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Invariant Theory

The theory of algebraic invariants was a most active field of research in the second half of the nineteenth century. Gauss's work on binary quadratic forms, published in the *Disquititiones Arithmeticae* dating from the beginning of the century, contained the earliest observations on algebraic invariant phenomena. However, the actual development of the theory began only in the 1840s in two different contexts, with works by George Boole in England and by Otto Hesse in Germany.

First, some general definitions. Boole's contribution arises from Lagrange's work on linear transformations of homogeneous polynomials. Assume an homogeneous binary form $f(x_1,x_2)$ of degree n is given - that is, a polynomial in x_1 and x_2 each term of which is of degree 2 - , with coefficients a_i ($1\leq i \leq n$), and let T be the transformation:

$$x_1 = a_{11}y_1 + a_{12}y_2$$

$$x_2 = a_{21}y_1 + a_{22}y_2.$$

T implicitly defines a new form $T(f)$ of degree n in the variables y_1, y_2, $T(f) = F(y_1,y_2)$ with coefficients b_i ($1\leq i \leq n$), which are rational functions, linear in the a_i's and of degree n in the a_{ij}'s. An algebraic n-ary relation I is called an invariant if

$$I(a_1,a_2,...,a_n) = \delta^r I(b_1,b_2,...,b_n),$$

where $\delta = a_{11}a_{22} - a_{12}a_{21}$, and r is any integer. This definition can be generalized to forms of any number m of variables and of any degree n, and also to more than two forms taken together. One can also consider invariants of the coefficients and the variables together; these are called co- and contravariants.

In 1841 Boole published a *memoir* in which he discussed, for the first time, a particular case of this question. He started from the binary quadratic form $Q = ax^2 + 2bxy + cy^2$, and used the system

$$\frac{\partial Q}{\partial x} = 0, \frac{\partial Q}{\partial y} = 0,$$

to define the relation $\theta(Q)$, which in this case is the discriminant $\theta(Q) = b^2 - ac$. If Q is now transformed to obtain $T(Q)$, Boole proved that $\theta(Q) = \delta^r \theta(\mathrm{T}(Q))$ (where, necessarily, the transformation has to be non-singular). He also proved a similar result for binary cubic forms.

Boole thus opened a new line of research that was taken up very soon by Arthur Cayley. In 1845 Cayley published the first of a long list of studies dedicated to the calculation of algebraic relations among coefficients of forms of higher degrees, which satisfied conditions similar to those found earlier by Boole. It was Cayley who called such algebraic relations “invariants”. He developed effective, though somewhat cumbersome, techniques for finding different invariants of a given form, and also discovered that various invariants of the same form sometimes satisfy rational relations among them. The term “syzygies” was introduced in 1853 by James Joseph Sylvester to denote these relations among invariants.

For instance, given the binary quartic

$$A = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

Cayley discovered the invariant

$$u = ae - 4bd + 3c^2,$$

and independently Boole discovered the following one:

$$v = ace - ad^2 - eb^2 - c^3 + 2bcd.$$

A third invariant of this form is a polynomial combination of the former two:

$$(ae - 4bd + 3c^2)^3 + 27\,(ace - ad^2 - eb^2 - c^3 + 2bcd)^2.$$

Cayley and Sylvester were lifelong friends and collaborators, and together they established a very idiosyncratic and prolific tradition of research on invariants, that also included Edwin B. Elliot, H.W. Turnbull, George Salmon in Britain, Charles Hermite and Camille Jordan in France, Francesco Brioschi in Italy, and Fabian Franklin in the USA.

Early on, Cayley formulated the main problem that would attract the efforts of the mathematicians involved in research in this new domain: to find all the invariants of a given form, and, more specifically, to find a minimal set of invariants that would allow one to construct, using the associated syzigies in a thoughtful way, the whole system in question. The work of the British invariant-theorist focused on developing algorithms for finding the desired invariants in particular cases. The German invariant-theorists, in their turn, pursued a task similar to that of their British counterparts, but they did so using quite different —and as it turned out, more effective— techniques.

The main line of research on invariants in Germany derived from the work of Hesse in geometry. However, somewhat before Hesse, similar ideas were also developed by Gotthold Eisenstein in the framework of number theory. Eisenstein's ideas arose as a direct attempt to generalize Gauss's theory of quadratic forms into the third degree. He thus provided a procedure for associating to a class of cubic forms certain classes of corresponding quadratic forms. The invariants properties of the latter could then be used to deduce those of the former.

Hesse's approach had its starting point in geometry: the study of critical points in third-order plane curves with the help of the Hessian determinant ϕ of an homogeneous polynomial f, of degree m in n variables:

$$\phi = \left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| \ (1 \leq i, j \leq n).$$

From purely geometrical considerations to do with inflection points on the curve $f = 0$, and unaware of Boole's results, Hesse proved in 1844 that —if we formulate it in the terminology introduced above— if f is transformed into $T(f)$, then $\phi(f) = \delta^2 \phi(T(f))$.

Beginning in 1848, Siegfried Aronhold, Hesse's student, undertook to investigate this particular aspect of the latter's research in purely algebraic terms, while disconnecting it from its direct geometrical interpretation. Eventually Aronhold became acquainted with the works of Cayley and Sylvester, and adopted much of their terminology. The significantly different approach he developed for finding and dealing with invariants led him and his followers to substantial achievements in this domain over the coming decades. The most important among those followers were Rudolf Alfred Clebsch, Paul Gordan and Max Noether, who established an influential research school in Germany.

The approach followed by this German school became known as "symbolic", and it was meant to circumvent, at least partially, some of the tedious, explicit calculations that characterized the works of Cayley, Sylvester and their followers. Whereas the latter would write a binary form of the degree *m* as

$$a_0x^m + a_1\binom{m}{1}x^{m-1}y + a_2\binom{m}{2}\,x^{m-2}y^2 + \ldots + a_m\binom{m}{1}y^m,$$

and then operate directly on it in order to find its invariants, the symbolic notation would allow focusing on more succinct, equivalent expressions such as the following:

$$(a_1x_1 + a_2x_2)^m,$$

and sometimes even on the more concise a^m. A main related theorem would then assert that any invariant of this latter binary form can be written as :

$$(ab)^\alpha(ac)^\beta(bd)^\gamma \ldots a^pb^qc^r \ldots$$

Here a,b,c,d, etc., are various different symbolic representations of the same form and (ab) represents the determinant $a_1b_2\text{-}a_2b_1$. The number of different factors a,b,c etc, appearing in this product equals *m* or a multiple of *m*. Whereas Aronhold was the first to express the relation between symbolic products of this kind and the ternary cubic forms arising from Hesse's work, it was Clebsch who proved, in 1861, its validity for the general case of arbitrary degree and number of indeterminates.

Within the German symbolic approach it is possible to derive certain formal expressions that highly simplify the manipulation of invariants. Thus, for instance, using the above notation it is easy to prove that

$$a(bc) + b(ca) + c(ab) = 0.$$

A more complex, and highly useful, identity derivable from the above is

$$b^2(ca)^2 + c^2(ab)^2 - a^2(bc)^2 = 2bc(ab)(ac).$$

The most important results of the theory of invariants after 1860 were achieved on the basis of symbolic manipulations of this kind. The foremost example of such results is the proof of the so-called “finiteness theorem” for binary forms. This proof was published in 1868 by Paul Gordan, who was the leading authority in the discipline for many years. Gordan proved that given any system of binary forms of arbitrary degree, a finite sub-system of it can be chosen, such that any invariant of the whole system may be written as a rational combination of forms of the sub-system. Using the symbolic approach, Gordan was also able to provide, through laborious calculations, “smallest possible systems of groundforms” for the case of binary forms of the fifth and sixth degrees. In the years following its publication, various limited generalizations of Gordan’s theorem were proved, and improved proofs of his original theorem were also given. At the same time, techniques for calculating invariants were significantly improved.

And yet, a full generalization of Gordan’s theorem remained an open problem for many years: to prove the existence of a finite basis for any system of invariants of arbitrary degree and with an arbitrary number of indeterminates. This problem was tackled by David Hilbert, in a series of works that implied a significant turning point in the development of both this mathematical domain and his own professional career.

In a series of short announcements in 1888-89, and then in a more detailed fashion in 1890, Hilbert published the first work that brought him international recognition, when he proved a general version of the finite basis theorem, for arbitrarily large collections of forms of any degree in any number of variables. Gordan himself was among the referees of Hilbert’s article, and he voiced serious concerns about his proof. Hilbert did not actually show how the desired basis could be constructed, as Gordan had done for the binary case: he proved its existence by a *reductio ad absurdum* argument, the legitimacy of which was far from being unanimously accepted. But this initially negative reaction changed soon, if only because already in 1893 Hilbert published a much more constructive proof of his finiteness theorem. In fact, Gordan himself published a simplification of Hilbert’s proof. Another important innovation of Hilbert’s approach was its reliance on ideas originally introduced by Leopold Kronecker in his number-theoretical research, on the one hand, and by Richard Dedekind and Heinrich Weber in a joint article of 1882 on algebraic functions, on the other.

Hilbert himself was the first to assess the historical significance of his own work on invariant theory. In a review article read in his name at the International Congress of Mathematicians held in Chicago in 1893, Hilbert mentioned three clearly separated stages, that in his view mathematical theories usually undergo in their development: the naive, the formal and the critical. In the case of invariant theory, Hilbert saw the works of Cayley and Sylvester as representing the naive stage and the work of Gordan and of Clebsch representing the formal stage. In Hilbert's assessment, his own work was the only representative of the critical stage in the theory of invariants. Moreover, in his 1893 article on invariants Hilbert explicitly claimed to have fulfilled all the major tasks of the discipline, and he in fact abandoned all research in invariant theory thereafter. Hilbert's assessment has often been echoed and accepted at face value. Invariant theory and the algorithmic approach characteristic of its practitioners have been pronounced dead following Hilbert's achievements, thus finally clearing the way to the rise of the new structural algebra.

On the other hand, however, some invariant-theorists saw Hilbert's results as having opened new avenues of research for the discipline, rather than bringing it to a dead-end. Although the number of mathematicians involved in research on invariants and the quantity of published research works on the subject considerably diminished by the turn of the century, it did never completely disappear. In fact, as late as 1933 a new textbook on invariant theory was published which explicitly opposed the approach introduced by Hilbert. The author of that book, Eduard Study, claimed that abstract generalization alone, as instantiated in Hilbert's work, could be no substitute for the "real mathematics", namely the algorithmic one represented by the methods of Aronhold-Clebsch. It is likewise noteworthy that in Hilbert's 1900 list of twenty three problems, the fourteenth proposes to prove the finiteness of certain kinds of systems of invariants. Relatively few efforts, however, seem to have been directed towards the solution of this particular problem until the fifties, when a system of the kind stipulated by the problem was found, for which there is no finite basis.

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