Structures in Mathematical Theories

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Reflexive Thinking in Mathematics
Formal and Non-formal Aspects

Two sorts of questions can be asked concerning any scientific discipline: substantive questions of the discipline, and questions *about* the discipline *qua discipline,* or second-order questions. We can accordingly distinguish two layers related to any scientific field – “body” of knowledge and “images” of knowledge. The body of knowledge includes theories, ’facts,’ methods and open problems. The images of knowledge deal with second-order questions as: What is the legitimate methodology of the discipline? What is a good theory? Which are the burning issues of the discipline? etc. The study of the interaction between these two layers is central to the understanding of the growth of scientific knowledge.

Recent studies in the philosophy of mathematics have increasingly stressed the social and historical dimensions of mathematical practices.[1] Although this new emphasis has fathered interesting new perspectives, it has also blurred the distinction between mathematics and other scientific fields. This distinction can be clarified by examining the special interaction between the body and images of mathematics.

In most scientific disciplines, facts and theories are continually added to and deleted from the body of scientific knowledge, while the images of knowledge are affected by these and by a wide variety of other factors. But in contrast, claims that enter the body of mathematics through proof are seldom if ever rejected. As a rule, new theorems and new proofs of old theorems do not falsified old theorems and proofs. Still, the process of mathematical change is not one of linear accumulation.

It is not enough to discover a new theorem, proof or concept in order to say that mathematical knowledge has changed. It is the images of knowledge (which are determined by social and philosophical factors, by the interaction with other sciences, etc.) that determine, in mathematics and elsewhere, the way in which a new item will be integrated to the existing picture of knowledge; whether it will be considered important or whether it will be ignored. Eventual changes in the images of knowledge may later transform the status of existing pieces of knowledge and produce a different overall picture of mathematics. Change proceeds not only quantitatively, by addition of new results or concepts. These additions are, of course, fundamental to the growth of mathematics, but real change occurs only and insofar as the quantitative growth is accompanied by a *qualitative* new understanding of the body of knowledge.

But not only the body of mathematics presents a particular behavior. Mathematics has also “reflexive” capacities unlike those of any other exact science. No other exact science affords the possibility of using the standard methodology of the discipline in order to study the nature of the discipline itself. An additional layer of “reflexive knowledge” peculiar to mathematics, addresses problems of the discipline *qua* discipline, while yet remaining fully a part of the body of mathematics.

It may sometimes be hard to distinguish between the pure body of knowledge and pure images of knowledge. These constitute the extremes of a continuum and “reflexive knowledge” stands somewhere in between. Reflexive thinking is clearly part of the body of mathematical knowledge and justified by proof. On the other hand, it is produced by concentrating on purely second-order problems, and hence it is related to the images of knowledge. Mathematical knowledge includes *all* the layers, and their separation is done for analytical purposes only and usually in hindsight. The historical process of the growth of mathematics is that of the continuos interaction of the different layers.[2]

Reflexive mathematical theories have contributed enormously, specially in the present century, to enhance our understanding of the nature of mathematical knowledge. The elucidation of several central ideas of mathematics which happen to be second-order ideas has increasingly been relegated to its formal, mathematical aspects. The important results attained in this way have lead to an absolute lack of attention to additional, non-formal features underlying these reflexive ideas. The two most outstanding examples of this are “Proof” and “Structure”. I propose to discuss both concepts in terms of the body/images duality in order to reject the incorrect identification of those ideas with their formal counterparts.

It is generally accepted that true mathematical knowledge is knowledge justified by proof. Being a claim *about* knowledge, this is an image of knowledge. There was a considerable body of mathematical knowledge the Greeks. The introduction of mathematical proof by the Greeks changed the overall conception of mathematical knowledge no less than it enlarged the quantity of knowledge. Since the time of the ancient Greeks, the reliance on proof as a criteria for legitimate knowledge persisted as one of the central images of mathematics. But the nature and role of mathematical proof remained a debatable philosophical question, and the criteria for deciding what a legitimate proof is changed through the centuries.

By the turn of the century a formal, reflexive concept of mathematical proof was introduced through the works of Freges and Hilbert. The metamathematical theories developed following these works produced many deep results which sensibly improved our understanding of the nature and limitations of proof. The success of such theories brought about a further image of mathematics, according to which claims about proof (and, more generally, about mathematical knowledge) are relevant and legitimate only inasmuch as they appear within the framework of forma, mathematical theories. But on the other hand, mathematicians working in all fields of mathematics continued to use proofs similar to those used in their respective fields before the formal concept of proof was available. Thus, the formal concept of proof became an ideal. It was supposed that any mathematical theory may be completely formalized and axiomatized, following the standards of Russel and Whitehead's *Principia,* and that any proof accepted as legitimate by the relevant mathematical community can, in principle at least, be formalized within the purported system. The actual process of formalization is seldom done, since it is considered unilluminating and straightforward, but it is the certainty that such a process is possible which affords the basis for the legitimization accorded to the proof.

The existence of a formal concept of proof, then, seems to solve the problem of providing criteria for deciding when a proposed proof is legitimate. Indeed, this is the idea which dominated the philosophy of mathematics until recently. However, over the past two decades two opposed directions of research have seriously questioned the truth of this view. On the one hand, historical research such as advanced by Imre Lakatos has stressed the need for distinguishing between the concept of proof throughout history and its modern, formal counterpart[3]. On the other hand internal developments in several branches of mathematics have introduced new forms of proof which do not, and cannot even in principle, fit the ideal picture provided by the formal definition of proof. These include 'computer-assisted proofs', 'very long proofs', and proofs which now show that a given theorem has an 'extremely high probability of being true'[4] . Although proofs of this kind are not considered today as mainstream, they raise serious doubts as to the possibility of defining proof as a chain of unquestioned deductive inferences within a formal axiomatic system.

By stressing the non-formal aspects of the idea of proof we attain a much wider perspective of its nature and function. As pointed out above, the fact that proof constitutes the demarcation criterion between *legitimate and illegitimate knowledge* in mathematics is and historical one. It has remained steady throughout the centuries, yet it is not itself part of the body knowledge, but rather an image of knowledge determined by historic and sociological factors. Now, something similar happens with the demarcation criteria for *legitimate proof.* At a given stage of history there usually exists a high degree of agreement among mathematicians about what constitutes an acceptable proof. However, the criteria of demarcation for rigorous proof are seldom formulated explicitly. Rather, they constitute a tacit code shared by practitioners of a given branch of mathematics and they are subject to debate and change. They belong entirely to the non-formal images of mathematical knowledge, and they having nothing to do with a formal, reflexive concept of proof.

To summarize then, when talking about "mathematical proof" we are actually referring to two different objects: on the one hand a formal concept defined within the framework of a formal, metamathematical theory and, on the other, a general, non-formal concept, which meaning is understood and tacitly shared by the practitioners of a given mathematical field, although there is no clear-cut characterization of it. The existence of a formal, reflexive Theory of Proof has not brought the second-order debate about the nature of proof to an end. This is a debate about images of mathematics and such debates can not always be settled within a formal theory.

So much for the concept of proof. The need to discuss the non-formal aspects of proof in order to understand its actual role in mathematics has been already stressed by many authors [5]. The claim being made here is that similar qualifications must be made in other second-order debates on mathematics.This is specially true of the debate on the nature and role of ”mathematical structures”.

The claim that mathematics is a science dealing with structures is a claim about knowledge and it is therefore an image of mathematics. In fact, it is a rather new image of knowledge which took root around the 1903’s.This image of knowledge comprises a *non-formal* idea that deeply influenced several decades of mathematical research, not by providing definite concepts and results, but rather by suggesting questions to be worked out and by providing a standard repertoire of techniques leading to the proper answers. In addition to its direct influence on the day-to-day work of mathematicians in many branches, this non-formal idea became, soon after its emergence, the focus of second-order concerns. Several formal, reflexive mathematical theories have been proposed in whose framework the idea of structure can be abstractly considered and elucidate. Bourbaki’s theory of structures, Category theory, and the now forgotten attempt by Oysten Ore to provide foundations for abstract algebra upon a general, lattice-theoretical concept of *structure* [6] although stemming from dissimilar motivations, are examples of such theories. However, none of these formal theories succeeded in exhaustively encompassing the meaning of the non-formal idea of mathematical structure.

The so called “structural approach to mathematics, then, is based on a non-formal idea, namely that mathematics deal with structures. What is exactly meant by this can be better understood by cursorily examining the context of the first mathematical textbook generally considered to have espoused a structural approach, van der Waerden’s *Modern Algebra (*1930). This book presented a unified and systematic exposition of a great number of results which had been obtained in the previous decades in various branches of mathematics. But the innovate character of the book cannot be found in any specific concept or theorem presented in it, but rather in the new overall conception of algebra it suggested.

The aim and contents of the book may be roughly characterized through a formulation which will seem rather trivial nowadays: to define the diverse algebraic domains and to attempt to fully elucidate their structure. What is meant by “defining an algebraic domain”? Algebraic domains are defined by van der Waerden in two different ways: either by endowing a non-empty set with one or two operations defined abstractly (such as in the case of groups,rings, hypercomplex systems,etc.),or by taken a given domain and defining a procedure which begets a new abstract construct based upon the former (field of quotients of a domain of integrity, rings of polynomials of a given ring, extension of a field, etc.).But what is meant by “elucidating the structure of an algebraic domain”? There is not clear-cut answer to this question and, in fact, it is never explicitly discussed in *Moderne Algebra*. Chapter V, for instance opens with the claim: ”The aim of this chapter is to give a general view of the structure of commutative fields, and of their simplest subfields and extension fields." What must we know of a field (or of any other algebraic domain, for that matter) in order to claim that we know its structure? Although this question is not explicitly asked in van der Waerden's book, we can indeed find an implicit, tacit answer to it by examining the questions and answers that repeatedly arise in the actual discussion of the different domains.

For lack of space, a detailed inventory of those recurring questions and answers cannot be presented here [7],yet a few words can be said about the issue. An account of the problems presented by addressed by van der Waerden in the different domains considered in his book, and of the standard solutions adopted may seem trifling to anyone who received his mathematical training in the last fifty years. However, it is only by appraising such an inventory vis-a-vis the central image of nineteenth century algebra (i.e.,algebra as the branch of mathematics dealing with polynomial equations), that we can understand what the structural approach of *Moderne Algebra* is and what is novel about it. Although it is a commonplace that *Moderne Algebra* was the first text in which the "structural approach" was fully adopted, the meaning of the latter expression has been usually taken for granted and it has often been tacitly identified with the "modern axiomatic method"/ The abstract axiomatic formulation of algebraic concepts was indeed a necessary condition for the rise of the structural approach, but a detailed inventory as proposed above will show that the structural approach comprise many aspects other than the axiomatic method. Among these aspects, we find many which cannot be defined in formal terms. Thus, the new approach became standard in algebra from the early thirties *even though no formal concept of "algebraic structure was available or necessary to bring about a change in the images of algebraic knowledge*.

The structural approach soon extended to other fields of mathematics as well. As noted above, the very idea of mathematical structure became itself a second-order issue in need of further, specific elucidation. Now this elucidation has been carried out both at the non-formal and the reflexive, formal level. At the non-formal level we find a score of expository articles where mathematicians explain the essence of the structural approach, philosophical analysis of the role and nature of structures in mathematics, and historic accounts of the development of the concept. It is noteworthy, that the meaning given to the concept of structure in such writings significantly varies from author to author and even among different texts of the same author.

From among the formal, reflexive attempts at elucidating the concept of mathematical structure, Bourbaki's theory of <u>structures</u> probably is the most widely known. It is also the primary source of the incorrect belief that there exists a single, clear-cut formal definition of mathematical structure. The initial stages of Bourbaki's ambitious program for the unification of mathematics around the concept of structure were driven by the belief that "structure" can be defined within a formal theory, within which significant results may be derived for the different branches of mathematics. This outlook was announced in the early writings of the group in the late thirties but the formal theory of <u>structures</u> was actually published only in 1957. In spite of the high expectations attached to the theory. It actually played no significant role at all, neither within Bourbaki"s own treatise nor within any mathematical book or research paper. Yet Bourbaki's work, considered as a whole, is a paradigm of unified, thorough structural treatment of an impressive quantity of mathematical branches. What bestows it structural character upon the treatise, however is not its being based upon a formal concept of structure, but the recurring occurrence of certain features which can only be characterized like van der Waerden's approach mentioned above, at the non-formal level.

Neither Bourbaki's theory nor any similar attempt have succeeded to to the present day in fully elucidating the idea of mathematical structure. However, since mathematical knowledge is usually granted special status of unquestioned certainty, beyond that of other forms of human knowledge, the very existence of formal theories dealing with structures has often been interpreted at face-value, by mathematicians and non-mathematics alike, as though this concept would be unequivocally understood within mathematics. Thus, it is not unusual to find, in a general text about structuralism, the claim that:

> The discipline employing the notion of 'structure' can be ordered along a continuum, ranging from mathematics through to physical sciences, the life sciences, the social sciences, to the humanities… Mathematics, first, possesses the structure by definition. (Lane 1970,20)

Moreover, some images of mathematics openly held by Bourbaki, and which involve claims about the role of structure (and structures) mathematics have mistakenly been understood by many to be claims deductively derived within the theory of structures and, therefore possessing mathematical certainty. Thus, for instance, Bourbaki's claims about the centrality of the mother-structures, were enthusiastically greeted by Piaget, who saw in those claims a most important corroboration of certain aspects of his own theory of developmental psychology. These claims, however, are based only on the (admittedly authoritative) personal opinion of some of Bourbaki's members, and they are in no sense deductively obtained results.

Like "proof, "structure" refers to two different kind of meanings—formal and not formal. But it should be stressed , to conclude, that the identification of a non-formal idea with its formal counterpart is even more misleading in the case of "structure" that in the case of "proof", since the formal theory of proof produced a body of mathematical results which positively improved our Understanding of this second-order idea, while nothing of the same magnitude has yet been attained by the diverse formalization of the idea of structure.

NOTES:

1. Cf., e.g., [Tymockzko 1985]
2. For a more detailed discussion of these ideas see [Corry 1990]
3. See specially [Lakatos 1976]

4. The philosophical implications of the existence of such proofs has been widely discussed. Cf., e.g., [Kitcher 1983, 40 f.f.] or [Davis 1972]
5. Besides the above mentioned collection edited by Tymozcko, see also [Aspray & Kitcher 1988] especially the introduction.
6. The fact that several authors use the term structure to mean different concepts is somewhat confusing. Bourbaki's technical term is denoted here <u>structure</u> (underlined) while *structure* (italics) is used for Ore's term Ore's program was exposed in [Ore 1935;1936}
7. For a comprehensive inventory see [Corry 1990a]
8. For a detailed discussion of this issue see [Corry 1990a)

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