Chapter 8
Number and Equations in the Works of Descartes, Newton, and their Contemporaries

Let us complete now the overview on the world of numbers and equations as it emerged at the time of the scientific revolution, by looking at the all-important work of René Descartes (1596–1650) and some of his British contemporaries, including the giant figure of Isaac Newton (1642–1727). Before entering into details about Descartes's ideas, however, it is important to stress that his entire scientific enterprise, including his views about numbers, should be best understood in the framework of a broader discussion of his philosophical system. As a matter of fact, there are not many cases in the history of mathematics where the connection between a philosophical doctrine and the development of scientific ideas is as strong and unmistakable as in the case of Descartes. For the purposes of our account here, suffice it to say that the text whose contents we will examine more closely in order to understand his views on arithmetic and geometry, *La géométrie*, appeared in 1637 as one of three appendixes to one of Descartes' well-known philosophical treatises, *Discours de la methode*. For Descartes, mathematics was in the first place an invaluable tool for educating the mind so that it could be fit for penetrating the secrets of nature and the true grounds of metaphysics. Above all, the mathematical ideas presented in the appendix to his philosophical book were intended as a well-focused and particularly important illustration of the philosophical system discussed in the main text.

8.1 Descartes' New Approach to Numbers and Equations

A convenient way to understand Descartes' original ideas on numbers and equations is by comparison to Viète. First, like Viète, Descartes saw his ideas as part of a general method for "solving any problem" in mathematics. Descartes also took for granted the same widespread assumption about the putative method of "analysis" that the ancient Greeks had maliciously hidden from us, and that should be renovated now. Descartes thought that Viète had already made an important contribution in that direction but there was still need to do more.

Viète, as we saw, devoted himself to developing the old-new analysis from a hands-on mathematical perspective: he started from the algebraic methods known in his time and tried to extend the use of the symbolic methods to all possible quantities, both discrete and continuous. Descartes' starting point, to the contrary, was a philosophical perspective that looked at scientific problems in general and attempted to systematically classify them so as to be able to determine beforehand the appropriate solution for each and every one of them. Algebra was his tool of choice for approaching this task. In general he adhered to the widespread view that algebraic methods such as taught by Viète involved a rediscovery of techniques the ancients had systematically concealed. But in some places he was eager to emphasize the novelty implied by his own methods.

Both Viète and Descartes faced the need to translate geometrical situations into symbolic language, but each of them made this translation in his own way. Viète defined the multiplication of two segments as a rectangle formation (that is: an operation between two magnitudes of the same kind yielding a third magnitude of a different kind). But when it came to equations he always adhered to strict dimensional homogeneity. Descartes, on the contrary defined multiplication of two lengths in way that yield a third length, as we will immediately see. He did as much for the other algebraic operations: dividing one length by another to yield a length and extracting the root of a length to yield a length. In addition he made the first steps in the newly conceived idea of analytic geometry, whereby he established a direct link between geometric figures, such as a straight line and a parabola, and a certain well-defined class of equations representing each type of them.

In retrospect, one can find some of these ideas in the works of Descartes' predecessors (Bombelli defined a multiplication of lengths that yield a length; Fermat came up with some of the basic ideas of analytic geometry), but what appeared in previous works as hesitant or sporadic, as a first step that was not carried through, or perhaps as a passing comment, becomes with Descartes an organically interconnected whole, which is systematically pursued with far-reaching consequences. Descartes' views on the interconnection between algebra and geometry led to a more general and abstract understanding of the idea of an equation under which the traditional requirement of dimensional homogeneity would eventually (but as always, only hesitantly) be abandoned. Descartes himself, for one, did try to preserve homogeneity in many of the equations he considered. But his approach allowed in principle, and in practice also led to, the possibility of ignoring, once and for all, this burdensome requirement that was a legacy of centuries and that hindered the full understanding of what is involved in working with a polynomial equation. Let us see some details of how he worked out these ideas.

Descartes' definition of the multiplication of two segments, *BD* and *BC*, is presented in Figure 8.1. The most important feature of this multiplication is that its outcome is not an area but rather a third segment, *BE*. The procedure is based on defining a certain length, *AB*, which is considered to be a "unit length", namely, a segment whose length is 1. The segment *BE* is constructed by placing the three segments *AB*, *BC*, and *BD*, as indicated in the figure.

In the figure, segment *AC* is first drawn and then *DE* is drawn from *D* and parallel to *AC*. A simple consideration of similarity in the triangles yields the proportion *AB*:*BC* :: *BD*:*BE*. And since the length of *AB* is 1, then we obtain that the length

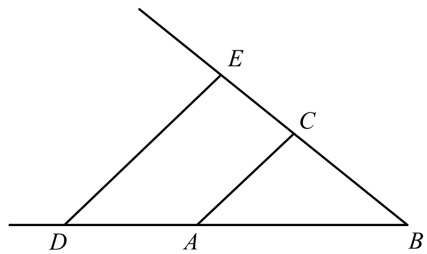

Fig. 8.1 Descartes' multiplication of two segments.

BE equals the product of the lengths *BC* and *BD*, as requested. It is clear how this same diagram can be used to construct the segment *BC* as the division of two given segments *BD* and *BE*, with the help of the same unit length*AB*.

In a different example, shown in Figure 8.2, Descartes showed how to obtain a segment which is the square root of a given segment. Given a segment *GH*, we extend it to reach the point *F*, with *GF* being the unit-length segment. The segment *FH* is bisected at *K*, and we trace the circumference *FH* with center at *K* and radius *KF*. At *G* we raise a perpendicular that cuts the circumference at *I*. A simple theorem about circles, that was known to Greek geometers, states that the square built on *GI* equals the rectangle built on the segments *GF*, *GH*. As the length of *GF* was set to be 1, we get GI^2 =*GH*, or, in other words, *GI* is the square root of the given segment *GH*.

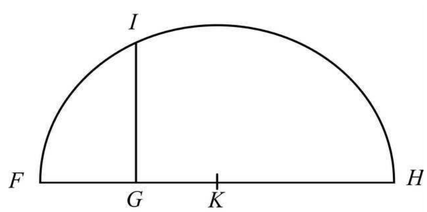

Fig. 8.2 Descartes' extraction of a square root of a given segments.

To the reader of the present book these two examples may seem utterly trivial and as requiring no additional explanation. What have we done here, after all? No new knowledge seems to be involved, other than simple theorems about similarity of triangles and a property of the circle. And indeed, Descartes' *La géométrie* is the first among the texts we have examined up to this point in the book, that a modern-day mathematical reader can approach with concepts, terminology, notation and methods that are essentially known to him. But this is precisely the point I am trying to emphasize here, and that highlights the striking innovations involved in Descartes' work. Descartes' approach to solving problems via geometrical constructions, where magnitudes are considered as numbers without restrictions, and where one freely operates with these magnitudes in the framework of algebraic equations, is absolutely close to our understanding because it has left behind the bulk of the previous, more limited views on numbers. Descartes wrote explicitly that it is possible to find the appropriate construction to solving any geometrical problem by finding the lengths of some segments. His definition of operations with segments was aimed precisely at fulfilling this task.

If Descartes defined the operations with lengths on the basis of theorems that were known to the Greeks, it is evident that not a technical difficulty prevented any of his predecessors to take the step he took in defining operations between segments the way he did. Rather, what was at stake were more fundamental questions of principle about mathematics, about the relationship between geometry and algebra, and about the question of what are numbers and what is their role in mathematics. At the technical level, the key for defining the various operations lays in one step which is almost imperceptible, and certainly almost insignificant from the point of view of modern mathematics, but which is the crucial one here: the use of a segment of "unit length" in each of the operations above (*AB* in the case of multiplication, *FG* in the case of root extraction). It is this unit segment that allows, at the bottom line, to finally overcome the need for distinguishing between magnitudes of different dimensions and for abiding by the homogeneity among terms appearing in an equation. Unit lengths, to be sure, had appeared in geometric texts from the time of Islam. They had also appeared more recently in texts that explored in a tentative manner the relations between geometry and algebra in more modern terms. But it was only its systematic use by Descartes in the framework of his innovative treatment of geometry—and of the operations with segments as part of it—that turned it into a fundamental piece of a new, overall conception of numbers and magnitudes.

The systematic introduction of the unit length afforded the possibility to abandon the need to strictly abide by dimensional homogeneity, and from his explanations it is clear that Descartes was aware of it. Still he did not immediate give up the habit to do so. He used a symbolic language similar to that of Viète, with one stylistic difference that has remained in use up to our times: the first letters of the alphabet are used for the known quantities and the last ones for the unknown quantities. In this regard he wrote in the opening passages of Book I of *La géométrie*:[1]

> Often it is not necessary thus to draw the lines on paper, but it is sufficient to designate each by a single letter. Thus to add the lines *BD* and *GH*, I call one a and the other b, and write $a+b$. Then $a-b$ will indicate that b is subtracted from a; ab is that a is multiplied by b; a/b that a is divided by b; aa or a^2 that a is multiplied by itself; a^3 that this result is

[1] (Descartes 1637 [1954], p. 5).

> multiplied by a, and so on, indefinitely. Again, if I wish to extract the square root of $a^2 + b^2$, I write $\sqrt{a^2 + b^2}$, if I wish to extract the cube root of $a^3 - b^3 + abb$, I write $\sqrt{C.a^3 - b^3 + abb}$, and similarly for other roots.

Notice another interesting, if minor stylistic difference between Descartes and current usage, namely that Descartes indicates the cubic root with a letter C inside the root symbol, rather than as an index outside it. This follows from a fact that is interesting in itself in the context of the history of the concept of numbers, namely, that roots are not yet conceived as a fractionary power, and as a matter of fact not as a power at all. The square root is taken here of a quadratic expression, whereas the cubic root is taken of an expression which is a sum of cubes. But from the explanation one can easily understand that this kind of homogeneity is not necessary thanks to the use of the unit length. And indeed, after the above passage Descartes added the following:

> Here it must be observed that by a^2, b^3 and similar expressions, I ordinarily mean only simple lines, which, however, I name squares, cubes, etc. so that I make use of the terms employed in algebra ... It should also be noted that all parts of a single line should as a rule be expressed by the same number of dimensions, when the unit is not determined in the problem. Thus a^3 contains as many dimensions as abb or b^3, these being the components of the line which I have called $\sqrt{C.a^3 - b^3 + abb}$. It is not, however, the same thing when the unit is determined, because it can always be understood, even where there are too many or too few dimensions; thus if be required to extract the cube root of $a^2b^2 - b$, we must consider the quantity a^2b^2 divided once by the unit, and the quantity b multiplied twice by the unit.

In order to understand the full significance of this new possibility of bypassing the traditional demand for homogeneity, thanks to the introduction of the unit length and the steps taken by Descartes based on it, you will find it relevant to revisit briefly some passages in §3.7 (especially those related with Figure 3.12 and Figure 3.13), where we discussed the lack of length measurements in synthetic geometry as presented in the *Elements*. This lack of measurement continued to influence the mainstream of Greek geometry and thereafter, but all of this changed now with Descartes.

By adopting a thoroughly abstract algebraic approach in geometry, based on an appropriate symbolism and on the use of a unit length, Descartes came up with truly novel constructions that could be used in solving longstanding open problems, as well as in providing new solutions to problems that had previously been solved. A straightforward example is that of the quadratic equation (see Appendix 8.5).

A more complex example concerns Descartes' solution of the geometric locus of four lines. This problem had remained unsolved since the time of Pappus (see Figure 4.4). It was as a result of Descartes' efforts to solve this problem (as he understood it at the time) that he introduced the basic ideas of analytic geometry. The techniques he developed allowed him to tackle, with the same method he used for the 4-lines locus, also the general, n-lines locus problem. His solution represented a crucial milestone in the history of mathematics, but for lack of space, it will not be possible to discuss it in this book.[2]

From his treatment of geometric constructions with the help of algebraic methods, Descartes was also led to focus on the investigation of equations and of polynomials as objects of intrinsic mathematical interest. In this context he systematically developed some important ideas that had already incipiently surfaced in the works of Cardano and others. One of them is the relationship between the solutions of an equation and the possibility of factorizing the corresponding polynomial into elementary factors. Descartes, by the way, did not yet clearly distinguish between the polynomial (say, $x^2 - 5x + 6$) and the equation ($x^2 - 5x + 6 = 0$).

Descartes also analyzed the relationship between the signs of the solutions and those of the coefficients of the polynomial. When we replace the unknown x with a certain value a, and the value of the polynomial expression is zero, Descartes called that value a "root" (and we continue to do so). If the roots of a given polynomial are, say, 2 and 3, Descartes showed that the polynomial is obtained as a product of two factors (or, as he said, of two equations): $(x - 2)$ and $(x - 3)$, and hence the equation in question is $x^2 - 5x + 6 = 0$. If we want to add the root 4, then we need to multiply by $(x - 4) = 0$, thus obtaining the equation $x^3 - 9x^2 + 26x - 24 = 0$.

Descartes was very clear in his attitude towards negative solutions: he considered them as possible roots, but he called them "false" (*faux*) roots, as Cardano had also done much earlier. "If we suppose x to represent the defect [*defaut*] of a quantity 5—he said—we have $x+5 = 0$, which multiplied by $x^3-9xx+26x-24 = 0$, gives $x^4 - 4x^3 - 19xx + 106x - 120 = 0$." This was for him an equation with four roots, three of which are "true roots" (2,3,4) and one is a "false" root, 5. In addition, the number of "false" roots equals the number of times that the signs of the coefficients of the equation remain the same when passing from a power of the unknown to the one immediately under it (in this case, this happens only once: $-4x^3 - 19x^2$). These are known nowadays as "Descartes' rules of signs". The number of "true" roots equals the number of times that the signs of the coefficients of the equation do change when passing from a power of the unknown to the one immediately below it.

It was unavoidable that as part of this kind of investigation, Descartes would have to deal with equations leading to the appearance of roots of negative numbers. His position on this issue is interesting and it turned out to be very influential. To see what it was, we need to comment on the so-called "fundamental theorem of algebra", already mentioned in Chapter 1. It states that every polynomial equation (with real or complex coefficients) of degree n has exactly n roots, some or all of which may be complex numbers.

[2] See (Bos 2001, pp. 273-331).

Descartes was aware, as we just saw, of the relationship between the fact that a is a root of the polynomial and that the latter can be divided exactly by the factor $(x - a)$. It was surely natural for Descartes, then, to somehow come up with the basic idea behind the fundamental theorem. Indeed, the idea had already been hinted at in various ways by Descartes predecessors, such as Cardano. It had appeared quite explicitly in a book that was well-known at the time, *L'invention en algebra*, published in 1629 by Albert Girard (1595–1632).

Descartes pointed out, in the first place, that the number of roots of a polynomial equation cannot be greater than the degree of the equation. Later on, in Book III, he wrote explicitly:[3]

> For the rest, neither the false nor the true roots of the equation are always real, sometimes they are only imaginary, that is to say that one may always imagine as many in any equation as I have said, but that sometimes there is no quantity corresponding to those one imagines.

In other words, Descartes asserted the existence, beyond "true" roots, also of roots which are false and also of others which are only "imaginary" or "imagined". Only if one takes these roots into consideration, he emphasized, then their number equals that of the degree of the polynomial. Even if at this point Descartes was not yet willing to bestow on all these kinds of roots equal status as *legitimate* kinds of numbers, the identical role they played from the point of the polynomial was now a major mathematical consideration that could not be easily overlooked.

Descartes' views on polynomials, including the important insight on the number of roots, implied a significant breakthrough, especially since it appeared in the framework of an influential book on geometry. From now on mathematicians would refer to a new kind of entity, autonomous and abstract, the polynomial, to which much attention needed to be directed. Many questions that arose previously in different contexts, and that were investigated separately, would be treated now from a unified point of view based on the new knowledge developed in relation with the polynomials and their roots. Rather than speaking about specific questions with unknown quantities, each separately treated according to its type and to the degree in which the unknown appears in it, all of them were now seen as particular cases of a more general theory, the theory of polynomials. The availability of a flexible and efficient symbolism such as Descartes developed in his work, as a high-point of a long and hesitant process that preceded it, played a crucial role in this development. Many important additional developments over the following centuries in mathematics in general, and in algebra in particular, derived from this new perspective whose clear origin was with Descartes.

But at the same time, we should pay close attention to the interesting nuances that appear in Descartes' approach to numbers of the various kinds. When he spoke about "imaginary" roots, for example, Descartes meant it quite literally, that is to

[3] Quoted in (Bos 2001, p. 385).

say, he saw them as numbers that are only in our imagination, and hence they neither represent a geometric quantity nor are similar to other kinds of numbers that appear as roots. Under the marked influence of this text, terms like "imaginary" and "real" were sweepingly adopted by mathematicians of the following generations. Nonetheless, this was not really of great help in giving some coherent meaning or in understanding the nature of expressions that comprised square roots of negative numbers. Even his attitude toward the "false" roots cannot be seen as true progress towards a more systematic incorporation of the idea of negative numbers. Descartes did include the "false" roots under those that are "real", but he never considered the possibility of speaking about a negative number in isolation: the false roots appear as part of the expression $(x + a)$ that divides the polynomial, and in the polynomial one may find coefficients that are preceded by the sign "$-$", as we saw.

Evidently, this kind of practice involving the continued use of all such kinds of numbers as part of the theory of polynomials made it easier and more natural to accept them gradually as part of the general landscape of arithmetic. But the full acceptance was not part of Descartes' own view, as one may see in his approach to solving purely geometrical problems. When explaining the geometric way to solve quadratic equations (as shown in Appendix 8.5), Descartes specifically refrained from dealing with the equation $z^2+az+bb = 0$, precisely because a and b represent here lengths (that is, positive quantities), while the only solutions that one obtains are negative, which are devoid of geometrical significance. For the same reasons, also in his analytic geometry we only find positive coordinates. The idea that coordinates may be either positive or negative appeared somewhat later, simultaneously with the increasing acceptance of the legitimacy of negative numbers.

8.2 Wallis and the Primacy of Algebra

The year 1631 was important in the history of algebra in the British Isles. It saw the publication of two books that marked the beginning of an increased pace in algebraic activity in the British context: *Clavis Mathematicae* ("The Key of Mathematics") by William Oughtred (1575–1660) and *Artis Analyticae Praxis* ("The Practice of the Analytic Art") by Thomas Harriot, posthumously published. This increase in algebraic activity was accompanied by debates about the question of the relationship between algebra and geometry, and in particular about the nature and role of numbers of various kinds. Following Oughtred and Harriot, no one played a more significant role in helping assimilate and develop algebraic ideas among British mathematicians than John Wallis (1616–1703).

Wallis contributed many of his own original ideas to the continued expansion of the concept of number and the continued blurring of the borderline between numbers and abstract quantities. Unlike many of his predecessors and many contemporaries that continued to abide by more classical views, Wallis unequivocally attributed full conceptual precedence to algebra over geometry. Likewise he actively put forward many attempts to finding coherent ways to legitimize the use of negative and imaginary numbers. It is therefore important to devote some attention to his ideas as part of our account here.

Wallis' serious involvement with mathematics came at a relatively late age and in a rather non-systematic way. He was formally trained in the classical tradition which included mainly the study of Aristotelian logic, theology, ethics, and metaphysics, and in 1640 he was ordained priest. Like Viète, also Wallis developed a keen interest in cryptography. During the English Civil Wars of 1642–1651 he exercised his skills in decoding Royalist messages for the Parliamentarian party. It was only in 1647, at the age of 31, that he studied Oughtred's *Clavis* for the first time. This marked the beginning of his highly creative mathematical career. In 1649 he was appointed to the Savillian Chair of Geometry at Oxford.

Wallis' most original contributions relate to calculations of areas and volumes, as well as tangents. At the time, such calculations started to involve geometric situations of increasing complexity that, over the next few decades, became the core of the infinitesimal calculus. The problems discussed as part of this trend occupied the minds of the leading mathematicians of the period. But while most of them tried their best using methods that were essentially geometric and followed the Greek indirect method for dealing with the infinite (see Appendix 3.10), Wallis went his own way and introduced many new *arithmetic* methods for dealing with infinite sums and products. It was here that Wallis displayed the full power of his mathematical ingenuity and developed truly original methods.

One of his most stunning results, published in 1656 in his *Arithmetica Infinitorum*, involved an innovative method for approximating the value of π. Like Viète's calculation almost eighty years earlier, also Wallis' method involved an infinite product. However, this one was based on arithmetic considerations, rather than on geometrical approximation, and hence it was much more powerful. It can be symbolically represented as follows:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \quad \ldots$$

Wallis also expanded the concept of power to include negative and fractional exponents, being the first to work out useful insights such as $a^{1/2} = \sqrt{a}$ or $a^{-n} = \frac{1}{a^n}$ (even though his notation was somewhat different from ours).

One of the more impressive displays of the power of algebraic methods in the work of Wallis appeared in his treatment of conic sections. The recent development of Cartesian methods had helped characterize a parabola with the help of a quadratic equation $y = ax^2 + bx + c$. Wallis was the first to do something similar for the ellipse and the hyperbola. Because of his success in providing algebraic tools for dealing with a topic that, in the purely geometrical treatment of Apollonius had traditionally been considered to be of extreme difficulty, Wallis saw himself as implanting simplicity in a field that had previously deterred many. Apollonius' original treatment was still the only one available at the time of Wallis, and Wallis was truly proud of the deep change he brought into the field of research on conic sections.

Wallis joined those who believed in the existence of a lost analytic “method of discovery”, which “was in use of old among the *Grecians*; but studiously concealed as a Great Secret”. Therefore, like Viète and Descartes before him, he saw his work as both continuation and an improvement of that putative analytic method.

From the perspective afforded by his strongly arithmetic approach Wallis also advanced a further significant step in the direction of looking at proportions as no more than equalities between two fractions. In doing so, he simply dismissed off-hand, in a more decisive and unequivocal fashion than anyone prior to him, the ages-old separation between ratios and numbers. Wallis spoke of a ratio between two numbers or two magnitudes simply as a division of the first by the second. Plain and simple, as we would consider it nowadays. In this view, four magnitudes are said to be in proportion if the ratio of the first to the second, *seen as a number*, equals the ratio of the third to the fourth, *also seen as a number*. In other words, for Wallis the proportion $a:b :: c:d$ was not different from the identity $a/b = c/d$. Remarkably, Wallis did not emphasize that he was changing an accepted view rooted in a centuries-old tradition and based on a completely different definition.

In considering arithmetic rather than geometry as the more solid conceptual basis for mathematics at large, Wallis had few constraints in using all kinds of numbers in various mathematical contexts. Nevertheless, he did not always promote a full acceptance of numbers of all kinds. Wallis’ views on the legitimate use of negative, rational and irrational numbers were somewhat fluctuating. Negative numbers were for him a necessity, but he did not always consider them as legitimate in all situations, because it is not possible that a quantity “can be Less than Nothing, or any number fewer than None”. A ratio between a positive and a negative number he initially considered as devoid of meaning, but later on he came up with a strange argument that proved—so he thought—that dividing a positive by a negative number yields a result “greater than infinity”.

And yet, since the idea of negative number was so useful and it is not “altogether absurd”, Wallis suggested that these numbers should be given some kind of interpretation via a well-known physical analogy. More generally, for a mathematician like Wallis, it was imperative to provide some kind of underlying conceptual consistency to arithmetic and to avoid “impossible” situations that might arise in operating with natural numbers: subtracting a greater number from a smaller one, dividing a number by another number which is not a factor, extracting square root of a non-square number or a cube root of a non-cubic number, or coming up with equations whose roots are square roots of negative numbers.

In spite of his own definition of ratio as a division of numbers, Wallis had doubts about the legitimate status of fractions and of irrationals. Still, given their practical usefulness in the solutions of many mathematical problems (including the kinds of innovative solutions that he himself had been developing with infinite series and the like), he did not limit their use. He chose to consider them as approximate values expressible in terms of decimal fractions as Stevin had taught. Concerning negative numbers, he was not able to come up with any definitive argument to justify their legitimacy, and all he was able to gather was a series of more-or-less convincing claims.

So, somehow along the lines of Descartes, Wallis defined both negative numbers and their roots as "imaginary", in the sense that negative numbers represent a quantity which is "less than noting". With this definition his intention was to promote the view that whoever accepts the legitimacy of the negative numbers has no real reason to reject that of their square roots. This was a wise move. He suggested extending to imaginary numbers the kinds of arguments typically used for providing legitimacy for the negatives, namely, some well-conceived physical analogy. The details of his argument are worthy of discussion here.

In his *Treatise on Algebra*, published in 1685, Wallis came up with the following original account of imaginary numbers (Figure 8.3): on a straight line where a starting point A is indicated, a man walks a distance of 5 yards in the direction of B, and then he retreats a distance of 2 yards in the direction of C. If asked what is the distance he has advanced, one will have no hesitation in answering 3 yards. But if the man retreats from B 8 yards to D, what is the answer to the same question? Clearly -3, and Wallis said "3 yards less than nothing".

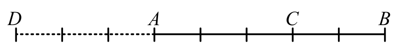

Fig. 8.3 Wallis' graphical representation of negative numbers on a straight line.

This obvious argument is presented just in preparation for Wallis' original idea on how to interpret in a similar, graphic way the square roots of those same negative numbers. What happens, he then asked, if on a certain place on the seashore we gain from the sea an area of 26 units and loose to the sea, in some other place, and area of 10 units? How much have we gained, all in all? Clearly an area of 16 units. If we assume this area to be a perfect square, then the side of this square is of 4 units of length (or -4 units, if we admit the negative roots of positive squares). Nothing special or new thus far. But now, what happens if we gain from the sea 10 units and loose in some other place 26 units. In analogy with the previous case, we may say that we lost 16 units, or gained -16 units, and if the area lost is a perfect square, what is then its side? It is, he concluded, the square root of -16. This was Wallis' point: negative numbers and imaginary numbers are equally legitimate or equally illegitimate, and there is no reason to accept the former and reject the latter in mathematics.

Wallis tried yet some other, possible geometrical interpretations of the imaginary numbers. One of his ideas was based on the construction of the geometric mean of two (positive) magnitudes b, c (we can denote it here as $\sqrt{bc}$). A classical construction of this means is embodied in the diagram of Figure 8.4.

This construction is based on an elementary theorem on circles (already mentioned above in relation with Descartes), stating that if AC is the diameter and PB is orthogonal at any point B on the diameter, then the square built on PB equals in area the rectangle built on AB,BC. Wallis suggested to consider the square root

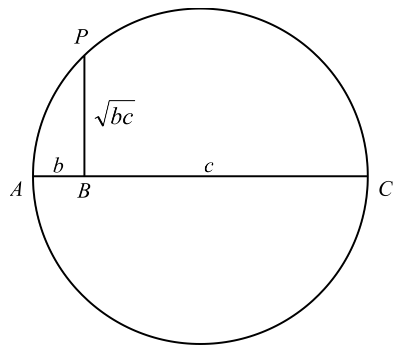

Fig. 8.4 The geometric mean of two quantities, b, c.

of a negative number as the geometric mean of two lengths, one of them positive, the other negative: for instance, $-b$, c or b, $-c$. Graphically, this is represented in Figure 8.5.

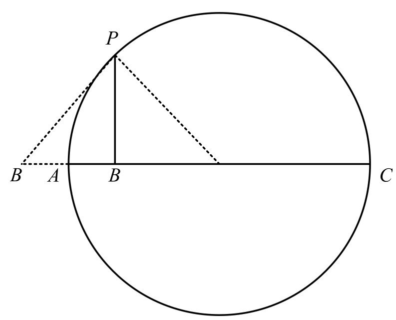

Fig. 8.5 Wallis' representation of an imaginary number as the geometric mean of a positive and a negative quantity.

Indeed, if we take the quantity b to the left of A, so that $AB = -b$, and then the quantity c to the right of B, so that $BC = c$, and hence $AC = -b + c$, then it is easy to see by a simple geometric argument, that PB, the tangent to the circle on P, represents the geometric mean $\sqrt{-bc}$.

Wallis was not really satisfied with this, and he went one step further, suggesting yet another geometric interpretation, which comes very close to the one that will eventually turn into the accepted interpretation of complex numbers. We will speak in greater detail in Chapter 9 about this later interpretation and its origins in the 18th century, but as a brief reminder for the readers at this point, I just want to emphasize that the geometric interpretation is based on extrapolating the representation of real numbers on a straight line into the entire plane as a way to representing the complex numbers. Wallis' last and very original attempt to interpret imaginary numbers geometrically also extended the representation from the straight line to the plane but the way he followed turned out to have serious limitations. It appeared as part of an explanation of the geometric meaning of solutions to quadratic equations $x^2+2bx+c^2=0$, where b and c are positive quantities. The solutions are obtained, of course, through the formula:

$$x = -b \pm \sqrt{b^2 - c^2}.$$

Wallis drew up a diagram in which the solutions appear as two points P_1, P_2, and in which one sees that real solutions may exist only when $b \geq c$ (Figure 8.6).

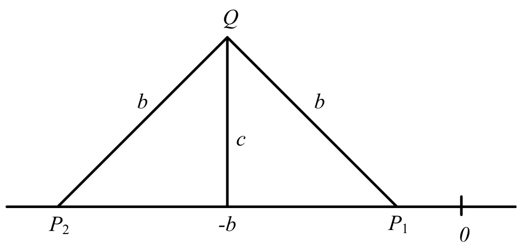

Fig. 8.6 Wallis' graphic representation of two solutions to the quadratic equation. The sides of the triangle are of length b and the same length is taken on the horizontal axis to the right of O.

But what happens here when $c > b$? From the algebraic point of view, the formula says that the solution would involve square roots of negative numbers. In terms of the diagram, what we see is that the points P_1, P_2 would lay outside the line that was chosen to represent the numbers, and yet they lay on the same plane (Figure 8.7).

This appears indeed as a possible representation of these roots of negative numbers. But a significant problem arises immediately: if b is taken to decrease continually, then P_1, P_2 will approach each other on the plane. If b finally becomes zero, then $P_1 = P_2$. The meaning of this is that $\sqrt{-1} = -\sqrt{-1}$, which is clearly unacceptable.

Seen against the background of contemporary debates on the nature of number, and in particular of Wallis' own uncertain views about the negative numbers, one is not surprised to realize that he tried hard and had some brilliant starts, but that he did not succeed in forming for himself a coherent view of a possible geometric representation of imaginary numbers. As I already pointed out, analytic geometry started its way more or less at that time in the work of Descartes (and independently also in the work of Fermat). Also the more general implications of the

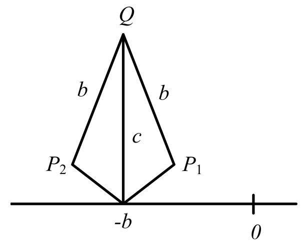

Fig. 8.7 Wallis' graphic representation of imaginary solutions to the quadratic equation on the plane.

relationship between geometrical forms and algebraic expressions embodied in this mathematical discipline took time to be fully worked out. Negative coordinates, for example, did not appear from the beginning, among other things because of the uncertain status of the negative numbers. The work of Wallis, precisely because of his acknowledged accomplishments in successfully applying innovative and powerful arithmetical methods, is of special interest to highlight the difficulties still encountered at this time in dealing with such concepts that we deem nowadays so simple and straightforward. It also highlights the influence, still pervasive at that time, of views on numbers stemming from the ancient Greek tradition.

8.3 Barrow and the Opposition to the Primacy of Algebra

In the last part of the 17th century there developed in England a trend that took a more restrained attitude towards the rising tide of Viète and Descartes' kind of algebra. This trend mistrusted algebra as a possible source of certainty in mathematics and sought to restore primacy to synthetic geometry seen along the lines of the classical Greek tradition. Two main figures in this trend were Thomas Hobbes (1588–1679) and Isaac Barrow (1630–1677). The transparent structure of geometry was in their view the perfect paradigm of simplicity, certainty and clarity. Nothing like this could be found in either arithmetic or algebra, in their view. They opposed the use of algebraic arguments for solving geometrical problems, but they cared to state that their opposition did not imply a more general, negative attitude to the spirit of the new science and the mathematics of their times. Rather, they had very clear and specific arguments against the use of algebra in certain situations. So, some of the algebraic ideas and methods introduced by Viète and Descartes, and by their British followers, did find a way into their works and were naturally incorporated therein in spite of their declared opposition. This gave rise to an interesting and original synthesis, which is of particular interest for our story. Let us see some details as they are manifest in the work of Barrow.

Barrow was a profound scholar with a very broad background in classical and modern languages, and a deep interest in divinity studies. He was professor of Greek until 1663, when he was appointed the first Lucasian Professor of Mathematics at Cambridge. A few years thereafter he would renounce the chair on behalf of Newton, whose outstanding talents he was among the first to recognize while Newton was still a student.

One of Barrow's earliest mathematical publications was an abridged and commented Latin edition of Euclid's *Elements*, published in 1655. In its English translation of 1660 it became a widely used text in the British context up until the 18th century. Barrow incorporated into this text some clearly algebraic elements and combined them into his purist approach to geometry. At the same time he explicitly emphasized that in his presentation he was not deviating in any sense from the original. It is likely that he was sincere in this belief, even though in historical perspective the deviation is more than obvious. Let us take one specific example, namely, theorem II.5 of the *Elements*.

In many of the propositions discussed in his edition, instead of the classical accompanying diagrams, Barrow preferred to write the geometrical property to be proved in an idiosyncratic symbolic language. This did not translate the property into an algebraic equation, to be sure, and his symbols were not mean to be manipulated. But Barrow's symbolism allowed, if not actively suggested, a reading of Euclid in which the geometric magnitudes could also be seen as abstract quantities. He mixed without much constraint classical geometric constructions, symbolic expressions, and numerical examples. Still, he kept stressing that he followed this approach just in order to present the proofs (which he characterized as fully geometric in spirit) in a more condensed manner. This point is better understood by looking at a detailed example, which I have presented in Appendix 8.6.

This unique blend of a declared attitude that promoted the classical standards of Greek geometry, on the one hand, and, on the other hand, favored the adoption of a symbolic language as a way to allow for a clearer presentation of geometrical results stands also in the background of Barrow's attitudes towards numbers. His views are known to us via the texts of his lectures in Cambridge, beginning in 1664. Arguing for the primacy of geometry over algebra, Barrow put forward some philosophical statements, not always very convincing, about the way in which the objects of geometry are perceived through the senses. Quantities, he said, appear in nature only as "continuous magnitudes", and these are the only true objects of mathematics. Numbers, as opposed to magnitudes, are devoid of independent existence of their own and they are nothing other than names or signs with the help of which we refer to some magnitudes.

Barrow explicitly criticized Wallis' views on numbers and algebra. If for Wallis the formula "$2 + 2 = 4$" was true independently and previous to any geometrical embodiment of it, for Barrow it was arbitrary and devoid of autonomous meaning. Indeed, for him, it was constrained by the ability to apply it in some specific geometric situation. For example, when we add a line of length 2 feet to another line of same length, then we obtain a line of length four feet. But when we add a line of length 2 feet to a line of length 2 inches, we obtain a line neither of length 4 feet nor of length 4 inches, nor of any other 4 known units. So, in his view, the meaning of the sign 2 depended directly on the geometric context to which it is applied.

But if natural numbers are no more than signs for magnitudes, what can then be said about irrationals, negative or imaginary numbers? Irrational numbers were the easiest to adapt to Barrow's views, and indeed he used them to strengthen his position as opposed to that of Wallis. His claim was that there is no number, either integer or fractional, that when multiplied by itself yields 2, and from his own point of view there is no need to understand $\sqrt{2}$ in terms of natural or fractional numbers, or even approximations of decimal fractions (as was the case of Wallis). For Barrow, $\sqrt{2}$ was nothing but a name, or a sign, that indicates a certain geometrical magnitude, namely, in this case, the length of the diagonal of the square with side 2. And from here he also derived an additional criticism to Wallis, namely to the latter's arithmetic interpretation of ratios and proportions, that so strongly deviated from the classical Greek tradition, as we just saw above. Barrow admitted that certain ratios, *but by no means all of them*, can be expressed as fractions. The classical case of the diagonal of a square was for him the indisputable instance to think about in this regard. Ratios, in Barrow's views, could in no way be conceived as numbers, since numbers represent only magnitudes.

Negative numbers—Barrow suggested very much like Wallis—should be seen as differences between a smaller and a larger natural number. But then, how can one interpret the number -1, when 1 is no more than a sign to indicate a magnitude? Well, here Barrow admitted the difficulty of thinking about a number that is "less than nothing", but he illustrated the idea with the same kinds of physical-geometrical analogies adduced by Wallis. And concerning the roots of negative numbers: interestingly, Barrow did not mention them at all.

Wallis and Barrow are emblematic representatives of two trends in mid-seventeenth century British mathematics that laid their stress on different aspects of mathematical practice. These trends, however, were not diametrically opposed and they complemented each other in various respects. Wallis' concerns with algebraic methods as new instruments for discovery, for example, did not imply a disregard for Barrow's insistence on classical rigor. On the other hand, Barrow's preference for geometry should not be seen just as a stubborn refusal to adopt "modern" methods. At the time, only a rather limited kind of curves could be treated with the help of algebraic methods (curves that we call nowadays "algebraic curves"). Other kinds of curves, such as spirals and cycloids (which we refer to nowadays as "transcendental curves") could not be covered by algebra. A mathematician like Barrow aimed at developing mathematical methods of a clarity and generality that algebra could not deliver at the time the way geometry did.

8.4 Newton's *Universal Arithmetick*

The trends of ideas embodied in the works of Wallis and Barrow interacted at the heart of a process where innovative views on the relationship between algebra and geometry gradually consolidated. A modern conception of number was among the outcomes of this process. The intellectual stature of Wallis and Barrow and their acknowledged status within the British mathematical community turn their contrasting views and debates, as well as their points of convergence, into a highly visible milestone from which to analyze this significant crossroads in the history of mathematics. But at the bottom line, these processes, and indeed all significant processes that shaped British contemporary ideas in the exact sciences, crystallized under the towering shadow of Issac Newton and his pervasive influence.

I devote the last section of this chapter to a brief description of Newton's views on numbers. When examining his work, however, one must always keep in mind the complexity of the task involved. The entire 17th century, as we have seen thus far, is a truly transitional period in all what concerns the disciplinary identity of mathematics. While in the 16th century, Euclidean methods provided a stable reference model, and while in the 18th century mathematicians will refer to the calculus as the language and method that will provide an underlying unity to their field, Newton's time is precisely that of passage from the former to the latter. Questions about the interrelation between geometry and arithmetic, and the related ones about the nature of magnitudes and numbers, arose in this context along new questions about the applicability of mathematics to study of the natural world.

But on top of the difficulty generated by the originality and intrinsic depth of Newton's mathematical ideas in a time of deep changes, one cannot overlook the variety of methodological, institutional and personal considerations that keep affecting his work at different periods of his lifetime. We find interesting tensions between his declared intentions, his practice and his method. We must examine linguistic and publication choices related to the various dialogues and confrontations that he entertained with his contemporaries (and of particular interest are those with Descartes and Leibniz). We need also to consider the different kinds of intended readers he addressed in different texts that he wrote. In short, we should not assume that Newton's ideas on any topic, the idea of number included, can be summarized under a simple, coherent formula or description.

As a young student in Cambridge, Newton immersed himself in the study of the mathematical works Viète and Descartes, as well as those of Oughtred, Wallis and Barrow. He came up with an efficient and thoroughgoing synthesis of concepts and symbols introduced earlier in all active fields of mathematics. He also went on to develop many new fields of research while introducing highly innovative methodologies, the most important of which comprised the techniques of "fluxions and fluents", that later would become part of infinitesimal calculus (about which we will not speak here). Later in his life, Newton became increasingly critical of Descartes' methods and views, and he devoted efforts to reconsider his own earlier achievements against the principles of the classical tradition. Newton sought to consolidate a unified view of mathematics in which the calculus of fluxions could be reconciled with Euclid's *Elements* or with Apollonius' *conics*.

In 1669 Newton was appointed to the Lucasian Chair of Mathematics at Cambridge, following Barrow's resignation in order to take a position as chaplain to the King. Newton's lecture notes indicate that he devoted great energies between 1673 and 1683 to algebra, the field of knowledge that Barrow described a few years earlier as "not yet a science". It is not completely certain that the dates retrospectively added to the notes reflect the actual teaching of Newton during those years, but it is quite clear that the notes underwent many transformations, before being published as a Latin book in 1707. Several English editions of the book, *Universal Arithmetick*, were published over the following decades, and they were widely read and highly influential in 18th century England.

From the point of view of its intrinsic mathematical value, *Universal Arithmetick* is far from being one of Newton's most important texts. As a matter of fact, he did not really mean to publish his notes. Retrospectively he even manifested his discontent when the book was published thanks to the efforts of William Whiston (1667–1752), Newton's follower in the Lucasian chair. Between 1684 and 1687 most of Newton's efforts were devoted to the writing of *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), the real climax of his scientific opus (and it must be said that, also in the case of this epoch-making book, Newton was not at all enthusiastic about its publication for fear of criticism that it might attract. Publication became possible in the end thanks to the continued intercession of the famous astronomer Edmond Halley (1656–1742)).

In the following years Newton was very busy with debates that arose in the wake of the publication of the *Principia*. At this time, all plans for a possible publication of his algebraic notes remained unattended. But then, in the 1705 elections to the British Parliament, when Newton presented his candidacy but his campaign did not show signs of taking-off, some of his colleagues at Cambridge promised their support in exchange for a considerable donation on his side to Trinity College, and a final permission to publish the notes, after these would be revised and edited by Whiston.

I take the trouble to tell all these details in the background to the publication of *Universal Arithmetick*, just in order to stress the almost incidental character of its appearance. If we compare the published version with some of the manuscripts found in Newton's scientific legacy, it is easy to recognize the many hesitations and continued changes throughout the years. This should come as no surprise, of course, given that these were drafts and teaching notes, rather than a text prepared carefully for publication. But those who prepared the various editions for print did not always pay close attention to all nuances and changes. Accordingly, different points of emphasis are noticeable within the published texts as well as ideas that conflict with those appearing in earlier versions.

The point is that, whatever the background to its publication, the many readers of the book saw its contents as expressing, in all respects and without qualifications, the ideas of the great Newton. No doubt, beyond the intrinsic mathematical assets or setbacks of the ideas exposed in the book, the very authority of Newton as their perceived supporter gave them an enormous legitimation that would help disseminating and assimilating them in the mainstream of ideas about algebra and arithmetic in Europe.

The focus of *Universal Arithmetick* was in algebraic practice, and there was little room in the text for debates on the foundations of the discipline. The central concepts were only briefly explained and the rules of calculation were presented without any kind of comments or arguments for legitimation. By contrast, every technique that was explained was accompanied by many examples that were worked out to the details. The influence of Viète is clearly visible throughout the text, but even more pervasive is the presence of Cartesian algebraic methods for problem solving. While the book as a whole is an implicit way to fully legitimize the methods of algebra and its use as a tool for solving geometrical problems, Newton used every available opportunity to stress his own preference for the classical methods of synthetic geometry and continued to praise its virtues and to support it as the example to be followed everywhere in mathematics.

Newton's attitude towards Descartes' ideas was complex and ambivalent at best. In the margins of Newton's copy of *La géométrie* we find many critical annotations: "*Error*", "*Non probo*", "*Non Geom*", "*Imperf*". They may have been written while Newton was still a student at Cambridge and they referred to Descartes' use of algebra in a geometrical context. Later on, however, as he himself began to teach algebra, and after having been exposed to the kind of ideas developed by Wallis, Newton was more open to admit the advantages of applying algebraic methods to geometry.

In the opening chapter of the book, Newton provided a concise definition of number, combining together ideas that had appeared in the various traditions from which he was taking inspiration:[4]

> By *Number* we understand, not so much a *Multitude of Unities*, as the abstracted ratio of any Quantity, to another Quantity of the same Kind, which we take for Unity. And this is threefold; integer, fracted, and surd: An *Integer*, is what is measured by Unity; a *fraction*, that which a submultiple Part of Unity measures; and a *Surd*, to which Unity is incommensurable.

This synthesis is extremely interesting. On the one hand, following Barrow, also Newton tended to eliminate the separation between continuous and discrete magnitudes. On the other hand, like Wallis, he identified ratios with numbers, but he abode by the classical demand that the ratio be between "quantities of the same kind". The unit, the integers, the fractions and the irrational numbers appear here—perhaps for the first time and certainly in an influential text in such clear-cut terms—all as mathematical entities of one and the same kind, the differences between them being circumscribed to a single feature clearly discernible in terms of a property of ratios: either the ratio with unity is exact (integer), or the ratio with a part of unity is exact (fraction), or there is no common measure between the two quantities in the ratio

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[4] There are various editions of this book. I cite here from the 1769 edition: *Universal arithmetick: or, A treatise of arithmetical composition and resolution. Written in Latin by Sir Isaac Newton. Translated by the late Mr. Ralphson; and rev. and cor. by Mr. Cunn. To which is added, a treatise upon the measures of ratios, by James Maguire, A.M. The whole illustrated and explained, in a series of notes, by the Rev. Theaker Wilder*, London: W. Johnston. This passage is on p. 2.

(surd). Moreover, and very importantly, numbers are *abstract* entities: themselves they are not quantities, but they may represent either a quantity or a ratio between quantities.

The influence of Newton's definition is clearly visible in many eighteenth-century books throughout Europe, in which it is sometimes repeated verbatim. But the remarkable fact is that this definition is not put to use within Newton's own book. As already said, Newton focused in the book on the practice of problem-solving and he gave little attention to philosophical or methodological questions concerning the central concepts of algebra and arithmetic.

Newton also introduced the negative numbers without much comment or philosophical considerations, while indicating what is it that characterizes them as quantities: quantities may be "affirmative", i.e., larger than nothing, or "negative", i.e., less than nothing. Newton did not adopt the terminology of Wallis, who had called them "fictions" or "imaginary quantities". Rather, he relied on analogies, relating the negative numbers to "debts" or "subtraction of a larger number from a smaller one". Where Wallis had spoken of "impossible subtraction", Newton just spoke of a subtraction to whose outcome we anticipate a "−" sign, and without further distinguishing between positives and negatives. He also presented the rules of multiplication with signs without further ado, with no explanations or justifications, and simply providing numerous examples of their use.

There was no new element in Newton's presentation that had not previously appeared in some British book on algebra, but the systematic and simple picture arising from the book, and—perhaps more importantly—the fact that this picture carried with it the authoritative legitimation stamp of the great Newton, endowed it with a special status that helped turning it into the standard point of necessary reference for both concepts and terminology all around Europe over the decades to come.

Newton's attitude towards imaginary numbers is of particular interest because of the hesitation and lack of final decision that arises from the published text. This attitude reflects the remaining weaknesses in the concept of number, seen as either a quantity, or a ratio of quantities. Newton's inability to take a final stance on this matter derived from the difficulty involved in considering square roots of negative numbers as quantities of some specific kind, like the rest of the numbers. Imaginary numbers appear in Newton's book in the section where he discusses Descartes' rules for counting roots of polynomials (explained above) and the relationship between roots and coefficients.

In his early lectures in Cambridge, Newton spoke—following Descartes—about the possibility that a polynomial equation may have roots that exists "only in our imagination", but to which no quantity can be associated. In this sense the term "imaginary" described quite literally the way these roots were conceived. The manuscripts of the lectures show that he gradually changed this view and the associated terminology. In one place, Newton formulated a version of the fundamental theorem of algebra as the assertion that the number of roots of a polynomial equation cannot surpass the highest order of the unknown in the equation, but these roots may be either positive, or negative or "impossible" (rather than "imaginary"). And what he meant by "impossible" he explained by reference to the solution of the following equation:

$$x^2 - 2ax + b^2 = 0.$$

Here, we obtain two roots, namely,

$$a + \sqrt{a^2 - b^2} \quad \text{and} \quad a - \sqrt{a^2 - b^2}.$$

Now, when a^2 is greater than b^2—Newton wrote—then the roots are "real". In the opposite case, when b^2 is greater than a^2, then, of course, the root is "impossible". But interestingly, Newton nevertheless went on to stress that both expressions are roots of the polynomial, for the simple reason that, when introduced in the equation in place of the unknowns, then the equation is satisfied because "their factors eliminate each other". In other words: a square root of a negative number is an impossibility and hence it does not represent a number in the proper sense of the word, but expressions containing such impossible entities are legitimate roots of an equation and they allow for an appealing formulation of the fundamental theorem of algebra, as Newton conceived of it.

We have already seen these kinds of ambiguous attitudes appearing in algebraic texts at least since the time of Cardano and Bombelli. The fact that by the time of Newton the ambiguity has not been fully bridged is highly indicative of the pervasiveness of certain basic ideas that in retrospective we see as completely inadequate. Newton's formulations make patent the continued tension between what the existing concepts of number implied and what the actual practice required. More than a century of intense mathematical activity would still be needed before truly satisfactory definitions of imaginary numbers would appear, as we will see in the next chapters.

No less confusing for the reader could be Newton's remarks on the relationship between algebra and geometry. As I already said, Newton made extended use of Cartesian methods that combine algebra and geometry, and nevertheless, in some places he specifically refrained from using algebra, while stressing that even in apparently difficult geometrical problems algebra is not the adequate tool for finding the solution. In the printed edition of *Universal Arithmetick* there is a well-known passage in which Newton declared that Cartesian methods endanger the purity of geometry. He wrote:[5]

> Equations are Expressions of Arithmetical Computation and properly have no Place in Geometry, except as far as Quantities truly Geometrical (that is, Lines, Surfaces, Solids and proportions) may be said to be some equal to others. Multiplications, Divisions, and such sorts of Computations, are newly received in geometry, and that unwarily, and contrary to the first Design of this Science ... Therefore these two Sci-

[5] *Universal Arithmetick*, p. 470.

> ences ought not to be confounded. The Ancients did so industriously distinguish them from one another, that they never introduced Arithmetical Terms into Geometry. And the Moderns, by confounding both, have lost that Simplicity in which all the Elegancy of Geometry consists.

These passage was repeatedly cited in many European mathematical texts over the following decades. The mathematicians who cited it were actually those who sought to preserve the primacy of geometry over algebra. It is quite ironic to contrast what Newton wrote with the approach which is dominant in *Universal Arithmetick*, where the prominence of algebra *in practice* is so blatant. From reading the manuscripts of the various versions of the book, one readily realizes that Newton continually hesitated and changed his views on this important point. The clear-cut statement cited above is what in the end was included in all editions of the book, and this is what the readers came to associate with the name of Newton.

I want to stress that in the late 1670s, the time when he was involved with these texts, Newton had just begun reading Pappus. He came to the conclusion that the putative method of discovery of the ancients (the "analysis" that I have already mentioned) was superior to Cartesian algebra. At this time, Newton began conceiving Descartes and Cartesians of all sorts as his personal enemies, while at the same time he began also to conceive himself as direct heir of the ancients. The view of algebra that emerged in this context saw it as a heuristic method that could be used for discovery, but that was not adequate for publication. Algebra—Newton thought at this time in line with Barrow's views—lacks the clarity of geometry, and it is also philosophically misleading since it makes us believe that non-existent things actually exist.

The underlying relationship between algebra and geometry is even more complex in the case of Newton's most famous and influential book, the *Principia*. From the vantage point of later developments in mathematics, it appears to us convenient to present this revolutionary book in the language of the calculus, a language whose initial stages Newton himself was instrumental in help shaping in some of his other important works. But a reader of the original text of the *Principia* will find its style, on the face of it, more reminiscent of classical Greek geometry than of a 19th century treatise of classical mechanics. This is, however, a kind of "classical façade" that Newton worked hard to bestow upon his text. Behind it, one can find in several places a wide variety of recently developed mathematical methods underlying the classical surface: infinite series, infinitesimals, quadratures, limit procedures, and also algebraic methods. There was plenty of evidence in Newton's text to create an image of him as an uncompromising champion of classical views about the primacy of geometry over algebra. In actual truth, however, especially when it came to developing a mathematical practice in relation with numbers and algebra, he followed a more flexible and variegated attitude.

The end of the 17th century marks a significant inflection point in our story. The face of science had profoundly changed, as had changed the place of science in society in most of Europe The consolidation of the new symbolic algebra, especially in the works of Viète and Descartes, and the rise of the infinitesimal calculus in the works of Newton and Leibniz, were a truly significant turning point in the history of mathematics. Many of the topics that we have been discussing thus far receded into the background. The influence of Euclid's *Elements* in the mainstream of advanced mathematical research declined. The importance of the Eudoxian theory of proportions almost disappeared. The divide between continuous magnitudes and discrete numbers lost in interest. The concept of number had undergone deep changes and the door was now open to a new stage in which significant additional changes would completely reshape this concept over the next two centuries. These changes will be described in the remaining chapters of the book.

Appendix 8.5 The Quadratic Equation. Descartes' Geometric Solution

A particularly illuminating perspective from which to understand the innovation implied by Descartes' approach to the relationship between algebra and geometry is afforded by a detailed examination of his treatment of the quadratic equation and the geometric solutions he suggested for them. In this appendix I bring some direct quotations from *La géométrie*. The reader may thus get a first-hand grasp of the way Descartes handled the various kinds of numbers and geometric magnitudes that appear in his equations. Of particular interest is the unhesitant transition from algebraic expressions to geometric interpretations. We are of course used to such transitions, but they implied a far-reaching innovation, even if Descartes did not particularly emphasized this in the book.

The following is quoted from the original text:[6]

> For example, if I have $z^2 = az + bb$, I construct a right triangle *NLM* with one side *LM*, equal to *b*, the square root of the known quantity *bb*, and the other side, *LN*, equal to ½*a*, that is to half the other known quantity which was multiplied by *z*, which I suppose to be the unknown line. Then prolonging *MN*, the hypothenuse of this triangle, to *O*, so that *NO* is equal to *NL*, the whole line *OM* is the required line *z*.

[6] (Descartes 1637 [1954], pp. 13–14).