular polygons whose angles can fill the space about a point O in the plane, viz. the triangle, the square and the hexagon. It is very likely that this proposition was related to their investigation of the regular polyhedra.

We recall that Plutarch, speaking of the distich "When Pythagoras discovered his famous figure, for which he sacrificed a bull", doubted whether the proposition, referred to, is the one about the hypothenuse, or whether it has to do with the adaptation of areas, or perhaps with the problem of constructing a plane figure, equal in area to a second figure and similar to a third. It seems probable to me, that these three subjects were not arbitrarily thrown together, but that

Plutarch, or the source upon which he draws, got these three important propositions from "The Tradition of Pythagoras". As a matter of fact, they are directly connected with one another. Let us start with the last one.

The general problem of constructing a polygon similar to a given polygon and equal in area to another one, can be reduced to the special case of constructing a square equal in area to a given rectangle (Euclid, II 14). This amounts to the construction of the mean proportional between the base and the altitude of the rectangle. In his duplication of the cube, which we shall discuss later on, Archytas uses twice, as an auxiliary, the construction of the mean proportional $x = \sqrt{ab}$, by means of a semi-circle as illustrated in Fig. 32. And, as will be seen presently, the same construction is needed in the "application of areas", which is, according to Eudemus, a find of the Pythagoreans.

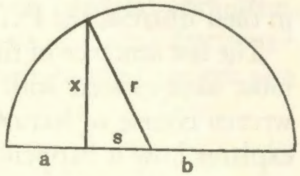


Fig. 32.

Euclid's proof of the proposition $x^2 = ab$ proceeds as follows:

$$x^2 = r^2 - s^2 = (r - s)(r + s) = ab.$$

We see that the "Theorem of Pythagoras" $r^2 = x^2 + s^2$, is applied here. I assume that this proof was taken from "The Tradition of Pythagoras".

But what does Plutarch mean by the "application of areas"? This very important subject deserves separate consideration.

"Geometric Algebra".

When one opens Book II of the Elements, one finds a sequence of propositions, which are nothing but geometric formulations of algebraic rules. So, e.g., II 1:

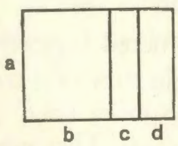


Fig. 33.

"If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments,

corresponds to the formula

$$a(b + c + \ldots) = ab + ac + \ldots$$

II 2 and 3 are special cases of this proposition. II 4 corresponds to the formula

$$(a + b)^2 = a^2 + b^2 + 2ab.$$

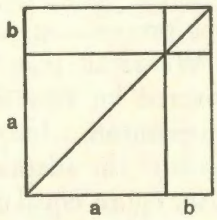


Fig. 34.

The proof can be read off immediately from Fig. 34. In II 7, one recognizes the analogous formula for $(a - b)^2$. We have here, so to speak, the start of an algebra textbook, dressed up in geometrical form. The magnitudes under consideration are always line segments; instead of "the product *ab*", one speaks of "the rectangle formed by *a* and *b*", and in place of a^2, of "the square on *a*".

Quite properly, Zeuthen speaks in this connection of a "geometric algebra". Throughout Greek mathematics, one finds numerous applications of this "algebra" The line of thought is always algebraic, the formulation geometric. The greater part of the theory of polygons and polyhedra is based on this method; the entire theory of conic sections depends on it. Theaetetus in the 4th century, Archimedes and Apollonius in the 3rd are perfect virtuosos on this instrument.

Presently we shall make clear that this geometric algebra is the continuation of Babylonian algebra. The Babylonians also used the terms "rectangle" for xy and "square" for x^2, but besides these and alternating with them, such arithmetic expressions as multiplication, root extraction, etc. occur as well. The Greeks, on the other hand, consistently avoid such expressions, except in operations on integers and on simple fractions; everything is translated into geometric terminology. But since it is indeed a translation which occurs here and the line of thought is algebraic, there is no danger of misrepresentation, if we reconvert the derivations into algebraic language and use modern notations. From now on we shall therefore quite coolly replace expressions such as "the square on *a*", "the rectangle formed by *a* and *b*" by the modern symbols a^2 and ab, whenever they simplify the presentation. Similarly, we shall put proportionalities into a modern dress, e.g.

$$a^2 : ab = a : b.$$

For our purpose it is not necessary to use the abbreviations $O(a, b)$, $T(a)$ and $A(a, b)$ introduced by Dijksterhuis for ab, a^2 and $a : b$, provided we take good care, ***not to use algebraic transformations, which can not immediately be reformulated in the Greek terminology***. When Cantor (Geschichte der Mathematik I, 3rd or 4th edition, p. 213) derives e.g. from the proportionality

$$a : x = x : y = y : b,$$

the result $\quad x^2 = ay, \quad y^2 = bx,$

and then $\quad x^4 = a^2y^2 = a^2bx,$

thus obtaining $\quad x^3 = a^2b,$

the line of thought is contrary to that of the Greeks; x^3 can still be interpreted geometrically, viz. as the volume of a cube, but not x^4. It *is* proper to derive

$$x^2 = ay, \quad xy = ab$$

from the continued proportion, and then to write

$$x^3 = axy = a^2b,$$

thus leading to

$$a^3 : x^3 = a^3 : a^2b = a : b.$$

In words: ***If x and y are two mean proportionals between a and b, then the cube on a is to the cube on x as a is to b.***

Let us now return from our digression to the geometric algebra of Book II.

As the next formula one might well expect

$$a^2 - b^2 = (a - b)(a + b) \tag{1}$$

or, since Euclid avoids the subtraction of areas,

$$a^2 = (a - b)(a + b) + b^2. \tag{2}$$

It does actually occur, but in a remarkable double form. Propositions II 5, 6 are formulated as follows:

II 5. If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between points of section is equal to the square on the half.

II 6. If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

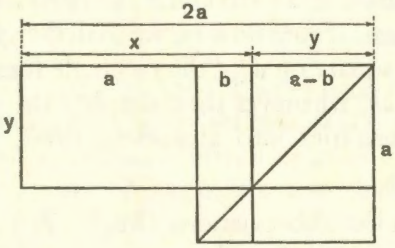


Fig. 35.

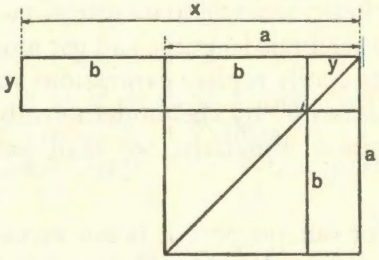


Fig. 36.

Designating the line given in II 5 by $2a$ and the segment between the points of division by b, we may express II 5 by formula (2). But, if the line given in II 6 is denoted by $2b$, and one half of it plus the extension by a, then this proposition is expressed by the same formula (2). But it can not have been the sole purpose of the two propositions to give formula (2) a geometric dress and to prove it in that way; for, why should two propositions be given for *one* formula?

Let us try something else! We call x and y the two unequal parts into which the line $2a$ is divided in II 5. Then II 5 can be expressed by means of the formula

$$\left(\frac{x+y}{2}\right)^2 = xy + \left(\frac{x-y}{2}\right)^2, \tag{3}$$

i.e. by

$$\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 = xy. \tag{4}$$

But, if x is used to designate the sum of the line given in II 6 and its extension, and y to denote the extension itself, then this proposition again leads to formula (3).

In both cases the proof is the same: the difference of the squares on a and on b is a "gnomon", a carpenter's square; this can be transformed into a rectangle xy by removing a piece (ay in the first case and by in the second) on the right side and placing it on the left side.

This gives again two propositions for one formula. Why? What was the line of thought of the man who formulated the propositions in this way?

The answer is found by following out, in the Elements themselves, and in Euclid's other works, the way in which propositions II 5 and II 6 are applied. In the Data (84 and 85), the question is considered how to prove that two segments, x and y, are known, when their sum $x+y$ or their difference $x-y$, and also the area of the rectangle formed by them, are given. When the sum $x+y=2a$ is given, then II 5 is applied to determine half the difference $\frac{1}{2}(x-y)=b$; if the difference is given, then II 6 is used so that half the sum is found from (3).

We see therefore, that, at bottom, II 5 and II 6 are not propositions, but *solutions of problems*; II 5 calls for the construction of two segments x and y of which the sum and product are given, while in II 6 the difference and the product are given.

The applications in the Elements themselves are consistent with this view. In II 11 a line has to be divided into mean and extreme ratio. In the terminology of area calculations, the problem receives the following form: "To divide a given straight line in such a way, that the rectangle, formed by the entire line and one of the parts is equal to the square on the other part". This leads to the equation

$$y^2 = a(a-y),$$

i.e.

$$y^2 + ay = a^2$$

or

$$y(y+a) = a^2.$$

This last equation calls for the construction of two segments y and $y+a$, of which the difference a and the product a^2 are given. Since the difference is given, the problem is solved by use of II 6.

This interpretation of II 5, 6 as solutions of problems is raised beyond all doubt by the generalizations VI 28, 29, which are quite openly formulated as problems:

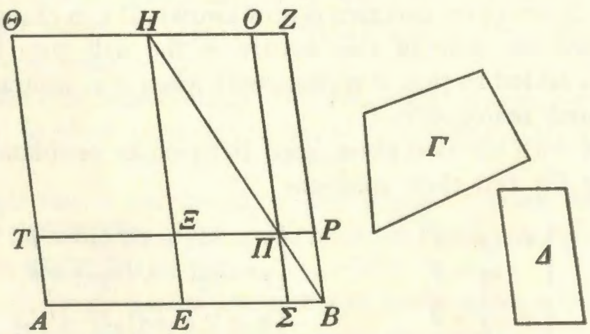


Fig. 37.

VI 28. To apply to a given line AB a parallelogram ($A\Pi$), equal in area to a given rectilinear figure Γ, lacking a parallelogram ($B\Pi$) similar to a given (parallelogram) Δ.

VII 29. To apply to a given line AB a parallelogram ($A\Xi$), equal in area to a rectilinear figure and in excess by a parallelogram, similar to a given (parallelogram) Δ.

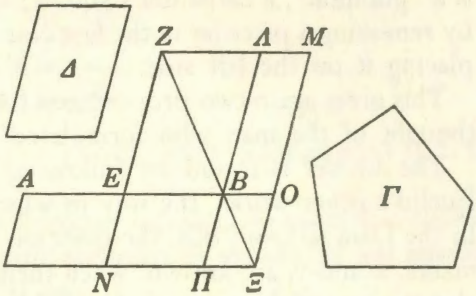


Fig. 38.

Figures 37 and 38 show clearly what is meant. In VI 28 the parallelogram $A\Pi$ (Euclid designates parallelograms by indicating two opposite vertices) must have the same area as the given polygon Γ, while the parallelogram $B\Pi$ on the remaining piece of the given line must be similar to the given parallelogram Δ.

Analogously we explain VI 29.

In the most important applications the given parallelogram is a square. The required parallelogram $B\Pi$, or $B\Xi$, must then be a square. If the base and altitude of the required parallelogram are denoted by x and y, the given area (of Γ) by F and the given line to which the rectangle has to be applied by $2a$, then the conditions in VI 28 are

$$xy = F, \qquad x + y = 2a, \tag{5}$$

and in VI 29

$$xy = F, \qquad x - y = 2a. \tag{6}$$

The solutions of problems (5) and (6), indicated in the generalizations VI 28, 29, is exactly the same as that supplied by propositions II 5, 6. In case (6), we have, by means of (3)

$$\left(\frac{x+y}{2}\right)^2 = xy + \left(\frac{x-y}{2}\right)^2 = F + a^2.$$

When F and $2a$ are given, this area is now known. If it is changed into a square, using II 14, then the side of this square is the half sum $\frac{1}{2}(x + y)$. When $a = \frac{1}{2}(x - y)$ is added to this, x is obtained; when it is subtracted, one finds y. Case (5) is treated analogously.

A comparison with the two normalized Babylonian problems B 1, 2 (see the end of Chapter III) and their solutions

$$\text{(B1)} \quad \begin{cases} x + y = s \\ xy = F \end{cases} \qquad \begin{cases} x = \frac{1}{2}s + \sqrt{(\frac{1}{2}s)^2 - F} \\ y = \frac{1}{2}s - \sqrt{(\frac{1}{2}s)^2 - F}, \end{cases}$$

$$\text{(B2)} \quad \begin{cases} x - y = d \\ xy = F \end{cases} \qquad \begin{cases} x = \sqrt{F + (\frac{1}{2}d)^2} + \frac{1}{2}d \\ y = \sqrt{F + (\frac{1}{2}d)^2} - \frac{1}{2}d, \end{cases}$$

shows clearly that these are entirely analogous to II 5, 6 and VI 28, 29.

Deficiency and excess, the parallelogram of given shape, which is lacking and which is left over, are called in Greek Elleipsis and Hyperbolè; the application to a line of a parallelogram of given area is called Parabolè. In his commentary on Euclid I 44, Proclus has the following to say about the history of these adaptations:

> These things, so says Eudemus, the peripathetic, are discoveries of the Pythagorean muse, namely the application of areas (parabolè), their deficiency (elleipsis) and their excess (hyperbolè). Later on these names were carried over to the three conic sections. [1]

The solution of problems (5) and (6) plays an extremely important part in Greek mathematics. We have already seen that they are dealt with twice in the Elements, and again in the Data. But, moreover, in Book X, which deals with the theory of irrational segments, an important construction (X 33—35) depends on the elliptic application of areas.

For the solution of quadratic equations, the Greeks reduced them to one of the forms

$$x(x + a) = F, \qquad x(a - x) = F, \qquad x(x - a) = F,$$

which are then solved by means of the application of areas. In the first and third cases, there are two line segments, x and $x \pm a$, of which the difference and the product are given, so that we have an application with a square in excess; in the second case, we have two line segments, x and $a - x$, of which the sum and the product are known, thus leading to an application with deficiency of a square. When the term in x^2 has the coefficient $\gamma = p : q$,

$$\gamma x^2 + xa = F, \qquad ax - \gamma x^2 = F, \qquad \gamma x^2 - ax = F,$$

the more general applications VI 28, 29 are applied, with an excess or a deficiency of given form (e.g. a rectangle).

According to Eudemus therefore, this important part of geometric algebra is a discovery of the Pythagoreans. However, in view of the fact that in both adaptations, with excess and with defect, it is always necessary to change a given area into a square (since a square root has to be extracted), proposition II 14, which solves this problem by use of the Theorem of Pythagoras, must also have been familiar to the Pythagoreans.

It occurs a second time that there are two propositions which express the same algebraic formula. In their formulation these propositions are quite similar to II 5, 6:

II 9. *If a straight line be cut into equal and unequal segments, the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section.*

II 10. *If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line*

[1] It will be made clear later on (in the discussion of Apollonius) what these adaptations have to do with conic sections.

both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line.

Both propositions lead to the formula

$$x^2 + y^2 = 2\left\{\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2\right\}.$$

Both can be taken to be solutions of problems. For II 9, the problem is: to determine x and y when $x + y$ and $x^2 + y^2$ are given. For II 10: given $x - y$ and $x^2 + y^2$, required x and y. Hence we are concerned with the solution of the systems of equations

(7) $$x + y = s, \quad x^2 + y^2 = F;$$

(8) $$x - y = d, \quad x^2 + y^2 = F.$$

The solutions, indicated in II 9, 10, amount of course to the Babylonian solutions

(9) $$x = \tfrac{1}{2}s + \sqrt{(F/2) - (s/2)^2}, \quad y = \tfrac{1}{2}s - \sqrt{(F/2) - (s/2)^2},$$

and

(10) $$x = \sqrt{F/2 - (d/2)^2} + d/2, \quad y = \sqrt{F/2 - (d/2)^2} - d/2.$$

The proofs make clever use of the "Theorem of Pythagoras".

The fact that all four of the normalized forms of systems of equations, which we have found in the cuneiform texts, are taken over by Euclid, with their solutions, gives clear evidence of the derivation of the geometric algebra of Book II from Babylonian algebra. *Apparently the Pythagoreans formulated and proved geometrically the Babylonian rules for the solution of these systems.*

We observe moreover that the simple linear equation

$$ax = F$$

leads, in geometric formulation, to the simple application of an area to a line, without excess or deficiency. The pure quadratic $x^2 = F$ amounts to the transformation of a given area into a square (II 14). The pure cubic $x^3 = V$, in geometric form, poses the problem of constructing a cube of given volume. The ancients were concerned with this problem as well; a special case is the famous "duplication of the cube", to which we shall return later on. The mixed cubics $x^2(x + 1) = V$ and $x^2(x - 1) = V$ were solved by the Babylonians with the aid of tables, and the numbers in these tables are exactly the spatial numbers $n^2(n \pm 1)$, which are called "Arithmoi paramekepipedoi" by Nicomachus (see Becker, Quellen und Studien B4, p. 181).

Thus we conclude, that *all the Babylonian normalized equations have, without exception, left their trace in the arithmetic and the geometry of the Pythagoreans.* It is out of the question to attribute this to mere chance. What could only be surmised before, has now become certainty, namely that the Babylonian tradition supplied the material which the Greeks, the Pythagoreans in particular, used in constructing their mathematics.

Why the geometric formulation?

Why did the Greeks not simply adopt Babylonian algebra as it was, why did they put it in geometric form? Was it their delight in the tangible and the visible, which turned them away from numbers, to occupy themselves with figures instead?

Unquestionably, the enjoyment of what can be seen was a strong motive power in the Greeks. The sculpture reproduced on plate 16, as well as numerous other immortal works of art, show this clearly. And there is ample further evidence in the marvelously vivid descriptions of Homer, and in Plato's image of the cave.

But all of this is insufficient to account for the complete elimination of algebra. We must remember that, according to the reliable report of Eudemus, it was particularly the Pythagoreans who laid the foundations of geometric algebra. But for the Pythagoreans, numbers were "the rock-bottom of the entire universe", the world was made "by imitation of numbers"; the heavens were for them "harmony and numbers". And, as Aristotle says (Metaphysics A5), they attained these views exactly because they applied themselves to mathematics. Would these worshippers of numbers have solved quadratic equations, not in terms of numbers, but by means of segments and areas, purely for the delight in the visible? This is hard to believe; there must have been another push towards the geometrisation of algebra.

Indeed this is not difficult to find: it is the discovery of the irrational, which, as Pappus tells us, actually originated in the Pythagorean school. The diagonal of the square is not commensurable with the side. But this means that, when the side is chosen as the unit of length, the diagonal can not be measured; its length can not be expressed, neither by an integer, nor by a fraction.

Nowadays we say that the length of the diagonal is the "irrational number" $\sqrt{2}$, and we feel superior to the poor Greeks who "did not know irrationals". But the Greeks knew irrational ratios very well. As we shall see later on, they had a very clear understanding of the ratio of the diagonal to the side of the square, and they were able to prove rigorously that this ratio can not be expressed in terms of integers. That they did not consider $\sqrt{2}$ as a number was not a result of ignorance, but of strict adherence to the definition of number. Arithmos means quantity, therefore whole number. Their logical rigor did not even allow them to admit fractions; they replaced them by ratios of integers.

For the Babylonians, every segment and every area simply represented a number. They had no scruples in adding the area of a rectangle to its base. When they could not determine a square root exactly, they calmly accepted an approximation. Engineers and natural scientists have always done this. But the Greeks were concerned with exact knowledge, with "the diagonal itself", as Plato expresses it, not with an acceptable approximation.

In the domain of numbers, the equation $x^2 = 2$ can not be solved, not even in that of ratios of numbers. But it is solvable in the domain of segments: indeed the diagonal of the unit square is a solution. Consequently, in order to obtain exact solutions of quadratic equations, we have to pass from the domain of numbers to that of geometric magnitudes. Geometric algebra is valid also for irrational segments and is nevertheless an exact science. It is therefore logical necessity, not the mere delight in the visible, which compelled the Pythagoreans to transmute their algebra into a geometric form.

Side- and diagonal-numbers.

In the Republic Plato speaks of the number 7 as the "rational diagonal", connected with the side 5. In explanation of this passage, Proclus gives the following definition of "side- and diagonal-numbers", which he attributes to the Pythagoreans and which, indeed, we meet with again in Theon of Smyrna and in Iamblichus.

"As the source of all numbers, unity is potentially a side as well as a diagonal. Now let two units be taken, one lateral unit and one diagonal unit; then a new side is formed by adding the diagonal unit to the lateral unit, and a new diagonal by adding twice the lateral unit to the diagonal unit." Thus we obtain the side-number 2 and the diagonal-number 3. From here on we proceed in similar manner with the numbers found thus far. In this way, we get

$$2 + 3 = 5, \qquad 2 \times 2 + 3 = 7,$$

etc., according to the formulas

$$a_{n+1} = a_n + d_n, \quad d_{n+1} = 2a_n + d_n. \tag{1}$$

The names side- and diagonal-numbers hint at the fact that the ratio $a_n : d_n$ is an approximation for the ratio of the side of a square to its diagonal. This follows from the identity

$$d_n^2 = 2a_n^2 \pm 1 \tag{2}$$

which, according to Proclus [1], was proved by use of II 10. Proposition II 10, which was quoted above, can indeed be expressed by the formula

$$(2a + d)^2 + d^2 = 2a^2 + 2(a + d)^2$$

If now $d^2 = 2a^2 \pm 1$, then subtraction of this relation from the preceding one gives

$$(2a + d)^2 = 2(a + d)^2 \pm 1.$$

Thus, if (2) is valid for a particular value of n, then it also holds for $n + 1$, but with the opposite sign. But, if we set $a_1 = d_1 = 1$, then (2) holds for $n = 1$; hence it holds for $n = 2$, etc.

We see from this that the Pythagoreans knew the principle of mathematical induction, certainly in essence, and that they applied geometric algebra to problems of the theory of numbers.

[1] Commentary on Plato's Republic II, chapters 23 and 27.

But how did they get the recursion formula (1)? I venture the following conjecture:

Greek mathematics knew the method of successive subtractions (antanairesis) for determining the greatest common measure of two commensurable magnitudes a and b: the smaller one, say a, is subtracted from the larger one, thus giving two new magnitudes a and $b - a$; then the smaller of these magnitudes is again subtracted from the larger one, etc. If a common measure exists, the process leads ultimately to two equal magnitudes $c = d$, which equal the greatest common measure. In Book VII of the Elements, this method is applied to numbers for determining the G.C.D. and, at the beginning of Book X, to arbitrary magnitudes to decide whether a common measure exists and to determine it, in case it does exist. If applied to two incommensurable magnitudes, the process continues *ad infinitum*.

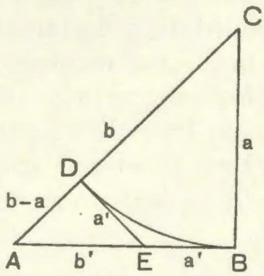


Fig. 39.

For example, if a is the side and b the diagonal of a square, then one can subtract a from b once (see Fig. 39). The remainder $b - a = AD = DE = EB = a'$ can now be subtracted from $a = AB$, leaving a remainder $b' = AE$. Now, a' and b' are again the side and the diagonal of a smaller square, and from

$$b' = a - a', \qquad a' = b - a$$

follows

$$(3) \qquad a = a' + b', \qquad b = 2a' + b'.$$

Formulas (3) already present the same form as the recursion formulas (1). Repetition of this same process of subtraction gives again a smaller side a'' and a smaller diagonal b''. If the process is continued until the difference between, say a''' and b''' has become too small to be observed and if one approximates by setting $a''' = b'''$, then, choosing a''' as the unit of length, a'' and b'', a' and b', and finally a and b are represented by means of (3), in the form of the successive side- and diagonal-numbers.

The problem of approximating to the ratio of the diagonal and the side by means of rational numbers, was proposed and solved by the Babylonians. But the Pythagoreans carried this old problem infinitely farther than the Babylonians. They found a whole set of approximations of indefinitely increasing accuracy; moreover they developed a scientific theory concerning these approximations and they proved the general proposition by complete induction. Again and again it becomes apparent that there were excellent number-theoreticians in the Pythagorean school.

Now we are going to take a look outside this school.

Anaxagoras of Clazomenae

is known chiefly as a natural philosopher. He was held in high esteem in Athens;