KEN SAITO

BOOK II OF EUCLID'S *ELEMENTS* IN THE LIGHT OF THE THEORY OF CONIC SECTIONS

INTRODUCTION

This paper proposes an alternative to the prevailing interpretation which regards the second book of the *Elements* (hereafter *Elem.* II) as basic part of the "geometric algebra". Chapter I of this paper is dedicated to an examination of the *Conics* of Apollonius. Though the central part of the "geometric algebra" is usually explained as a translation of the Babylonian algebraic techniques, it is not reasonable to attempt a determination of the nature of Book II by inconfirmable conjectures regarding its origin. The significance of *Elem.* II should be sought by studying applications of the propositions there. Thus we should examine how Euclid utilizes his propositions in *Elem.* II, when arguing about this book. The propositions in the *Elements* have been thoroughly examined by Ian Mueller.[1] But Mueller's study is not sufficient for purposes of the present paper, precisely because he limits his study to the *Elements*; other works of Euclid as well should be examined. Hence, the study of the *Conics* is necessary, since compilation of the fundamental part of the theory of conic sections is attributed to Euclid. The examination of the *Conics* to shed light on *Elem.* II can also be justified by the fact that the term "geometric algebra" originates in Zeuthen's study of Apollonius. Throughout my examination, geometric intuition in the *Conics* will be emphasized. In the second chapter, I examine *Elem.* II itself. Overall, this study is a refutation of the common interpretation of *Elem.* II, and an attempt to advance Mueller's study a step further.

CHAPTER I. THE "GEOMETRIC ALGEBRA" IN APOLLONIUS'S CONICS

Apollonius's *Conics* is one of the greatest works in Greek mathematics, well known for its difficulties to modern readers. The *Conics* were thoroughly studied by Zeuthen, and his *Die Lehre von den Kegelschnitten im Altertum*[2] remains the standard work. Zeuthen characterized the argument of Apollonius by its two | major auxiliary 3
methods[3] (Hülfsmittel), namely, the theory of proportions (*Elem.* V and VI) and the use of propositions in *Elem.* II. The latter was "geometric algebra", which is now a

1 I. Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (MIT Press, 1981).
2 H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum* (Kopenhagen, 1886; reprint, Hildesheim, 1966) (hereafter *Die Lehre*).
3 *Die Lehre*, 1er Abschnitt.

J. Christianidis (ed.), Classics in the History of Greek Mathematics, 139–168.

standard appellation for this book. Zeuthen insisted that these auxiliary methods were algebraic in essence, and that the whole argument of Apollonius was equivalent to modern algebraic operations. This view has been widely accepted. For example, van der Waerden states:

> ... Apollonius proves geometrically all the algebraic transformations performed on the equation. The line of thought is mostly purely algebraic and much more "modern" than the abstract geometric formulation would lead one to think. Apollonius is a virtuoso in dealing with geometric algebra and also a virtuoso in hiding his original line of thought. ...[4]

That the Greeks had algebraic modes of thought and hid their original (algebraic) line of thought under the guise of geometric formulations, is a leitmotiv of this interpretation. Is this idea, which can be traced back to the 16th- and 17th-century mathematicians, to Viète and Descartes, and even to Ramus, justified? Are Apollonius's arguments so tortuous that they cannot be understood without attributing to him some other line of thought (viz. algebraic) than that derived from the text itself?

In the following examination of some propositions in the *Conics*, I will make it clear that Apollonius's thought can be better understood if we assume that crucial steps of his argumemts depend on geometric intuitions.

34 My examination concentrates on propositions concerning the interchangeability | of the diameters of the conics (*Conics*, I 42–51), and one of the so-called power propositions (III 17) together with its lemmata (III 1–3),[5] leading me to hope that my choice will not be criticized for being arbitrary. Propositions I 42–51 are recognized as the aim of the first book of the *Conics*,[6] and III 17 as one of the propositions used in the solution of the four line locus,[7] of which Apollonius was so proud in his preface.[8]

Before minute examinations of these propositions, a rough sketch of the contents of the first book is in order. Propositions 1–10 are preliminary, and are followed by the derivation of the symptoms of the three conics (or four, if we count the opposite sections—hyperbola with two branches—as another conic section in the case of the hyperbola).

4 Van der Waerden, *Science Awakening*, tr. by A. Dresden (Groningen, 1959), p. 248 (hereafter *SA*).

5 All the propositions concerning the opposite sections (hyperbola with two branches) are omitted in my examination. The reasons for this are twofold: first, the characteristics of Apollonius's argument are sufficiently revealed through those propositions not involving the opposite sections; second, arguments involving the opposite sections are not included in the theory of the conic sections at the time of Euclid, and thus are not adequate for my purpose as stated in the introduction.

6 For example, see *SA*, p. 249.

7 *Die Lehre*, pp. 126–154. T. L. Heath reproduces Zeuthen's arguments in his *Apollonius of Perga* (Cambridge, 1896; reprint, 1961), pp. cxxxviii–cl. (hereafter Heath, *Apollonius*).

8 J. L. Heiberg ed., *Apollonii Pergaei quae graece exstant* (1891; reprint, Stuttgard, 1974), Vol. 1, p. 4. Note that there are sufficient reasons to believe that the propositions chosen here had been known before Apollonius, perhaps by Euclid. Archimedes cites the same proposition as *Conics* III 17 in his *Conoids and Spheroids* prop. 3, and states "This is included in the *Conic Elements*", which is usually identified with the lost work of Euclid. In Apollonius's *Conics*, III 17 is based, through III 1–3, on I 42, 43, which form the substantial part of the theorem on the interchangeability of diameters. As a result, it is likely that I 42, 43 were also included in the *Conic Elements* and thus natural to assume that the whole theorem on the interchangeability of the diameters was also included therein. As for a detailed argument on the contents of the *Conic Elements* of Euclid, see Heath, *Apollonius*, pp. xxxiv–xxxvi.

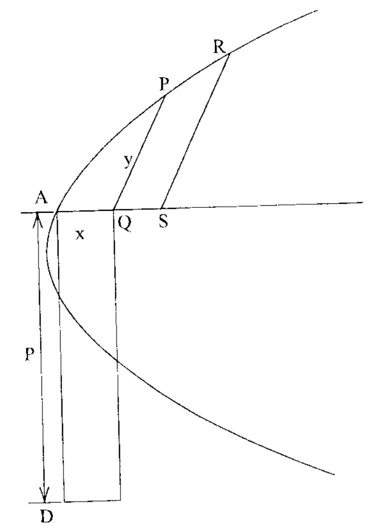

Fig.1

I 11 $\mathbf{T}(PQ) = \mathbf{O}(AD, AQ)$[9] (Fig. 1)

I 12,13 $\mathbf{T}(PQ)$ = rectangle AQEF (Fig. 2, 3 respectively)[10]

AQ is the diameter of the curve and bisects any chord parallel to the ordinate PQ. The point A is called the vertex. Called the *latus rectum*, line AD is drawn perpendicular to the diameter AQ at the vertex A. The length of *latus rectum* is determined to satisfy a certain proportionality for each conic section. A′ is the point where the diameter cuts the cone again. AA′ is called *latus transversum*.[11]

Though Apollonius's symptoms are different from those of his forerunners such as Archimedes, the difference is not substantial. First, Apollonius's symptom allows the diameter and the ordinate to be oblique to each other, while his predecessors made them orthogonal. But this does not mean that Apollonius was the first to discover the oblique symptom. Second, Apollonius's symptoms are stated | as the equalities 35
between the square of the ordinate and the rectangle applied on the *latus rectum*. According as the application is accomplished either without excess and defect, or with excess, or with defect, the conic sections produced are called *parabole, hyperbole*, and *elleipsis*, respectively. Van der Waerden seems to believe that this expression of the symptom is a sign of Apollonius's algebraic line of thought:

> The diagrams of Apollonius, here reproduced, consist of two unequal parts; one could speak of a geometric and an algebraic diagram. The geometric diagram

9 Hereafter I will use **T**(XY) for "the square on XY", and **O**(XY, YZ) for "the rectangle contained by XY and YZ".

10 Proposition I 14, the case of the opposite sections, is omitted.

11 For a detailed description, see Heath, *Apollonius*, pp. 8 12.

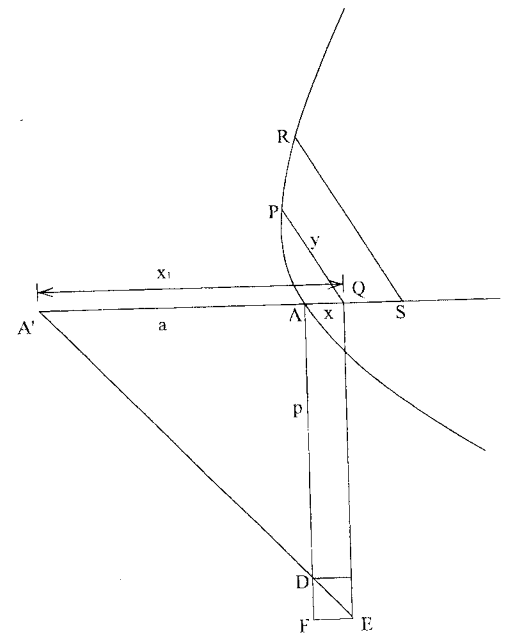

Fig. 2

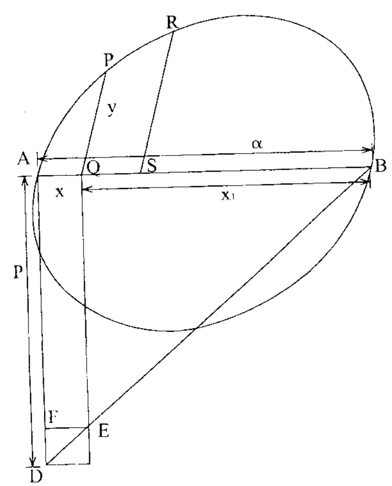

Fig. 3

> consists of the conic section, in which we are actually interested, the oblique axes, the abscissa x and the ordinate y. The algebraic diagram is not oblique, but rectangular; it expresses the algebraic relation between a, p, x, and y. The areas which are here added and substracted correspond to the terms of an equation in modern analytical geometry. . . .[12]

Van der Waerden's claim seems, however, deceptive. We should note that these allegedly algebraic symptoms play very little role in the propositions of the *Conics* except in the case of the parabola. In this case the symptom before Apollonius was already stated in virtually the same manner, *i.e.* the equality between the square of the ordinate and the rectangle contained by the abscissa and a line of constant length. In the cases of the hyperbola and the ellipse, Apollonius usually uses other forms of the symptoms which he derives later, namely in

Conics I 20: **T**(PQ): **T**(RS) = QA: AS (Fig. 1. For the parabola)

Conics I 21 (a): **T**(PQ): **O**(AQ, QA′) = *latus rectum: latus transversum* (Fig. 2, 3, for the hyperbola and the ellipse, respectively)

(b): **T**(PQ): **O**(AQ, QA′) = **T**(RS): **O**(AS, SA′)

For the hyperbola and the ellipse, Apollonius almost always uses I 21 (a), which is virtually equivalent to that of Euclid and Archimedes,[13] though Apollonius extends it to oblique conjugations. Thus, *Conics* I 11–14 are stated only as formal definitions of the conic sections, not as substantial bases for the later descriptions. Moreover, it appears that the orthogonal diagram of Apollonius which van der Waerden claims to be algebraic, is in fact a *geometric* expression (or visualization) of the symptom which can be sufficiently stated in terms of proportionality without any additional new diagram, and thus that Apollonius's intention is to visualize the relation expressed by the symptom.

Proposition I 17 states that the line parallel to the ordinate at the vertex is tangent to the section. Propositions I 18–31 deal with the intersections of conic sections and straight lines. I 32 is the converse of I 17, showing the uniqueness of the tangent at the vertex. I 33–34 are concerned with tangents of the conic sections at any point. For the 3
parabola (I 33), the tangent at point C is drawn as follows | (Fig. 4): Draw ordinate CD through C, and take a point E in the extension of the diameter AD, letting EA=AD. EC is thus the tangent of the parabola at the point C. For the hyperbola and the ellipse, point E is taken to satisfy the proportionality BE: EA = BD: DA (Fig. 5, 6), where B is the opposite end of the diameter. This proportionality is transformed in I 37 as follows:

I 37 (a) **O**(DZ, ZE) = **T**(ZA)

I 37 (b) **O**(AD, DB) = **O**(ZD, DE), or

T(CD): **O**(ZD, DE) = *latus rectum: latus transversum.*

---

12 *SA* pp. 247 248. Here van der Waerden describes the symptoms in the following manner. Let AD = p PQ = y, AA′ = a, AQ = x, A′Q = x_1 and the ratio p: $a = \alpha$. Then,

I 11: $y^2 = px$.

I 12, 13: $y^2 = x\alpha x_1 = x(p \pm \alpha x)$. (the + sign applies to the hyperbola and the − sign to the ellipse)

13 See Heath, *Apollonius*, pp. xxxv, l–li.

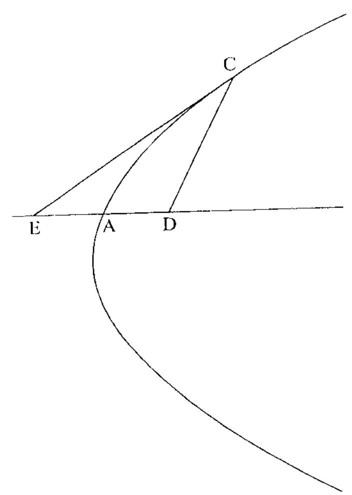

Fig. 4

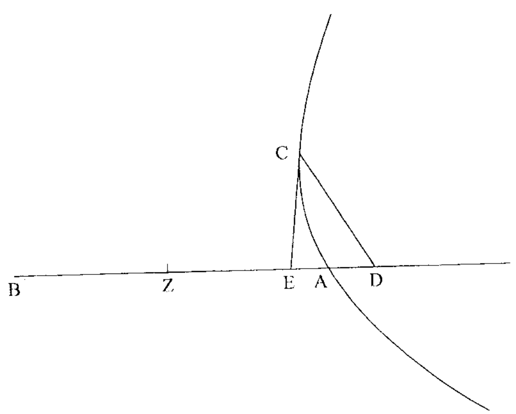

Fig. 5

37 | I 35–36 are the converses of I 33–34 respectively, and state that if CE is the tangent of the conic section, the above equality or proportionality holds. I 39–41 can be regarded as lemmata for the theorems of interchangeability of diameters for the hyperbola and ellipse, while I 42–51 form propositions concerning such interchangeability. I 52–60 demonstrate the existence of a cone and plane generating a conic section given in terms of the symptom. According to these propositions, any curve

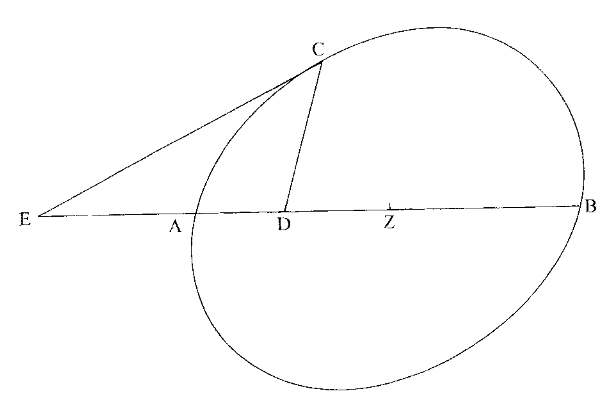

Fig. 6

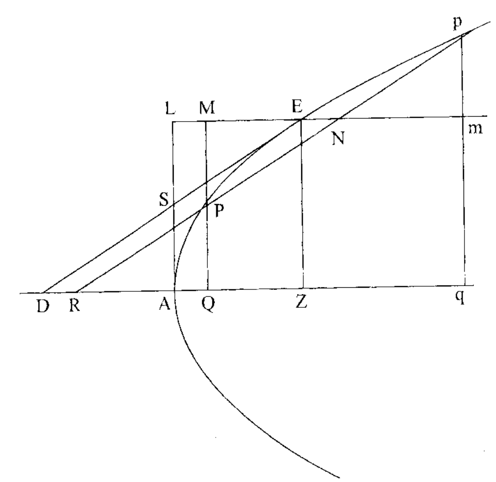

Fig. 7

expressed by a symptom of conic sections can be actually produced by cutting a cone with a plane.

The following is a detailed examination of propositions concerning the interchangeability of diameters. I begin with the case of the parabola. The results can be summarized as follows:

Let AE be a parabola with diameter AZ (Fig. 7). Any line EN parallel to AZ can be taken as a new diameter of the parabola; *i.e.* (1) there exists a group of parallel chords, each one of which (*e.g.* pP) is bisected by EN (*i.e.* pN = NP), (2) there exists a line l' (the *latus rectum* for the new diameter) such that $\mathbf{O}(l', EN) = \mathbf{T}(pN)$ for any pP. Apollonius proves (1) in I 46 and (2) in I 49. These two propositions depend entirely on

> I 42: Let a parabola be given through the origin A with diameter AD, and draw, at an arbitrary point E of the parabola, the tangent line ED and the ordinate EZ. Through an arbitrary point P on the parabola, draw the lines PR and PQ, parallel to the tangent line ED and the ordinate EZ, to their intersections with the diameter AD. Complete the parallelogram AZEL. Then we have
>
> triangel PQR=parallelogram ALMQ.[14]

The proof is as follows:

Since DA = AZ (I 33, 35),

tri. EZD = par. ALEZ.

38 | And ALEZ: ALMQ = ZA: AQ

= $\mathbf{T}$(EZ):$\mathbf{T}$(PQ)(I 20)

= tri. EZD: tri. PQR.

Therefore tri. PQR.= par. ALMQ.

This proposition is often explained as a transformation of the symptom of the parabola. Indeed if we interpret the symptom (I 11) and this proposition algebraically, they are essentially the same. In this point, van der Waerden states:

> Thus we see that I 42 is nothing but a transformation of the equation of the parabola ... But the formulation is already of such a character, that it contains two diameters, AD and EL, and two tangent lines AL and ED. This prepares for interchanging the roles played by the points A and E.[15]

His remark is completely understandable in the context of modern mathematics, but remains misleading. While it is true that I 42 is derived from the symptom of the parabola, the discovery of this proposition is by no means a result of such a derivation. Its necessity is first due to another reason, and its truth is later reduced to the relation already known, *i.e.* the symptom. To make this point clear, let us look at I 46, and try to reconstruct its analysis (in the sense in Greek mathematics).

> I 46: In the same parabola as in I 42, any chord Pp parallel to the tangent ED is bisected by the line EN; *i.e.* PN = pN.

This proposition is clearly an indispensable part of the theorem of the interchangeability of the diameters. The proof goes as follows:

From I 42,

triangle PQR = parallelogram ALMQ

triangle pqR = parallelogram ALmq.

[14] I have cited van der Waerden's paraphrase of this proposition. *SA*, pp. 252–253.

[15] *SA*, p. 253.

39 | By substraction,

par. pqQP = par. qmMQ.

Take away the common part PQqmN, then

tri. pmN = tri. PMN.

Since the two triangles pmN and PMN are similar,

pN = PN.

From this proof, we can see that I 42 does not precede I 46. It is natural to assume that the effort to discover a proof for I 46 led to the recognition of I 42. The process of the analysis can be easily reconstructed. To prove that EN (one of the parallels to the original diameter AD) is a diameter, one must find a group of parallel chords which are bisected by EN. I 17 and 32 suggest that these chords are parallel to the tangent at the new vertex E. Then, a chord Pp, parallel to ED, is drawn as a candidate for a new ordinate. Next, it must be proved that PN = pN. This simple equality is "reduced" to the equality between the areas of triangles (PNM and pNm). This in turn is transformed into the equality of qmMQ and pqQP. The next step, in which the relation of I 42 is induced, seems hard to discover, although the consideration of a special case in which P coincides with A may give a hint to this transformation.

As a result, we arrive at I 42 through the analysis of I 46. Though this analysis is a mere reconstruction from the existing proof, it is likely that I 42 is first recognized through this analysis. At very least, it would be surprising if I 42 were derived from the symptom first, without any definite purpose, and its application only later found by chance. As a result, we should be wary of overemphasizing its nature as a transformation of the symptom.

The reconstructed analysis of I 46 reveals another character of Apollonius's argument. There, the equality of two lines is transformed to that of two triangles, then the latter equality into that of other figures through geometric process, *i.e.* adding or taking away the same area. This argument is puzzling at first sight, since we tend to assume that the equality between lines is simpler than that between areas. But in the analysis of I 46, the reduction of the former to the latter is a crucial step. This reduction enables Apollonius to transform the desired equality freely through the observation of the diagram (not through alleged algebraic operations). In other words, it is impossible for Apollonius to connect the symptom of the parabola and its tangent (I 11, 33, 35) to I 46 (PN = Np) without the aid of geometric intuition. The two auxiliary methods, the "geometric algebra" and the theory of proportions, are insufficient to solve this problem by themselves. In the *Conics*, many results are written in terms of "geometric algebra" and the theory of proportions. The theorems on the interchangeability of the diameters are also written in these terms, as we shall see later (propositions I 49–51). But these are not always convenient to the investigation of new results, for though Apollonius has some means of treating the relations in these terms, geometric intuition is indispensable for his purpose. As a result, he is required to transform the symptom (I 11) into the equality between "visible" areas (though the symptom is already ex|pressed as an equality between square and rectangle, these 40 figures have been added afterwards and have no geometric relations with the conic section and the tangent lines, as noted by van der Waerden), in order to arrive at the desired result (I 46). Most of his argument is indeed algebraic in the sense that it can be translated into modern algebraic operation, but crucial steps of his argument depend on geometric intuition, and some of the operations are used to transform the relations already known or desired into a form in which one can rely on geometric intuition.

Neither Zeuthen nor van der Waerden is willing to admit this point, and they fail to pay due attention to I 46. Instead, they analyze I 42, explaining it as a derivation of the symptom through "algebraic" transformation, and thus emphasizing its character as another form of the symptom.[16] According to these scholars, it would appear that I 46 is merely a by-product of the algebraic operation. But as I noted earlier, it is not I 46 but I 42 that is a by-product of Apollonius's investigation. It strongly appears that scholars like Zeuthen were so convinced of the algebraic character of the *Conics* that they tended to overlook the significance of arguments which did not conform to the algebraic interpretation. Thus far I have pointed out one of the characteristics of the *Conics*, the intention of visualization. In the following argument it will become clearer that many other propositions in the *Conics* also confirm this characteristic.

Let us proceed to I 49 (Fig. 7). As PN = Np has been proved in I 46, it remains to prove that for some line l',

$$\mathbf{O}(l', \mathrm{EN}) = \mathbf{T}(\mathrm{pN}) \quad \cdots\cdots\cdots\cdots (※)$$

The proof is simple:

tri. SDA = tri. SLE, since AD = AZ = LE (I 33, 35)

Add to each figure ESAqm. Then,

EDqm = ALmq = tri. pqR (I 42)

Take away the common part NRqm, then

tri. pmN = DENR.

This is a "visualized" form of the desired symptom (※), just as I 42 is a visualized form of I 11, and the proof has been substantially accomplished; it remains merely to express the new *latus rectum* l' in terms of proportionality. The result is

SE : EL = l' : 2ED.

Here again, geometric transformation is the key of the proof. The process of expressing l', which may be called algebraic, does not involve an important step. It is a finishing touch to the result already found and confirmed by geometric intuition, and its aim is to bring the result into conformation with the standard style of statement as a proposition.

41 Let us examine the case of the hyperbola and the ellipse. I 43, 47 and 50 correspond respectively to I 42, 46 and 49 in the case of the parabola.

Neugebauer correctly pointed out that these propositions and their proofs of the case of the hyperbola were first found by analogy to those of the parabola.[17] Let us

16 *Die Lehre*, pp. 102–103. *SA*, pp. 252–256.

17 O. Neugebauer, "Apollonius Studien", *Quellen und Studien*, Bd. 2, pp. 215–254. Though Neugebauer refers only to the interrelation of I 42 and I 43, I extend this analogy to I 46 and I 47, since, as I have argued, I 42 should be regarded as a lemma for I 46.

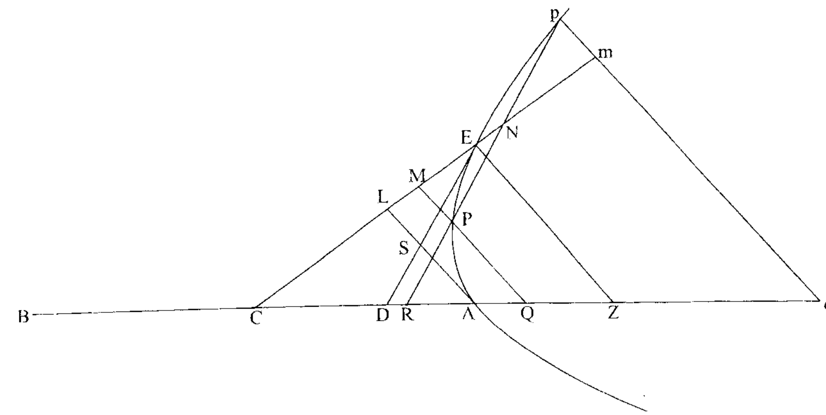

Fig. 8

recapitulate his argument. Fig. 7 is the diagram for I 42, 46 and 49, and Fig. 8 for I 43, 47 and 50.[18] I 43 states that

tri. PQR = ALMQ.

I 47, PN = Np.

Since these results correspond precisely to those in I 42 and 46, it is natural to assume that not only the result but also the proof for the theorem of interchangeability of diameters of the hyperbola has been modelled after that of the parabola. Indeed, one can see that the proof of I 47 corresponds literally to that in I 46.

Thus it is indisputable that Apollonius utilizes such analogies based on the observation of the diagrams. This fact is another support for Apollonius's dependence on geometric intuition. It also suggests that I 43, which is often interpreted as "symptom referred to the center", is nothing but a lemma for I 47. This proposition I 43 is an end result of the analysis of I 47, so that its truth is first expected as a sufficient condition for I 47. It can no more be justified to regard I 43 as if it were investigated for itself, calling it a "symptom referred to the center", than to regard I 42 as a mere transformation of the symptom of the parabola.

Let us examine Apollonius's argument in detail. As noted above, the proof of I 47 is completely conformable to that of I 46. On the other hand, the case for I 43 is far more complicated.[19] According to the extant text, this proposition is proved through two lemmata, namely, I 39 and 41. The former is derived from the second result of 42

18 Note that in the case of the hyperbola and ellipse, the candidate for new diameter is the line which passes through the center of the curve.

19 Neugebauer gives an algebraic reconstruction for the proof of I 43, but it becomes more complicated than the original text. Moreover, as he admits, the reconstruction is not faithful to the text, and I find it unconvincing.

I 37 (which I have called I 37 (b)), and the latter is based on the symptom. But several parts of these propositions overlap with Pappus's lemmata for the *Conics*, leading to the conclusion that the original Apollonius's text probably lacked these parts, and the text which has come down to us is the result of an inclusion of Pappus's lemmata.[20] This makes it difficult to reconstruct Apollonius's line of thought. Moreover, Eutocius's commentary to the *Conics* complicates the situation even more, for he cites another proof of I 43, which he claims he found in other manuscripts.[21] This proof conforms better to that of I 42, the parallel proposition for the parabola.

Rather than seeking the original form of the proof of I 43, I would like instead to point out some crucial steps which must have been indispensable, whatever Apollonius's proof may have been.

In I 42, which is indisputably the model for I 43, the ratio ALMQ: ALEZ can be at once reduced to the ratio AQ: AZ, since LM is parallel to AQ. In I 43, where LM and AQ are not parallel, but intersect at C, one must find some other means to treat the area of ALMQ. The analysis based on the proof which Eutocius transmits us is as follows (Fig. 8): the expected equality

ALMQ = tri. PQR

is reduced to

ALEZ = tri. EZD,

because tri. PQR: tri. EZD
= **T**(PQ): **T**(EZ)
= **O**(AQ, QB): **O**(AZ, ZB) (I 21)
= **T**(CQ) − **T**(CA): **T**(CZ) − **T**(CA) (*Elem.* II 6)
= tri. CMQ − tri. CLA: tri. CEZ − tri. CLA
= ALMQ: ALEZ.

And the reduced equality (ALEZ = tri. EZD) is transformed into the form

tri. ESL = tri. ASD

then, into

tri. CED = tri. CAL,

which can be easily proved by

I 37 (a): **T**(CA) = **O**(DC, CZ).

It is clear thet *Elem.* II 6 plays a crucial role in connecting the expected equality to the symptom of the hyperbola. And in the proof which we see in the text of the *Conics*, the same proposition in the *Elem.* II is used in a similar way.

The examination of I 43 and 47 confirms my view that it is geometric intuition that plays a central role in the *Conics*. But this is not all. I have already argued that the

[20] See J. L. Heiberg ed. *Apollonii Pergaei etc.*, Vol. 2, p. LIX.
[21] *Ibid.*, Vol. 2, pp. 254–264.

process of visualization is a necessary element in the reliance on geometric intuition. The proof of I 43, examined above, suggests that the propositions in *Elem.* II play an important role in this process. Let me illustrate this point. The symptom of the hyperbola:

| **T**(PQ): **O**(AQ, QB) = *latus rectum* : *latus transversum* (Fig. 8) 43

is expressed in terms of "invisible" figures such as **T**(PQ) and **O**(AQ, QB). By *Elem.* II 6, **O**(AQ, QB) can be replaced by **T**(CQ) − **T**(CA), which in turn can be transformed into the difference of two similar triangles, *i.e.* a trapezium. Thus, by aid of the theory of proportions, *Elem.* II 6 makes an invisible rectangle **O**(AQ, QB) "visible". From this point of view, the significance of the two main auxiliary methods mentioned by Zeuthen is completely different. They are not methods of treating the lines and areas as general quantities in a way similar to modern algebra, but they are the means for transformation between "visible" and "invisible" forms of areas. The former is indispensable because it makes geometric intuition available, while the latter is adapted to the formal statement of results as propositions and is used in the expression of the symptoms.

Proposition I 43 of the *Conics* provides another important suggestion regarding the character of *Elem.* II. The equality

O(AQ, QB) = **T**(CQ) − **T**(CA) (Fig. 8)

is guaranteed by *Elem.* II 6 in the case of the hyperbola. On the other hand, in the case of the ellipse, the equality

O(AQ, QB) = **T**(CA) − **T**(CQ) (Fig. 9)

holds, because the point Q falls between A and B, and this equality is based on *Elem.* II 5. Here, we encounter a situation in which both of II 5 and 6 are naturally required. This is a notable result, for the significance of these two propositions has been polemical, this because the one seems to make the other unnecessary. This point will be discussed in the next chapter, where I argue that such alternate (mutually complementary) uses of propositions (according to the arrangement of the points)

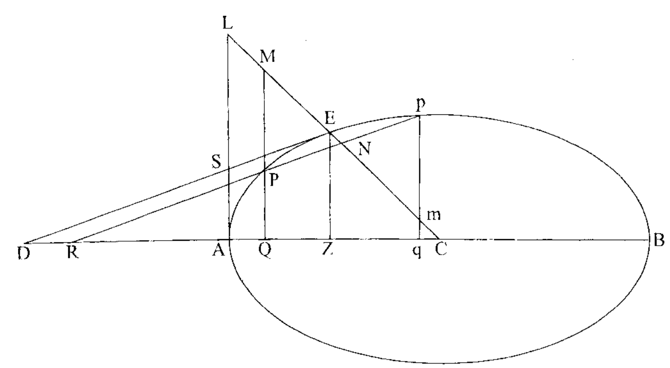

Fig. 9

have already been taken into account in the compilation of the *Elements*. For the moment, I confine myself to add that the alternate use of *Elem.* II 5. 6 is seen very often in the *Conics*.

44 As noted ealier, Proposition III 17 of the *Conics* is one of those used in the | solution of the four line locus, and the only one not involving the opposite sections. First, an examination of its preliminaries, namely, III 1–3.[22]

45 III 1 (Fig. 10, 11): P, Q being any two points on a conic, if the tangent at P and the diameter through Q meet in E, and the tangent at Q and the diameter | through P in T, and if the tangents intersect at O, then

tri. OPT = tri. OQE

III 2: (Keep the notation of III 1) If R be any other point on the conic, le tRU be drawn parallel to QT to meet the diameter through P in U, and let a parallel through R to the tangent at P meet QT and the diameters through Q, P in H, F, W respectively. Then

tri. HQF = HTUR

III 3: (Keep the notation of III 1,2) Take two points R′, R on the curve with points H′, F′, *etc.* corresponding to H, F, *etc.* And if RU, R′W′ intersect in I, and R′U′, RW in J, then

F′IRF = IUU′R′

and FJR′F′ = JU′UR.

We note that III 2 is a result stemming directly from I 42, 43 and that III 3 is proved from III 2 through purely geometric operations, *i.e.* adding or removing the equal areas. As for III 3, Zeuthen is likely correct in asserting that the case of the hyperbola and the ellipse can be interpreted as a proposition to the effect that the quadrilateral CFRU is constant. It is probable that Apollonius understands III 3 in this way. But it is absurd and anachronistic to argue, as Zeuthen did, that the proposition represents a symptom of conics referred to the oblique axes,[23] for there is no evidence at all that Apollonius regards the quadrilateral CFRU in that way.

The significance of these preliminary propositions become fully clear in their applications in the power propositions. Let us examine III 17 (Fig. 10, 11):

III 17: If OP, OQ be two tangents to any conic and Rr, R′r′ two chords parallel to them respectively and intersecting in J, then

T(OP): **T**(OQ) = **O**(RJ, Jr): **O**(R′J, Jr′).

The outline of the proof is as follows:

T(JW): tri. JWU′ = **T**(WR): tri. RWU,

then **T**(RW) − **T**(WJ): RJU′U = **T**(WR): tri. RWU

∴ **O**(RJ, Jr): RJU′U = **T**(WR): tri. RWU (*Elem.* II 5, 6)

= **T**(OP): tri. OPT.

22 The paraphrase below is based on Heath, *Apollonius*, pp. 84–87. I have omitted the diagram in the case of the ellipse.

23 *Die Lehre*, p. 98. Van der Waerden follows Zeuthen on this point. See *SA*, p. 256.

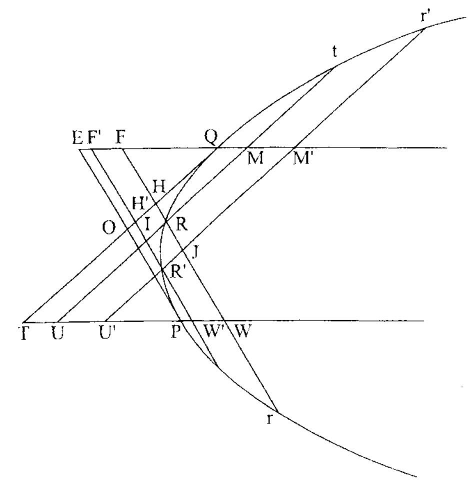

Fig. 10

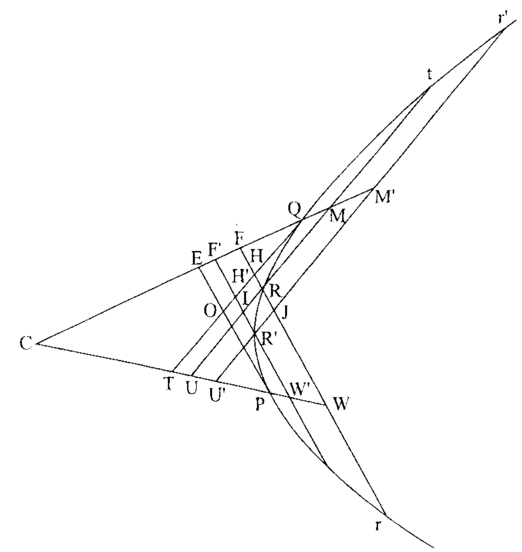

Fig. 11

We have already proved RJU′U = R′JFF′ (III 3)

and tri. OPT = tri. OQE (III 1),

therefore **O**(RJ, Jr): R′JFF′ = **T**(OP): tri. OQE.

Similarly, **O**(R′J, Jr′): R′JFF′ = **T**(OQ): tri. OQE.

The remainder of the proof is simple and straightforward.

In this proof, the intention of visualization is manifest. **O**(RJ, Jr) is a rectangle contained by two line segments which lie in a line, making the rectangle invisible. The case is the same for **T**(OP). These invisible figures are transformed into visible ones: trapezium RJU′U and triangle OPT. This transformation is made possible by *Elem.* II
46 5, 6 and the theory of proportions. It is also noteworthy that both | *Elem.* II 5, 6 are necessary depending on whether the intersection J falls inside or outside of the curve.[24] Here, we have another example of the mutually complementary uses of these two propositions in *Elem.* II.

The above examination of the *Conics* has revealed an important characteristic in Apollonius's investigation, namely the essential dependence on geometric intuition. The alleged algebraic methods, *i.e.* "geometric algebra" and the theory of proportions, which have long been assumed to be the backbone of Apollonius's thought, bears only peripheral importance to the course of the investigations of new results. Their role can be summed up as follows:

(1) To transform known or desired relations into a form suitable for the use of geometric intuition;
(2) To prove the relation already anticipated by geometric intuition;
(3) To transform the results into a form suitable for formal statements as a proposition.

I 42 and the analysis in I 46 provide the example for (1). The proof of I 43 (together with I 37, 39, 41) is an example of (2). As an example of (3), we have seen in I 49, that the relation tri. pmN=DENR (Fig. 7) has been transformed into the form

O(l', EN) = **T**(pN), l' being such a line that SE: EL = l': 2ED

To make the situation clearer I have introduced the words "visible" and "invisible". Apollonius's investigation depends on geometric intuition which is valid only for "visible" figures, *i.e.* lines and areas as they are. But the results of the investigation thus obtained are not suited for statement as a proposition, and need to be transformed into a concise, general form, which is the expression in terms of "geometric algebra" and the theory of proportions. The main feature of this formal expression is the "invisibility" of the figures involved, such as "the square of the ordinate" and "the rectangle contained by two line segments which lie in a line". When the results are used again in later investigations, they are visualized to make the use of geometric intuition possible. This is the process I have summed up above as (3) and (1).

---

24 Apollonius does not explicitly refer to the case in which the point J falls outside of the curve. In III 16, however, the limiting case of a chord coinciding with the tangent, J does fall outside the curve, and *Elem.* II 6 is required.

In the next chapter I will further examine the mutually complementary uses of *Elem.* II, showing that the object of the treatment in *Elem.* II was never "general quantities", but the lengths of lines conceived together with their positions and arrangements.

CHAPTER 2. ON THE INTERPRETATION OF THE *ELEMENTS*, BOOK II

The first ten propositions of the *Elements* II have been ususally interpreted as geometric expressions of algebraic theorems. As mentioned above, this interpretation began with Zeuthen, who has called the book "geometric algebra". Follow|ing Mueller 47
I will refer to this explanation as the "algebraic interpretation."[25]

The core of the algebraic interpretation consists in the identification of the Euclidean propositions with algebraic equalities. But this simple identification involves difficulties, since two propositions may correspond to the same algebraic equality. For example, both II 5 and II 6 can be expressed by the equality:

$$(a + b)(a - b) = a^2 - b^2$$

and some other pairs of propositions which can likewise be represented by a single algebraic equality.[26]

II 5 and 6 are interpreted as solutions of the following sets of equations.

$$\begin{cases} x + y = p \\ xy = q \end{cases} \quad \text{(II 5)} \quad (1) \qquad \begin{cases} x - y = p \\ xy = q \end{cases} \quad \text{(II 6)} \quad (2)$$

It is well known that propositions of the *Data* (84, 85) support this explanation, and the application of areas (*Elem.* VI 28, 29), which is attributed to the Pythagoreans, can also be regarded as an extension of these problems.

This interpretation was first raised by Tannery,[27] who accepted the algebraic character of the propositions though he was unable to furnish evidence for the existence of the algebraic equations (1) (2). Half a century later, the discovery of Babylonian mathematical texts seemed to confirm this view, for these texts were thought to contain numerical solutions of these equations. Thus the "geometric algebra" was regarded as a geometrized version of Babylonian mathematics, and the algebraic interpretation became virtually an established assumption.[28]

This interpretation, however, remains controversial. It seems clear that II 5, 6 (as well as other propositions) are solutions to some other problems, since they make little sense when considered alone. But it remains questionable whether the problems involved are algebraic. The *Elements* does include cases in which II 5, 6 are utilized.

---

25 Mueller, *op. cit.* (see note 1), p. 42.

26 II 9 and 10 are another pair. The existence of these pairs are known as "double form" in Book II. I will refer to the propositions making up a pair "twin-propositions".

27 P. Tannery, "De la solution géométrique des problèmes du second degré; avant Euclide" (1882), in his *Mémoires scientifiques*, J. L. Heiberg and H. G. Zeuthen eds. (Toulouse and Paris, 1912), pp. 254–280.

28 I assume readers' knowledge of the course of events surrounding the interpretation of the "geometric algebra" after Zeuthen and Tannery. See *SA*, pp. 118–124.

For example, II 14 and II 11 use II 5, 6 respectively. On the other hand, although the algebraic interpretation assumes algebraic problems underlying the *Elem.* II, there is no direct evidence to confirm this assumption. Mueller has vigorously investigated this point. He has summarized the characteristics of the algebraic interpretation quoting Zeuthen and van der Waerden as follows:

1. The lines and areas of geometric algebra represent arbitrary quantities;
2. Geometric algebra is a translation of Babylonian algebraic methods;
3. The "line of thought" in much of Greek mathematics is "at bottom purely algebraic".[29]

48 | Mueller admits that the truth of any one of these would be sufficient to establish the algebraic interpretation excluding the geometrical one (that is, the interpretation of the propositions in book II as lemmata for geometric propositions). He examines these claims carefully and concludes that there is no conclusive evidence to establish any one of them. Mueller then asks whether geometric interpretation is possible. Attempting to find the propositions in the *Elements* where II 1–10 are used, he concludes after a minute and thoroughgoing examination, that II 5 and 6 can be interpreted as lemmata for II 14 and II 11 respectively.[30] According to Mueller, II 14 is a reworking of the proof of VI 13, and he argues convincingly that II 14 is a result of the effort to avoid the use of the theory of proportions in VI 13; the necessity of II 5 arises in this process of reworking.[31] For II 11, which is parallel to VI 30, Mueller provides a similar argument[32]. He reproduces the process in which the effort to construct the regular pentagon without the use of the theory of proportions led to the recognition of II 11 and II 6. Though Tannery has already pointed out the repetitions of the same propositions in the *Elements* II and VI (in the former in terms of "geometric algebra", in the latter in the language of the theory of proportions),[33] Mueller views this fact from completely different standpoint and proceeds to develop a successful argument.

But Mueller does not succeed in explaining all the propositions at issue, *i.e.*, II 1–10. He confesses that some propositions (II 1, 3, 8–10) are never or only implicitly used in the *Elements*, or only in places of questionable authenticity.[34] The existence of unexplainable propositions is more damaging to Mueller's than to the algebraic interpretation, for, according to the latter, these propositions can be viewed as examples to illustrate the method of the "geometric algebra".

In the following argument, I fundamentally follow Mueller in investigating the possiblity of the geometric interpretation.

My points are as follows.

1. For each of the propositions II 1–10, there exists evidence or, at least, probability that Euclid used them in other propositions, so that the difficulty Mueller has encountered disappears.

---

29 Mueller, *op. cit.*, p. 50.
30 Note that this claim has already been made by Arpád Szabó, in his *Anfänge der griechischen Mathematik* (1969).
31 Mueller, *op. cit.*, pp. 161 162.
32 *Ibid.*, pp. 168 170, 192–194.
33 Tannery, *op. cit.*, p. 274.
34 Mueller, *op. cit.*, pp. 300–302.

2. The double form in II 5, 6, II 9, 10 and II 4, 7 can be explained by the geometric interpretation. Their role is that of mutual complement in the geometric context, *i.e.*, they are used alternately depending upon the arrangement of points and lines.
3. In relation to 2 above, I claim that the object of the argument in the second book of the *Elements* is not quantities in general. The areas and length of lines are always considered and treated together with their positions.
4. *Elem.* II is meant to prove the equalities between areas of "invisible" figures, by reducing them to "visible" ones, and to afford a set of propositions for the treatment of "invisible" figures. 49

In the previous chapter, I have shown some examples of the mutually complementary use of *Elem.* II 5, 6. These examples are found in the *Conics* I 43 (more precisely, in its preliminary lemmata) and III 17, and both propositions originate in Euclid's *Conic Elements*. So it is probable that Euclid himself is responsible for the use of *Elem.* II 5, 6 in this way. Moreover, this mutually complementary use of II 5, 6 can be found in the *Elements* itself. In III 35, 36, which is a special case (for the circle) of the *Conics* III 17, *Elem.* II 5, 6 are used exactly in the same way as in the *Conics*.

We can find similar examples for other twin-propositions in the *Elem.* II. As is well known, II 4 and II 7 are used in II 13 and II 12 respectively. The distinction of the cases in these two propositions is completely parallel to that of III 35, 36. For II 9, 10, although we cannot find their application in the *Elements*, the *Conics* III 27, 28 offer examples of the mutually complementary use of these propositions. In III 27, for example, one of *Elem.* II 9 and 10 is used according to whether the intersection O of the two chords Rr, R′r′ falls inside or outside the ellipse (Fig. 12), and the equality

$$\mathbf{T}(RO)+\mathbf{T}(Or)=2\mathbf{T}(Rw)+2\mathbf{T}(wO)$$

is proved in both cases. For the same *Elem.* II 9, 10, Pappus provides us with a simpler example. This is the proof of the theorem concerning the median line of a

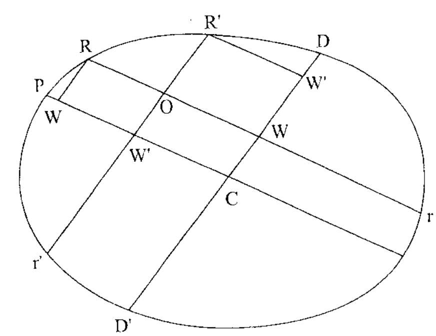

Fig. 12

triangle.[35] Though there is no evidence that Euclid did prove these propositions by *Elem.* II 9, 10, it is quite likely that Euclid knew some propositions for which II 9 and 10 are utilized in a mutually complementary way, and that Euclid inserted this pair of
50 propositions in the *Elements* for that reason. At very least, it is certain that | Euclid used the pairs II 5, 6 and II 4, 7 in this way, and he inserted another pair II 9, 10 in the same book of the *Elements*. If so, it would be rather unnatural to assume that Euclid could not find any case in which II 9, 10 are required in the same way as other pairs.

The case of *Elem.* II 2, 3 will make my argument more convincing. At first sight, these propositions appear to be nothing more than trivial special cases of II 1. Although these propositions have sometimes been explained as illustrations of the method of the geometric algebra (that is, some pedagogical intention is attributed to Euclid), it seems more reasonable to assume, with Heath, that their frequent necessity led Euclid to state them separately for sake of convenience.[36] But what Heath fails to mention is that II 2, 3 are a pair of twins the same as II 5, 6 *etc.* Pappus gives us an adequate example.[37] In one of the lemmata to the *Conics*, he proves

O(ZD, DE) = **O**(AD, DB) (I 37 (b), Fig. 5, 6)

from **T**(ZA) = **O**(DZ, ZE) (I 37 (a), Fig. 5, 6)

The proof is as follows.

For ellipse: take away **T**(ZD) from both sides of I 37 (a)

then, **O**(AD, DB) = **O**(ZD, DE) (*Elem.* II 5 and 3 are used)

For hyperbola: take away the both sides of I 37 (a) from **T**(ZD)

then, **O**(AD, DB) = **O**(ED, DZ) (*Elem.* II 6 and 2)

It is clear that *Elem.* II 2, 3 play mutually complementary roles in this proof. Of course, it was Pappus, who thus used these propositions, and there is no evidence that Apollonius proved *Conics* I 37 in this way. We can argue still less about Euclid since we have no certainty that Euclid's *Conic Elements* was even so constructed as to include the same proposition. But Pappus's lemma shows at least a possibility of using II 2, 3 in a way parallel to II 5, 6 *etc.* It is natural to assume that Euclid was aware of this possibility and so decided to insert II 2 and 3, even though the latter II 3 is not utilized in other parts of the *Elements*.

This leaves two propositions (propositions 1 & 8) unexplained. Before proceeding to these, the general significance of the mutually complementary use of twin-propositions merits discussion. The double form would appear to furnish evidence that Euclid did not treat the length of lines as general quantities. This fact can be illustrated by reference to II 5, 6. In both of these propositions the rectangle contained by two line segments is at issue. If Euclid had regarded these line segments as representations of general quantities, thus neglecting their arrangement as insignificant, he would not have distinguished the two cases, for, in both propositions, the contents would have been the same.

[35] Pappus, *La collection mathématique*, tr. by P. Ver Eecke (Paris, 1933, 1983), p. 662. T. L. Heath, *The Thirteen Books of the Elements*, Vol. 1, p. 401. (hereafter Heath, *Elements*)

[36] Heath, *Elements*, Vol. 1, p. 377–78.

[37] Pappus, *op. cit.*, pp. 723–24. In the following citation, I have put Pappus's proof into the context of the proposition in the *Conics*.

As Heath states, "The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference."[38] As a result, Euclid's motive in distinguishing the two cases must | have been the diversity of the arrangement of the 51
two line segments. In other words, he did not regard them as mere representations of quantities which can be placed arbitrarily, but as geometric existences the position and arrangement of which should also be considered. Euclid could not abstract the general quantity from line segments, or at least it was not convenient for him to perform such an abstraction. This leads to the conclusion that the first claim of the algebraic interpretation pointed out by Mueller cannot be supported.

Now for the two remaining propositions, *i.e.*, II 1 and II 8. We can find the application of II 8 in *Data* 86 which was cited by Tannery as an example of the solution of the equation by Euclid. Its content is as follows (Fig. 13):

Data proposition 86.[39] If two straight lines contain a given area in a given angle and the square of the one is greater than the square of the other by a given area as in ratio, each of those lines will be given.

Let the two straight lines AB and BG contain the given area AG in a given angle ABG and let **T**(GB) be greater than **T**(BA) by a given area as in ratio [*i.e.* the excess of **T**(GB) over a given area has a given ratio to **T**(BA)]. I say that each of AB and BG is also given.

Since **T**(GB) is greater than **T**(BA) by a given area as in ratio, let the given area **O**(GB, BD) be taken away. Then the ratio of the remainder **O**(DG, GB): **T**(AB) is given.

Since **O**(AB, BG) is given and **O**(GB, BD) is also given,
the ratio **O**(AB, BG): **O**(GB, BD) is given.

But **O**(AB, BG) : **O**(GB, BD) = AB : BD
Hence the ratio AB : BD is given.
Hence the ratio **T**(AB) : **T**(BD) is given.

But **T**(AB) : **O**(BG, GD) is given.
Therefore the ratio **O**(BG, GD) : **T**(DB) is also given.

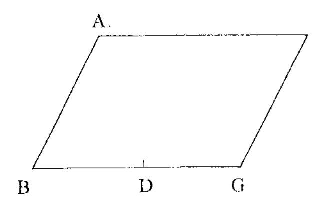

Fig. 13

[38] Heath, *Elements*, Vol. 1, p. 383.

[39] The translation of this proposition is based on S. Ito, *The Medieval Latin Translation of the Data of Euclid* (Tokyo, 1980). I have changed the notations in this translation, and corrected small deviations in the Latin translation according to the Greek text.

Hence the ratio 4**O**(BG, GD): **T**(BD) is also given.
Therefore the ratio 4**O**(BG, GD)+**T**(BD): **T**(BD) is given.

But 4**O**(BG, GD) + **T**(BD) = **T**(BG + GD) (*Elem.* II 8)
Therefore the ratio **T**(BG + GD) : **T**(BD) is also given.
52 | Hence the ratio (BG + GD) : BD is also given.

And *componendo*, the ratio 2GB : BD is given.
Hence the ratio GB : BD is given.

But GB : BD = **O**(GB, BD) : **T**(BD)
Therefore the ratio **O**(BG, BD) : **T**(BD) is given.

But **O**(GB, BD) is given.
Therefore **T**(BD) is also given. Therefore BD is given.
Hence BG is also given, for the ratio GB : BD is given and BD is given. And AG is given and the angle B is given. Therefore AB is also given. Therefore each of AB and BG is given.

Tannery interprets this proposition as the equation:

$$xy = A, \quad x^2 = my^2 + B$$

and explains its complicated solution, as the result of the avoidance of the introduction of the biquadratic term, $x^2y^2 = A^2$.[40]

But Tannery does not explain the reason why such a complicated equation appears in *Data*. *Data* contains three propositions (84–86) which can be construed as the solution of quadratic simultaneous equations. Propositions 84 and 85 are, as we have
53 noted, problem-versions of *Elem.* II 5, 6, respectively. Compared with | the simplicity of these two, the complexity of prop. 86 is striking. Assuming Euclid was developing techniques of "equations" in these propositions, the difference between 85 and 86 would represent a tremendous leap.

I believe that prop. 86 of *Data* was originally a problem concerning the intersection of two hyperbolas as follows (Fig. 14):

> Let there be given a hyperbola EP with the diameter BG and the secondary diameter BA. Let there be given another hyperbola PQ with asymptotes BA and BG. Then, the point of the intersection P is also given.

It is evident that *Data* 86 is mathematically identical to the above problem. Let the ordinate PG be drawn. As P is on the hyperbola PE, the ratio **T**(PG) : **O**(EG, GE′) is given.

As PG = AB and **O**(EG, GE′) = **T**(BG) − **T**(BE) (*Elem.* II 6)
the ratio **T**(BG) − **T**(BE) : **T**(AB) is given.

As **T**(BE) is given since the hyperbola PE is given, **T**(BG) is greater than **T**(AB) by the given area **T**(BE) as in ratio. On the other hand, as P is on the hyperbola PQ, **O**(AB, BG) is given.

Therefore it seems certain that *Data* 86 has its origin in the problem of finding the intersection of two hyperbolas. To assume this mode of interpretation is much more

[40] Tannery, *op. cit.*, pp. 262–63. See also *SA*, pp. 198–99.

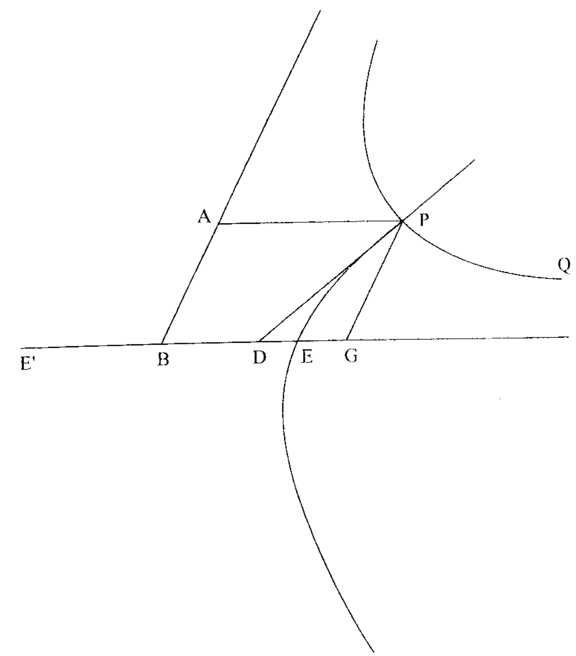

Fig. 14

natural than to regard the proposition as a simulataneous equation of obscure origin. There exists further confirmation for this interpretation. The propositions concerning the hyperbola gives hints to the solution of *Data* 86. The first step of the solution is to take away from **T**(BG) the given area by which it is greater than **T**(BA) as in ratio, and **O**(GB, BD) is made equal to this given area. In the hyperbola, the point D is the intersection of the tangent at P with the diameter because by *Conics* I 37 (a),

O(BG, BD) = **T**(BE)

and in the context of my interpretation, **T**(BE) is nothing but the given area by which **T**(BG) is greater than **T**(AE) as in ratio. In the next step, in *Data* 86, it is stated that the ratio of the remainder **O**(DG, GB) to **T**(AB) is given. According to my interpretation, this is nothing but

Conics I 37 (b): **T**(PG): **O**(DG, GB) = *latus rectum: latus transversum.*

This impressive coincidence is strong support for my interpretation. It is also noteworthy that this solution does not involve the use of any conic curve. Therefore *Data* 86 shows that a particular solid problem can be reduced to a plane problem. The remarkable nature of this result led Euclid to include it in the *Data*.

We have seen that the allegedly algebraic problem in the *Data* the solution of which requires *Elem.* II 8, is in fact a geometric problem. Further, it is quite likely that

Euclid's intention in inserting II 8 in the *Elements* was as part of the solution for geometric problems such as *Data* 86.

This leaves II 1. It seems most profitable to discuss this proposition from a different point of view, examining the proposition itself, since it appears that here is revealed Euclid's intention to elaborate those propositions which are now considered examples of the geometric algebra.

54 | *Elements* II 1 If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments.

> Let A, BC be two straight lines, and let BC be cut at random at the points D, E (Fig. 15);
>
> I say that the rectangle contained by A, BC is equal to the rectangle contained by A, BD, that contained by A, DE and that contained by A, EC.
>
> For let BF be drawn from B at right angles to BC;
> let BG be made equal to A,
> through G let GH be drawn parallel to BC,
> and through D, E, C let DK, EL, CH be drawn parallel to BG.
> Then BH is equal to BK, DL, EH.
>
> Now BH is the rectangle A, BC, for it is contained by GB, BC, and BG is equal to A;
> BK is the rectangle A, BD, for it is contained by GB, BD, and BG is equal to A;
> and DL is the rectangle A, DE, for DK, that is BG, is equal to A.
> Similarly also EH is the rectangle A, EC.
>
> Therefore the rectangle A, BC is equal to the rectangle A, BD, the rectangle A, DE and the rectangle A, EC. Therefore *etc.*[41]

This proposition appears quite strange, and one is puzzled about precisely what it is making claims. It seems to be a tautology, though it must have been a "proof" of

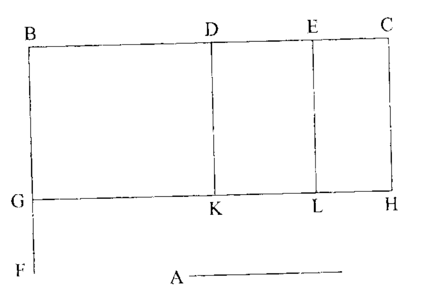

Fig. 15

41 Heath, *Elements*, Vol. 1, p. 375.

something unknown on the basis of something known or admitted.[42] If we summarize Euclid's proof, he seems to admit

$$BCHG = BDKG + DELK + ECHL \quad \ldots\ldots\ldots\ldots\ldots (3)$$

and from this equality to prove

$$\mathbf{O}(A, BC) = \mathbf{O}(A, BD) + \mathbf{O}(A, DE) + \mathbf{O}(A, EC) \quad \ldots\ldots (4)$$

(4) must have been to Euclid less evident than (3). But what is the difference between
| (3) and (4)? Here the notion of "visible" and "invisible" figures seems to be useful. 5
(3) is a equality between "visible" figures and (4) is that between "invisible" ones. What Euclid has done is to reduce the equality between "invisible" figures to that of "visible" ones. The latter equality is evident by geometric intuition and it can be safely conjectured that Euclid thought it a sound basis for the proof of the former. As a result, I claim that II 1 is by no means trivial, for it extends the "visible" equality to the "invisible".

The further examination of some propositions in the *Elements* supports this view. In the proof of I 47 (the Pythagorean theorem), II 1 (more precisely, II 2) seems to be used, for Euclid states that (Fig. 16)

$$BL + CL = BDEC \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Does Euclid hereby commit *petitio principii?*[43] If we distinguish the two levels of figures, visible and invisible, we will see at once that Euclid's proof is correct. (5) is

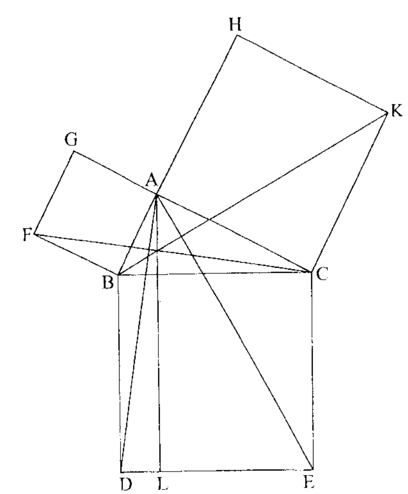

Fig. 16

42 According to the algebraic interpretation, II 1 is a statement of the distributional law.
43 W. R. Knorr claims it to be the evidence of diversity of the sources of Book I and II. See his *The Evolution of the Euclidean Elements* (Dordrecht, 1975), p. 179.

an equality between "visible" areas and equivalent to (3), the basis of the proof of II 1. As a result, there is no inconsistency in its use in I 47.

The case of XIII 10, the sole explicit example of the use of II 1–3 in the *Elements*, is in clear contrast to I 47, though the same equality appears in the proof. Euclid uses (Fig. 17)

$$\mathbf{O}(AB, BN) + \mathbf{O}(BA, AN) = \mathbf{T}(AB) \quad \ldots\ldots\ldots\ldots\ldots (6)$$

This is the same as (5) if one draws a square on AB. But Euclid does not bother to draw it. Why? Because he depends on II 2. Thanks to this proposition, he is freed from
56 the necessity of proving (6) by reducing the "invisible" figures such as | **O**(AB, BN) to "visible" ones.

Now the significance of II 2 in this context is clear. It is an equality between "invisible" figures, and it simplifies the arguments on "invisible" figures by making it unnecessary to reduce the relations to those between "visible" figures. This interpretation is valid also for other propositions in *Elem.* II.

The distinction between "visible" and "invisible" figures which I have made in the examination of the propositions in the *Conics* has turned out to be useful in the interpretation of the second book of the *Elements*. Euclid has two different classes of geometric objects in his study, visible and invisible figures. And the aim of *Elem.* II seems to afford some typical equalities between invisible figures. A criticism might be raised that the invisible figures and their sides are substantially the same as quantities in general, and thus that my interpretation is at bottom algebraic. But it is a mistake to view the invisible figures in this way. They retain their geometric properties, since depending on the arrangement of the figures, one of the pair of twin-propositions is necessary.

Elem. II contains the propositions concerning the "invisible" figures for the solution of geometric problems, and these propositions are usually stated in pairs, the two propositions being used in mutually complementary way to solve a problem. And

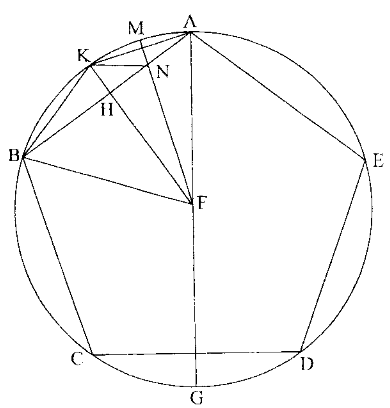

Fig. 17

this mutually complementary use of a pair of propositions is evidence that Euclid did not regard geometric magnitudes (areas and length of lines) as general quantities.

A word is on order regarding Zeuthen's remarks on twin-propositions. Pointing out that *Elem.* II 14 (he interprets this proposition as a problem of finding x for given a, b, such that $x^2 = ab$) could have been proved by II 6, as well as II 5, he states:

> Ob man den einen oder den anderen zu benutzen hat, beruht darauf, ob man—beim Beweise oder Herleitung, denn die Konstruktion ist dieselbe—damit be|ginnt eine der Strecken a und b entweder auf der anderen oder der Verlängerung der anderen abzutragen.[44] 5

This remark clarifies Zeuthen's views. His remark is, in a sense, correct. But his opinion fully relies on the assumption that II 14 is an algebraic problem concerning abstract general quantities, and that the lines which are the object of II 5, 6, 14 have been introduced afterwards to represent those quantities, so that their position and arrangement have nothing to do with the problem itself. In short, the lines can be "carried away" (abtragen) in Zeuthen's view because they were mere representations for quantities. But as noted earlier, the lines in II 5, 6 *etc.* cannot be "carried away". Evidence for the rectitude of this claim lies in the existence of the double form. Zeuthen's failure to recognize geometric significance in *Elem.* II was the result of his careless identification of the lines with quantities in algebra.

Finally, I discuss the following problem: those who argue for the algebraic interpretation of *Elem.* II tend to claim that propositions II 1–10 are illustration of the method of the geometric algebra by means of which other equalities are to be derived. Though the solution to this issue requires a thorough examination of a wide range of the Greek mathematical texts, it is the position of this paper that these propositions are meant to form a set of propositions necessary in Greek geometry and that no other equality is required. In support of this view, a refutation of Zeuthen's argument should be made.

Apollonius makes small skips in the course of his proofs in the *Conics*. Zeuthen argues that some of these skips involving the geometric algebra should be supplemented not by the propositions in *Elem.* II, but by a procedure illustrated in that book. Zeuthen presents the following example.[45] In the proof of *Conics* III 26, Apollonius assumes that (Fig. 18)

if AB = CD then

O(EC, EB) = **O**(AB, BD) + **O**(ED, EA)

| Zeuthen claims that this equality should be proved by drawing rectangles B′C and A′D, making EB′ and EA′ equal to EB and EA respectively (Fig. 19). Then the desired equality is apparent. On the other hand, Pappus affords a proof based on *Elem.* II 5, 6 for this equality. He takes the point Z in the midst of BC (Fig. 18). It is also the middle point of AD. By *Elem.* II, the following equalities hold: 5

$$\mathbf{O}(EC, EB) = \mathbf{T}(EZ) - \mathbf{T}(ZB) \quad (II, 6)$$
$$\mathbf{O}(AB, BD) = \mathbf{T}(AZ) - \mathbf{T}(ZB) \quad (II, 5)$$
$$\mathbf{O}(ED, EA) = \mathbf{T}(EZ) - \mathbf{T}(ZA) \quad (II, 6)$$

[44] *Die Lehre*, p. 15.
[45] *Die Lehre*, pp. 36–38.

Fig. 18

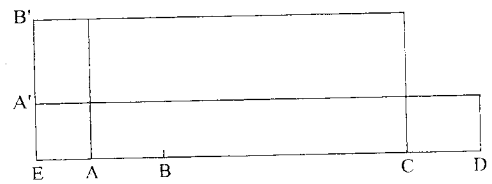

Fig. 19

The rest of the proof is relatively easy. Zeuthen criticizes Pappus's proof as "Pedantrie späterer Zeiten", and he approves Eutocius's commentray on which his own reconstruction is based.

Is Zeuthen right? Should we admit that Pappus did not understand the method of "geometric algebra" and stuck to the propositions themselves? On the contrary, I claim that Pappus's proof is most natural in this context, while Zeuthen's is akin to algebraic operations.

Let us examine the context in which this equality appears. In *Conics* III 26, the line ED is drawn to cut three branches of conjugate hyperbolas. The diameter OX, parallel to ED, is drawn. OY is the conjugate diameter to OX. Then OY bisects BC and AD by the propositions in Book II of the *Conics*. Though the point of intersection of OY and ED is not named in the text of the *Conics*, it must be at once clear for Apollonius that the rectangles such as **O**(EC, CD) can be transformed into the difference of two squares (Pappus later demonstrates) since Apollonius shows great mastery of *Elem.* II 5, 6 in the preceding arguments of the *Conics*. What Pappus needs is to name the midpoint of
59 BC, which has already appeared in the | diagram as the intersection of ED and OY.

For this reason it seems that Pappus's reconstruction is much more in conformation with the line of thought of Apollonius than the reconstruction of Eutocius and Zeuthen. Zeuthen's interpretation may have come from the following algebraic operations. Let EA $= x$, AB $=$ CD $= y$, BC $= z$. Then

$$\begin{aligned} \mathbf{O}(\mathrm{AB}, \mathrm{BD}) + \mathbf{O}(\mathrm{ED}, \mathrm{EA}) &= (y+z)y + (x+2y+z)x \\ &= (x+y+z)y + (x+y+z)x \\ &= (x+y+z)(x+y) \\ &= \mathbf{O}(\mathrm{EC}, \mathrm{EB}) \end{aligned}$$

This was likely the reason Zeuthen decided to approve this interpretation.

Thus Zeuthen's claim that Pappus's lemma contains improper use of the "geometric algebra" cannot be justified. In this example no equality is required except *Elem.* II 1–10. Though I cannot affirm that II 1–10 are sufficient for all the propositions, I have not, at least, seen any example for which II 1–10 are insufficient. Euclid's intention in

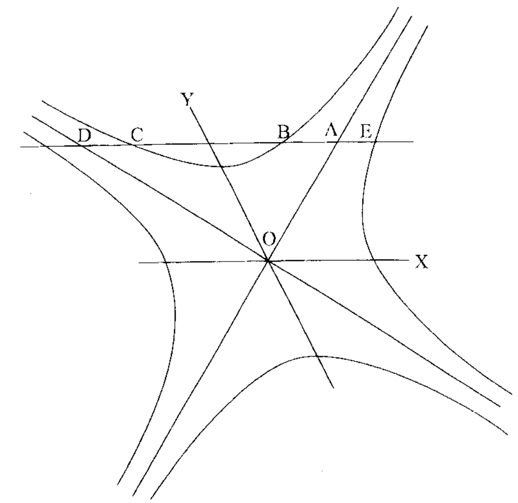

Fig. 20

the compilation of Book II was thus to afford a sufficient set of propositions concerning the "invisible" areas.

CONCLUSION

Book II of the *Elements* can be interpreted geometrically. The propositions II 1–10, those of the so-called geometric algebra, are lemmata for geometric arguments (the solution of geometric problems and the proof of geometric theorems). The necessity and significance of these propositions lies in the distinction of "visible" and "invisible" figures in Euclid. The propositions II 1–10 are those concerning invisible figures, and they must be proved by reducing invisible figures to visible ones, for one can apply to the latter the geometric intuition which is fundamental in Greek geometric arguments. Thus II 1–10 are a collection of the relations between "invisible" areas useful in works at the time of Euclid. We find some examples of their application in the *Elements*, but it was in the theory of the conic sections that these propositions were used repeatedly, and the significance of the distinction between visible and invisible figures was made clear. It is most likely that in the compilation of Book II of the *Elements*, Euclid intended to provide a set of propositions necessary to his *Conic Elements*. In connection with this claim, it is the assertion of this paper that interpretations attributing to Euclid some pedagogical intent in the compilation of *Elem.* II should be rejected. The view that some propositions in *Elem.* II are illustrations of the method of the "geometric algebra" is a contrivance to save the algebraic interpretation and has no positive evidence.

The double form *i.e.* the existence of twin-propositions, can be explained in the context of their application to the geometric arguments. They are used in mutually complementary ways according to the arrangements of points and lines in the problems and theorems to which they are applied. This also suggests that Euclid considered lines and areas not as representations of abstract quantities but as geometric entities, the arrangement of which is significant.

60 | Although I have avoided the argument regarding the origin of the "geometric algebra", I am inclined to think that Mueller's opinion which attributes its origin to the effort to avoid use of the theory of proportions, has much to commend it and I believe that the arguments presented here have given it even more substance by explaining the existence of propositions in *Elem.* II which Mueller left unexplained.

Since the four pairs of twin-propositions are used in mutually complementary ways in geometric context, any interpretation of the sources of these propositions which attributes different origins to the individual constituents of a pair, cannot be justified. Further, as Book II was compiled in order to afford a sufficient basis for geometric arguments involving invisible figures, we cannot assume that all the propositions II 1–10 can be traced back to the Pythagoreans. It might also be possible, for example, that some of the propositions such as II 5, 6, were recognized first, and others added later. At very least it would be meaningless to seek the origin of *each* proposition in some algebraic or arithmetic theory.

ADDITIONAL NOTES 2004

There is probably nobody who would not be tempted to add some after-thoughts if one's article is reprinted after almost twenty years. However, I have considered it better to use that time and energy to produce a new article, and I limit myself here to add some bibliographical notes. Page numbers are those in this volume.

—p. 157, note 35: Pappus's proposition is prop. 122 (Jones, 1986, p. 252; Hultsch, pp. 856-858).

—p. 158, note 37: prop. 170 (Jones, p. 300; Hultsch, p. 926).

—p. 159, note 39. Now [Taisbak 2003] provides English translation and commentary of the whole text of *Data*.

—p. 159ff.: The relation of *Data* 86 to hyperbola was already pointed out by Zeuthen in 1917. See [Taisbak 1996] and [Taisback 2003].

ADDITIONAL BIBLIOGRAPHY

Jones, A., 1986. *Pappus of Alexandria, Book 7 of the Collection.* 2 vols. New-York & Berlin: Springer.
Hultsch, ed. and trans., 1876-1878. *Pappi Alexandrini Collectionis quae supersunt.* 3 vols. Berlin.
Taisbak, Ch. M., 1996. 'Zeuthen and Euclid's Data 86: Algebra – or a Lemma about intersecting Hyperbolas?', *Centaurus* 38: 122-139.
Taisbak, Ch. M., 2003. *Euclid's Data or the Importance of Being Given.* Copenhagen: Museum Tusculanum Press.

G.E.R. LLOYD

THE MENO AND THE MYSTERIES OF MATHEMATICS

The principal object of the method of hypothesis introduced at *Meno* 86e ff is problem reduction. Vlastos,[1] p. 123, put it: 'the logical structure of the recommended method is entirely clear: when you are faced with a problematic proposition *p*, to "investigate it from a hypothesis," you hit on another proposition *h* (the "hypothesis"), such that *p* is true if and only if *h* is true, and then shift your search from *p* to *h*, and investigate the truth of *h*, undertaking to determine what would follow (quite apart from *p*) if *h* were true and, alternatively, if it were false.' That much is clear.[2] But almost everything else is obscure, thanks very largely to the obscurities in the mathematical example. This has generated an enormous secondary literature, and no interpretation can be said to be completely free from difficulty. The object of this paper is to attempt a new approach to that problem. I shall argue that the very obscurity of Plato's mathematical example is one of its *points*.

Let me first set out the text and a moderately straighforward rendering.

86e *λέγω δὲ τὸ ἐξ ὑποθέσεως ὧδε, ὥσπερ οἱ γεωμέ-*
τραι πολλάκις σκοποῦνται, ἐπειδάν τις ἔρηται αὐτούς, οἷον
περὶ χωρίου, εἰ οἷόν τε ἐς τόνδε τὸν κύκλον τόδε τὸ χωρίον
87 *τρίγωνον ἐνταθῆναι, εἴποι ἄν τις ὅτι "οὔπω οἶδα εἰ ἔστιν*
τοῦτο τοιοῦτον, ἀλλ' ὥσπερ μέν τινα ὑπόθεσιν προὔργου
οἶμαι ἔχειν πρὸς τὸ πρᾶγμα τοιάνδε· εἰ μέν ἐστιν τοῦτο τὸ
χωρίον τοιοῦτον οἷον παρὰ τὴν δοθεῖσαν αὐτοῦ γραμμὴν
παρατείναντα ἐλλείπειν τοιούτῳ χωρίῳ οἷον ἂν αὐτὸ τὸ
παρατεταμένον ᾖ, ἄλλο τι συμβαίνειν μοι δοκεῖ, καὶ ἄλλο
αὖ, εἰ ἀδύνατόν ἐστιν ταῦτα παθεῖν. ὑποθέμενος οὖν ἐθέλω
b *εἰπεῖν σοι τὸ συμβαῖνον περὶ τῆς ἐντάσεως αὐτοῦ εἰς τὸν*
κύκλον, εἴτε ἀδύνατον εἴτε μή."

| Knorr's version reads[3]: I say "from hypothesis" in the manner that the geometers often make inquiry, whenever someone has asked them, for instance about an area, whether this area here can be stretched out as a triangle in this circle here, one would say 'I don't yet know whether this is of such a sort, but I think that as a certain hypothesis the following will assist in the matter. If this area is such that the one who has 16

[1] Works cited by author alone are listed in the bibliography at the end of the article which offers a brief selection of the most important recent and earlier studies on the problem.
[2] One may compare already Aristotle's view, since he evidently has the *Meno*'s method of hypothesis in mind in his own account of *ἀπαγωγή* at *A Pr.* 69a20ff, 24ff.
[3] Knorr p. 71.

J. Christianidis (ed.), Classics in the History of Greek Mathematics, 169–183.