Mathematical Structures from Hilbert to Bourbaki: The Evolution of an Image of Mathematics

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The notion of a mathematical structure is a most pervasive and central one in twentieth-century mathematics. Over several decades of this century, many mathematical disciplines, particularly those belonging to the "pure" realm, developed under the conception that the aim of research is the elucidation of certain structures associated with them. At certain periods of time, especially from the 1950s to the 1970s, it was not uncommon in universities around the world to conceive the ideal professional formation of young mathematicians in terms of a gradual mastering of the various structures that were considered to embody the hard-core of mathematics.

Two important points of reference for understanding the development of twentieth century mathematics in general pertain the ideas of David Hilbert, on the one hand, and of the Bourbaki group, on the other hand. This is also true, in particular, for the development of the conception of mathematics as a science of structures. In the present article I explain how, and to what extent, the idea of a mathematical structure appears in the works of Hilbert and of Bourbaki.

My presentation will be based on defining the idea of a mathematical structure as a classical example of an image of mathematics. This image appeared for the first time in its full-fledged conception in 1930, in van der Waerden's classical book *Moderne Algebra*. In the work of David Hilbert we will find many of the basic building blocks from which van der Waerden's presentation came to be built. In the work of Bourbaki we will find, later on, an attempt to extend to all of mathematics what van der Waerden had accomplished for the relatively limited domain of algebra.

But before entering into all these works in more detail, it seems necessary to begin with some preliminary clarifications. The notion of an “image of mathematics”, parallel to that of the “body of mathematics”, is basic to my discussion here, and it is therefore necessary to explain how I use these terms in what follows.

Claims advanced as answers to questions directly related to the subject matter of any given discipline build the body of knowledge of that discipline. Claims which express knowledge *about* that discipline build its images of knowledge. The images of knowledge help us discussing questions which arise from the body of knowledge, but which are in general not part of, and cannot be settled within, the body of knowledge itself, such as the following: Which of the open problems of the discipline most urgently demands attention? What is to be considered a relevant experiment, or a relevant argument? What procedures, individuals or institutions have authority to adjudicate disagreements within the discipline? What is to be taken as the legitimate methodology of the discipline? What is the most efficient and illuminating technique that should be used to solve a certain kind of problem in the discipline? What is the appropriate university curriculum for educating the next generation of scientists in a given discipline? Thus the images of knowledge cover both cognitive and normative views of scientists concerning their own discipline.

The distinction between body and images of mathematical knowledge should not be confused with a second distinction that has sometimes been adopted, either explicitly or implicitly, by historians of mathematics, namely, the distinction between “mathematical content” and “mathematical form.” Inasmuch as it tacitly assumes the existence of an immutable core of mathematical ideas that manifest itself differently at different times, this distinction would seem to enhance the everywhere present, potential pitfall of anachronistic and historically misleading accounts of the development of mathematical ideas. The distinction between body and images of knowledge, on the other hand, is of a different kind. The borderline between these two domains is, by its very definition, somewhat blurred and always historically conditioned. Moreover, one should not perceive the difference between the body and the images of knowledge in terms of two layers, one more important, the other less so. Rather than differing in their importance, these two domains differ in the range of the questions they address: whereas the former answers questions dealing with the subject matter of the discipline, the latter answers questions about the discipline itself *qua* discipline. They appear as organically interconnected domains in the actual history of the discipline, while their distinction may be worked out by historians for analytical purposes and usually in hindsight.[1]

Analyzing the history of mathematics in terms of the distinction between body and images of knowledge helps attaining some insights that might otherwise pass unnoticed to historians. This is particularly the case concerning one specific, characteristic trait that we find in this area of science, but typically not in others. What I have in mind is the possibility offered by mathematics to formulate and prove, as part of the hard-core body of mathematical knowledge, certain statements *about* the discipline of mathematics, that is to say, the ability to absorb certain images of knowledge directly into the body of knowledge. In disciplinary terms, this ability is manifest in those branches usually grouped under the rubric of meta-mathematics, but by no means it is strictly circumscribed to them. The peculiar capacity of mathematics to become part of its own subject-matter is what I call here “the reflexive character of mathematics.”

The terminology introduced here will help us formulating the development of ideas covered by the present account. I will describe the introduction of the notion of structure as the adoption of a new image of mathematical knowledge, that essentially changed the conception of a specific mathematical discipline, namely, algebra. I will thus discuss the roots of this adoption and its early development. In particular, Bourbaki’s contribution will be described as an attempt to reflexively elucidate this image and make it an organic part of the body of knowledge.

“Pre-Structural” Algebra

I start with a very brief account of the classical, nineteenth-century, image of algebra, which the structural one eventually came to substitute. Throughout the eighteenth century, algebra was the branch of mathematics dealing with the theory of polynomial equations, including all the various kinds of techniques used to find exact or approximate roots, and

1. I have discussed this scheme in detail in Corry [1989].

to analyze the relationships between roots and coefficients of a polynomial. Over the nineteenth century these problems remained a main focus of interest of the discipline, and some new ones were added, including algebraic invariants, determinants, hypercomplex systems, etc. Parallel to this, algebraic number theory underwent a vigorous development, especially with the works of Gauss, Kummer, Dedekind and Kronecker, on problems of divisibility, congruence and factorization.

Between 1860 and 1930 an intense research activity was conducted in these mathematical domains, particularly in Germany, thus giving rise to certain, central trends that can be specifically noticed in it. Among them we can mention the following: [1] the penetration of methods derived from Galois's works into the study of polynomial equations; [2] the gradual introduction and elucidation of concepts like group and field, with the concomitant adoption of abstract mathematical definitions; [3] the improvement of methods for dealing with invariants of systems of polynomial forms, advanced by the German algorithmic school of Gordan and Max Noether; [4] the slow but consistent adoption of set-theoretical methods and of the modern axiomatic approach implied by the works of Cantor and of Hilbert; [5] the systematic study of factorization in general domains, conducted by Emmy Noether, Emil Artin and their students.

In 1895, the first edition of Heinrich Weber's *Lehrbuch der Algebra* [1895] was published. This book summarized the achievements attained in the body of knowledge of this discipline, and at the same time it embodied more than any other one the spirit of the classical, nineteenth century image of algebra. If one compares it with its most noteworthy predecessors —Serret's *Cours d'algèbre supérieure* [1849] and Jordan's *Traité des substitutions et des équations algébriques* [1870]— one discovers many important additions and innovations at the level of the body of knowledge, but the image of algebra it embodies remains essentially the same one: algebra as the discipline of polynomial equations and polynomial forms. Abstract concepts such as groups, in so far as they appear, are subordinate to the main classical tasks of algebra. And, most important, all the results are based on the assumption of a thorough knowledge of the basic properties of the systems of rational and real numbers: these systems are conceived as conceptually prior to algebra.

The first two decades of the twentieth century were ripe with new ideas, and towards the end of the 1920s, one finds a growing number of works that can be identified with recently consolidated algebraic theories. These works were usually aimed at investigating the properties of abstractly defined mathematical entities, as the focus of interest in algebraic research: groups, fields, ideals, rings, and others. Van der Waerden's *Moderne Algebra* appeared —like many other important textbooks— at a time when the need was felt for a comprehensive synthesis of what had been achieved since the publication of its predecessor, in this case Weber's book. *Moderne Algebra* presented ideas that had been developed earlier by Emmy Noether and Artin —whose courses van der Waerden had recently attended in Göttingen and Hamburg, respectively— and also by other algebraists, such as Ernst Steinitz, whose works van der Waerden had also studied under their guidance. Van der Waerden masterly incorporated a great deal of the important innovations accumulated over the last years at the level of the body of algebraic knowledge. But the originality and importance of his contribution is best recognized by focusing on the totally new way of conceiving the discipline it put forward. Van der Waerden presented systematically those mathematical branches then related with algebra, deriving all the relevant results from a single, unified perspective, and using similar concepts and methods for all those branches. This original perspective, which turned out to be enormously fruitful over the next decades of research —and not only in algebra, but in mathematics at large— is what I will call here the structural image of algebra

The Structural Image of Algebra

The approach put forward in van der Waerden's textbook is based on the recognition that a certain set of notions (i.e., groups, ideals, rings, fields, etc.) are in fact individual realizations of one and the same underlying idea, namely, the general idea of an algebraic structure, and that the aim of research in algebra is the full elucidation of the those notions. None of these notions, to be sure, appeared as such for the first time in this book. Groups, for instance, had been introduced since the mid-nineteenth century as a main tool in the study of the theory of polynomial equations, and they could already be found in mainstream textbooks on algebra as early as 1866 (i.e., in the third edition of Serret's *Cours*). Ideals and fields, in their turn, had been introduced in 1871 by Dedekind in his elaboration of Kummer's factorization theory of algebraic numbers. But the unified treatment they were accorded in *Moderne Algebra*, the single methodological approach adopted to define and study each and all of them, and the compelling, new picture it provided of a variety of domains, that were formerly seen as only vaguely related, all these implied a striking and original innovation.

The aim and contents of van der Waerden's book may be roughly described as an attempt to define the diverse algebraic domains and to elucidate their structure in detail. Of course, this characterization may appear banal and obvious to any mathematician nowadays, but by comparison to the earlier conception of algebra it is certainly not. At any rate, it raises two basic questions: [1] how is any algebraic domain defined? and [2], what does the complete elucidation of the structure of an algebraic domain entail? Algebraic domains are defined in *Moderne Algebra* in two different ways: either by endowing a non-empty set with one or more abstractly defined operations, or by taking an existing algebraic domain and constructing a new one over it, by means of a well-specified procedure (field of quotients of an integral domain, rings of polynomials of a given ring, extension of a field, etc.). In order to answer the second question, it is necessary to examine what van der Waerden actually did in his book for the various domains investigated. This description yields, in fact, an account of the essence of the structural image of algebra.

Basic to van der Waerden's analysis is the recurrent use of several fundamental concepts and questions. Among the most salient recurring concepts, one finds isomorphisms, homomorphisms, residue classes, composition series and direct products, etc. None of these concepts, however, is defined in a general fashion so as to be *a-priori* available for each of the particular algebraic systems. Isomorphisms, for instance, are defined separately for groups and for rings and fields, and van der Waerden showed in each case that the relation "being isomorphic to" is reflexive, transitive and symmetric. Thus the important notion, constitutive of the structural image of algebra, that two isomorphic constructs actually represent one and the same mathematical entity, appears prominently, yet only implicitly, in this textbook.

Perhaps one of the most fundamental innovations implied by van der Waerden's textbook is a redefinition of the conceptual hierarchy underlying the discipline which, in its classical image, generally granted priority to the foundations of the various number systems over algebra. In the opening chapters van der Waerden defined the natural numbers through a cursory review of Peano's axioms. He then extended them to the integers through the (informally presented) construction of pairs of natural numbers. But here he stopped: the rational and the real numbers he defined only in later chapters, and then with strict reference to their algebraic properties alone. Rational and real numbers have no conceptual priority in this book over, say, polynomials. Rather, they are defined as particular cases of abstract algebraic constructs. Thus, in Chapter III, van der Waerden introduced the concept of a field of fractions for integral domains in general, and he obtained the rational numbers as a particular case of this kind of construction, namely, as the field of quotients of the ring of integers.

Van der Waerden's definition of the system of real numbers in purely algebraic terms was somewhat more complicate. It was based on the concept of a "real field," recently elaborated by Artin and Schreier, whose seminars van der Waerden had attended in Hamburg. It is significant, however, that while van der Waerden was able to use this purely algebraic characterization, he did not give up completely the nineteenth-century, classical, image of algebra, and added a standard "analytic" definition of the real numbers using Cauchy sequences (and adopting the term originally introduced by Cantor to designate them: "fundamental sequences"). We thus find a single section in his book where ε–δ arguments appear, whereas in earlier textbooks it was standard to devote several chapters to discuss analytic questions concerning polynomials, including interpolation and approximation techniques. After van der Waerden, textbooks of algebra tended to exclude arguments of this sort. But in 1930, van der Waerden's attitude concerning the question whether a book on algebra can completely do without such considerations, even though they are not directly needed for the theories developed in it, still reflected his ties to the existing algebraic traditions.

This possibility of concentrating on determined features of a classical mathematical entity, while ignoring those that are considered momentarily irrelevant, is, no doubt, an important aspect of the study of algebraic "structures" and one of the most striking innovations reflected in the book. Naturally, this possibility was afforded by the abstract formulation of concepts, but the mere availability of such a formulation was not in itself a compelling reason to bring about a real change in the images of algebra. Weber, for instance, had been fully aware of the abstract formulation of groups and fields when he published his textbook; in fact, it was Weber himself who had published, back in 1893, the first article where the two concepts appeared as closely related from the point of view of their axiomatic definitions (Weber 1893). And yet, Weber's conception of algebra was absolutely subordinated to the system of real numbers, and thus he considered it necessary to begin any discussion on algebra by elucidating all of the properties of that system.

The task of finding the real and complex roots of an algebraic equation —the classical main core of algebra— was relegated in van der Waerden's book for the first time to a subsidiary role. Three short sections in his chapter on Galois theory deal with this specific application of the theory and they assume no previous knowledge of the properties of real numbers. In this way, two central concepts of classical algebra (rational and real numbers) are presented here merely as final products of a series of successive algebraic constructs, the "structure" of which was gradually elucidated. On the other hand, additional, non-algebraic properties such as continuity, density, were not considered at all by van der Waerden.

Within this conceptual setting, and together with the concepts which are repeatedly used in the book, one finds several problems that van der Waerden discussed when studying each algebraic system. Thus, for instance, factorization: for every algebraic domain, some specific kind of sub-domain is usually considered, which may be taken as playing a role similar to the role that prime numbers play for the system of integers. Accordingly, the elucidation of the structure of a domain involves the elucidation of the relationship between a given element of the domain and its "prime elements." This question is, in fact, a specific aspect of the broader question, central to the structural concerns of the book, of the relation between given domains and the systems of their subdomains. Galois theory, for instance, is also part of this question inasmuch as it "is concerned with the finite separable extensions of a field *K*... (and it) establishes a relationship between the extension fields of *K*, which are contained in a given normal field, and the subgroups of a certain finite group" [Van der Waerden 1930, p. 153].

An additional typical question that appears throughout the book concerns those new algebraic systems which are obtained from existing ones by performing certain standard constructs on them (e.g., fields of fractions of an integral domain, etc.). The question naturally arises to what extent specific properties of the original domains are reflected in the new ones. For example, if a given ring is an integral domain, the same is true for its ring of polynomials; if a given ring satisfies the base condition (namely, that every ideal in it has a finite generating set) then any quotient ring of that ring and also the polynomial ring associated with it satisfy the same condition; and so on. One may also ask which properties are passed over from a given algebraic domain to its subsystems or to its quotient systems. These kinds of questions are exemplified in van der Waerden's study of the structure of fields, which was modeled directly after the seminal work of Steinitz, who in 1910 had advanced a completely new approach for the treatment of abstract fields as an issue of intrinsic interest [Steinitz 1910]. Van der Waerden stressed the basic role played by "prime fields" as building blocks of the theory, and he claimed that after all the properties inherited from the prime field by its extension are known, then the structure of all fields is also known. Similar to Steinitz's had been the pioneering, but much less known, work of Abraham Fraenkel on abstract rings, a work that van der Waerden certainly knew from Emmy Noether.

Now, all these problems and all the concepts to which they apply are formulated in *Moderne Algebra* in terms of the "modern axiomatic method." It has been very common to consider this method as the most essential feature of the structural approach. And it is indeed the case that van der Waerden presented all algebraic theories in a completely abstract, axiomatic, way, like no other comprehensive exposition of algebra had ever done before. However, identifying the innovative character of the book with its use of the axiomatic method alone would be quite misleading. The real innovation implied by the book can only be understood by considering the particular way in which van der Waerden exploited the advantages of the axiomatic method in conjunction with all other components of the structural image of algebra which I mentioned above. The crucial issue concerns the clear recognition that all those concepts that deserve being axiomatically defined and studied in the framework of algebra are in fact different varieties of a same species ("varieties" and "species" understood here in a "biological", not mathematical term), namely, different kinds of algebraic structures. The central disciplinary concern of algebra becomes, in this conception, the systematic study of those different varieties through a common approach. In fact, this fundamental recognition appears in *Moderne Algebra* not only implicitly, but rather explicitly and even didactically epitomized in the *Leitfaden* that appears in the introduction to the book, and that pictures the hierarchical, structural interrelation between the various concepts investigated in the book.

Obviously, the new image of algebra presented by van der Waerden was enabled by the current state of development of the body of algebraic knowledge. However, the important point is that the former was *not* a *necessary* outcome of the latter, but rather an independent development of intrinsic value. This becomes clear when we notice that parallel to van der Waerden's, several other textbooks on algebra were published which also contained most of the latest developments in the body of knowledge, but which essentially preserved the classical image of algebra. Examples of these are Dickson's *Modern Algebraic Theories* [1926], Hasse's *Höhere Algebra* [1926] and Haupt's *Einführung in die Algebra* [1929]. But perhaps the most interesting example in this direction is provided by Fricke's *Lehrbuch der Algebra*, published in 1924, with the revealing sub-title: "*Verfasst mit Benutzung vom Heinrich Webers gleichnamigem Buche*". All these books were by no means of secondary importance. Dickson's book, for instance, became after publication the most advanced algebra text available in the USA and it was not until 1941 that a new one, better adapted to recent developments of algebra and closer to the spirit of *Moderne Algebra*, was published in the USA: *A Survey of Modern Algebra* by Garrett Birkhoff and Saunders Mac Lane [1941].

Hilbert and Structures

Having defined more precisely what the structural image of mathematics is, we can now more easily understand what was Hilbert's specific contribution to its rise and development. Hilbert's works in three domains are relevant to this account: invariant theory, algebraic number theory and axiomatics.

Hilbert's first important achievement in invariant theory concerns the proof of the generalized finite basis theorem [Hilbert 1889]. Hilbert's proof that any given system of invariants in *n* variables has a finite basis came to solve a main open problem in the discipline, after twenty years of unsuccessful efforts by its leading practitioners. But a more general significance of his achievement was that it implicitly asserted the legitimacy of a new kind of proof which was initially rejected by many, namely, a proof of existence by contradiction. Hilbert also developed arguments that amounted to an implicit use of the ascending chain condition, an idea which Emmy Noether explicitly formulated in her own later, seminal work on factorization in abstract rings [Hilbert 1890].

Hilbert's treatment of invariants was based on a conceptual analogy between the basic problems and the conceptual tools currently available in research on number theory and on the theory of polynomials, yet its setting remained the 'concrete' one of the theory of polynomials over the field of complex numbers. The various systems of numbers (complex, rationals, etc.) appear in Hilbert's work on invariants as the basic mathematical entities, while systems of polynomial forms are subsidiary constructs. The properties of the latter are always deduced from those of the former. Hilbert added with his work in this domain important ingredients to the future elaboration of the structural image of algebra, but he himself applied those ingredients without essentially changing the accepted images of the discipline.

The same can be said of Hilbert's work on the theory of algebraic number fields, his main domain of research between 1892 and 1899. In 1897 Hilbert published the influential *Zahlbericht* [1897], in which he presented and further developed the many recent achievements of this discipline since the times of Kummer. Dedekind and Kronecker were responsible for the bulk of these achievements, but they had addressed the whole issue from diverging points of view, whose basic orientations can be roughly characterized, respectively, as conceptual vs. algorithmic. Hilbert's adoption of Dedekind's point of view as his own leading one, together with the deep influence that his work exerted over the next generations of researchers, explains to a large extent the relative dominance of this approach over Kronecker's during the following decades.

Although this "conceptual" approach is closely connected the basic spirit of the structural approach, it is important not to confound the two. In his own work on algebraic number fields, and in his joint work with Weber on algebraic functions, Dedekind had treated groups, fields and ideals within a single, common formulation, yet these concepts played completely different roles in the theories where they appeared. Hilbert followed Dedekind in this distinction as well. For Dedekind, fields (not abstract fields, but rather fields of numbers) were the main *subject-matter* of research in both Galois theory and algebraic number theory. Groups, on the contrary, were only a *tool* — an effective and powerful one, it is true, yet subsidiary to the main concern of that research. In the *Zahlbericht*, while fields were ascribed a role similar to that conceded by Dedekind in his works, groups were barely mentioned at all, since results of group theory are much less needed here than, say, in Galois theory.

Hilbert also defined a ring as a system of algebraic integers in a given field, which is closed under addition, subtraction and product. An ideal of a ring is any system of algebraic integers belonging to the ring, such that any linear combination of them (with coefficients in the ring) belongs itself to the ideal. But Hilbert never describe a ring as a group endowed with an additional operation, or as a field whose division fails to satisfy a certain property. Neither did he present ideals of fields as a distinguished kind of ring. Hilbert's ideals are always ideals in fields of numbers. In spite of his interest in the theory of polynomials and his acquaintance with the main problems of this discipline, Hilbert never attempted —in the *Zahlbericht* or elsewhere— to use ideals as an abstract tool allowing a unified analysis of factorization in fields of numbers and in systems of polynomials. This step, crucial for the later unification of the two branches under the abstract theory of rings, will be taken only more than twenty years later by Emmy Noether. Obviously, the absence of such a step in Hilbert's work needs not be explained in terms of a lack of technical capabilities, but rather in terms of motivations, i.e., as an indication of the nature of his images of algebra, to which the idea of algebraic structures as an organizing principle (in the sense explained above for the case of van der Waerden) was foreign.

Finally, I mention Hilbert's axiomatic approach and its connection with the structural image of algebra. Hilbert conceived the axiomatic analysis of theories as a necessary stage in their latest stages of development, and *never as a starting point*. This conception was fully manifest for the first time in 1899 in *Die Grundlagen der Geometrie*. Geometry was the foremost example of a fully-fledged scientific discipline, amenable to a comprehensive and thorough axiomatic analysis. Until the 1920, when he dedicated much effort to study the foundations of arithmetic using axiomatic analysis, Hilbert occasionally discussed in his lectures arithmetic and logic in axiomatic terms, but whenever he dedicated some real effort to seriously discuss the application of this method beyond geometry and its deeper significance, it was always in connection with *physical* theories. What he never did, at any rate, was to discuss any of the notions later associated to algebraic structures — groups, fields, rings, ideals— in axiomatic terms.[1]

Standard axiomatic definitions of such algebraic concepts, and their analysis following the guidelines stipulated by Hilbert, were advanced over the first decade of the century in the USA, especially in the circle associated with Eliakim Hastings Moore at the University of Chicago. The activity of these American mathematicians, which came to be known as "postulational analysis",[2] was triggered by their attentive study of the *Grundlagen*, but it took the methods developed in this book in a direction slightly different from that originally envisaged by Hilbert. Whereas for Hilbert it was the 'concrete', elaborate mathematical and physical theories which mattered, and the axioms were only a tool to enhance our understanding of them, for Moore and his collaborators it was *the system of postulates as such* that became an issue of inherent mathematical interest. Moore asked how these systems can be formulated in the most convenient and succinct way from the deductive point of view, without caring much whether this axioms actually convey any

1. On Hilbert see Corry [1996, Chap. 3; 1996a; 1997].
2. On the American School of Postulational Analysis see Corry [1996, pp. 173-183].

intuitive, geometrical meaning [Moore 1902, 1902a]. His questions could equally be applied to a system of postulates defining Euclidean geometry, groups or fields, or, in fact, any arbitrarily defined system. No doubt, the point of view and the techniques introduced by Hilbert could in principle have been applied in this more general context as well, but Hilbert's actual motivations did never contemplate the realization of that possibility. Neither he nor his students in Göttingen published any work in that direction. Thus, Moore's perspective implied a subtle shifting of the focus of interest away from geometry and the other 'concrete' entities of classical, nineteenth century mathematics, and towards the study of a new kind of autonomous mathematical domain: the analysis of systems of postulates.

The accumulated experience of research on postulational analysis eventually brought about an increased understanding of the essence of postulational systems as an object of intrinsic mathematical interest. In the long run, it also had a great influence on the development of mathematical logic in America, since it led to the creation of model theory. But as a more direct by-product of its very activity, postulational analysis also provided a collection of standard axiomatic systems that were to become universally adopted in each of the disciplines considered. The works of Steinitz, Emmy Noether and Artin, and whatever van der Waerden took from them, relied heavily on the use of these systems. They are only indirect products, however, of Hilbert's conceptions of the axiomatic method in mathematics.

A further perspective from which to analyze the relation of Hilbert with the structural image of algebra is provided by the works of his many students and collaborators. Of the sixty-eight (!) doctoral dissertations that Hilbert supervised, only four deal with issues directly or indirectly related to Hilbert's first domain of research: invariant theory. Not one of them deals with problems connected with the theory of factorization of polynomials, although at that same time important works were being published by other mathematicians —such as Lasker [1905] and Macaulay [1913]— which elaborated on Hilbert's own. Nor is there any dissertation dealing with topics that later came to be connected with modern algebra —such as abstract fields, or the theory of groups in any of its manifestations— and that knew at the time intense activity throughout the mathematical world. Moreover, a close look at the contents of the dissertations shows that Hilbert's students did not depart from the master's images of algebra. Thus, for instance, in the work of the Swiss mathematician Karl Rudolf Fueter [1905; 1907] on algebraic number fields, one finds the same functional separation between fields, on the one hand, and groups, on the other hand, that characterized the earlier works of Dedekind or Hilbert.

Likewise worthy of mention here is that, although five among the twenty-three problems that Hilbert included in his 1900 list can be considered in some sense as belonging to algebra in the nineteenth-century sense of the word, none of them deals with problems connected with more modern algebraic concerns, and in particular not with the theory of groups [Hilbert 1901].

But on the other hand, if one looks at the works of Emmy Noether, for instance, one understands that a direct elaboration of many ideas that had been laid down by Hilbert led very soon to develop the basic elements of the structural image of algebra. For lack of space, we cannot discuss here the works of Emmy Noether (or of Artin, for that matter) to see how she elaborated those ideas and finally crafted the new image of algebra that van der Waerden then so masterfully cast into his textbook, applying it systematically and uniformly to so many domains of algebra.

What might have been, then, Hilbert's actual attitude towards the structural image of algebra—one may ask— as it consolidated around 1930, after he himself had already gone into retirement? In particular, what was Hilbert's attitude towards the treatment accorded to the discipline in van der Waerden's *Moderne Algebra*? What did Hilbert think, for instance, of van der Waerden's definition of the fields of rational and real numbers — those mathematical entities laying at the heart of Hilbert's approach to invariants, to algebraic number theory and to geometry— as particular cases of more general, abstractly defined algebraic constructs? Could in his view the conceptual order be turned around so that the system of real numbers be dependent on the results of algebra rather than being the basis for it? Unfortunately, we have no direct evidence to answer these questions, yet from what I have said here, it seems fair to conjecture that such answers would be, at best, not really straightforward and at least ambiguous.

We can thus summarize Hilbert's contribution to the rise of the structural image of algebra as follows: although Hilbert's works contain most of the materials needed for elaborating the structural image of algebra, Hilbert himself neither put forward this image in its completed form in his works, nor suggested that it, or something similar to it, should be adopted in algebra.

Bourbaki and Structures

No name has been more widely associated with the notion of structure in modern mathematics than the name of Nicolas Bourbaki. Under this pseudonym, a group of leading French mathematicians began in 1939 the publication of an influential, multi-volume treatise that eventually covered the main areas of pure mathematics. The title of the treatise was *Eléments de mathématique* [1939-] and its subtitle, "The Fundamental Structures of Analysis".

The initial aim of the group was to publish an up-to-date treatise of mathematical analysis, suitable both as a textbook for students and as reference for researchers, and adapted to the latest advances and the current needs of the discipline. The applicability of the topics discussed and their usefulness for physicists and engineers was a main concern during the early meetings of Bourbaki. However, given the more abstract inclinations of certain members and the way in which the writing of the chapters evolved over the first years of activities, something completely different from the classical French textbooks on analysis began to emerge. In fact, the titles of the various volumes of the treatise reflect one of Bourbaki's most immediate innovations, namely the departure from the classical view according to which the main branches of mathematics are geometry, arithmetic and algebra, and analysis. In Bourbaki's presentation, the "fundamental structures of analysis" were: set theory, algebra, general topology, functions of a real variable, topological vector spaces; integration, Lie groups and Lie algebras; commutative algebra, spectral theories, differential and analytic manifolds, and (later) homological algebra.

A main source of inspiration for the work of the group was the model put forward by van der Waerden in his textbook on algebra. As a matter of fact, the whole Bourbaki project in its mature form may be defined as an attempt to extend to the whole of mathematics the disciplinary image put forward in *Moderne Algebra*, namely, as a recognition that different mathematical branches such as algebra, topology, functional analysis, etc., are all individual materializations of one and the same underlying, general idea, i.e., the idea of a mathematical structure. Bourbaki attempted to present a unified an comprehensive picture of what they saw as the main core of mathematics, using a standard system of notation, addressing similar questions in the various fields investigated, and using similar conceptual tools and methods across apparently distant mathematical domains.

Bourbaki gathered an exceptional collection of leading mathematicians with perhaps relatively homogenous, but nevertheless distinctive and certainly well-consolidated, opinions about what mathematics is and should be. Thus, one has to exercise special care when analyzing the work of Bourbaki, and especially when advancing any claim about the images of mathematics associated with their work. In the very reduced space available here, this problem is only more pressing. Still, I will attempt to describe in general lines what can be taken to the group's collective conceptions, although to a large extent I will stress those articulated by Jean Dieudonné, the group's most outspoken member.

In 1950 Dieudonné published, signing with the name of Bourbaki, an article that came to be identified as the group's manifesto, "The Architecture of Mathematics". Dieudonné raised the question of the unity of mathematics, given the unprecedented growth and diversification of knowledge in this discipline over the preceding decades. Mathematics is a strongly unified branch of knowledge in spite of appearances, he claimed, and the basis of this unity is the use of the axiomatic method. Mathematics should be seen, he added, as a hierarchy of structures at the heart of which lie the so called "mother structures":

> At the center of our universe are found the great types of structures, ... they might be called the mother structures ... Beyond this first nucleus, appear the structures which might be called multiple structures. They involve two or more of the great mother-structures not in simple juxtaposition (which would not produce anything new) but combined organically by one or more axioms which set up a connection between them... Farther along we come finally to the theories properly called particular. In these the elements of the sets under consideration, which in the general structures have remained entirely indeterminate, obtain a more definitely characterized individuality. [Bourbaki 1950, 228-29]

This characterization of mathematics was inspired, as I said, by van der Waerden's presentation of algebra, but it went much further than the latter in many respects. For one thing, what van der Waerden had left at the implicit level, the centrality of the hierarchy of structures, became explicit and constitutive for Bourbaki. For another, Bourbaki did not limit itself to promote a conception of mathematics based on a notion that, fruitful and suggestive as it might be, could not be elucidated in strictly mathematical terms. Thus, the fourth chapter of Bourbaki's book on the theory of sets [1968], defines the concept of *structure* (I use italics to denote this particular, technical use of the term), which is assumed to provide the conceptual foundation on which the whole edifice of mathematics as presented in the *Eléments* is supposedly built.

The central notion of structure, then, has a double meaning in Bourbaki's mathematical discourse. On the one hand, it relates to a general organizational scheme, that conceives the whole discipline in hierarchical terms. Like in van der Waerden's book, one does not need to have a definite definition of this notion in order to be able to grasp its meaning and, more importantly, to put it to use. Rather, one has to master the various mathematical branches presented in the treatise, and one can thus see the force of the whole structural approach and how it can be applied to additional topics not considered in the various volumes of the treatise. On the other hand, one has a formal mathematical concept, *structures*, about which Bourbaki proved some results in the relevant chapter of the treatise. Surprisingly, however, this concept plays no role at all in any of the other books of the treatise, and those few places where it is invoked, only help us understanding how little relevant it is for the issues covered by Bourbaki.[1]

But the truly astonishing point concerning this duality is the fact that many pronouncements by Bourbaki (and especially by Dieudonné) concerning the centrality of structures have been often accepted *as if* they referred to a strictly mathematical notion (such as *structures*), even when they referred, in fact, to the more general sense of the term. Sometimes the blurring of the two meanings is as explicit as it can be, and some other times it is subtler. But the confusion it has brought about can be traced in many places. The mother structures mentioned in the above quotation, taken from the "Architecture" manifesto, is an interesting point in case. The description of the mother structures and their central role in the whole edifice of mathematics is not an integral part of the formal, axiomatic, theory of *structures* developed by Bourbaki. The classification of structures according to this scheme is mentioned several times in the book on set theory, but only as an illustration appearing in scattered examples. Many assertions that were suggested either explicitly or implicitly by Bourbaki or by its individual members —i.e., that all of mathematical research can be understood as research on structures, that there are mother structures bearing a special significance for mathematics, that there are exactly three, and that these three mother structures are precisely the algebraic-, order- and topological-structures (or *structures*)— all this is by no means a logical consequence of the axioms defining a *structure*. The notion of mother structures and the picture of mathematics as a hierarchy of structures are not results obtained within a mathematical theory of any kind. Rather, they belong strictly to Bourbaki's non-formal images of mathematics; they appear in non-technical, popular, articles, such as in the above quoted passage, or in the myth that arose around Bourbaki. Still, it is not uncommon that pronouncements about the mother structures are accepted, implicitly or explicitly, as results obtained in the framework of a standard mathematical discipline.

A second remarkable manifestation of the ambiguity inherent in Bourbaki's use of the term "structure" concerns its historical dimensions, and therefore it is particularly interesting for the present account. Bourbaki, and in particular Dieudonné, dedicated considerable efforts to historical writing, that produced an influential historiographical body of marked Whiggish spirit [Bourbaki 1969; Dieudonné 1978, 1985, 1987]. The idea of a mathematical structure appears in it as the culminating, comprehensive and definitive

1. For a detailed discussion of this issue, see Corry [1996, Chap. 7].

stage of historical development of the whole discipline: the structural presentation of mathematics as embodied in Bourbaki's treatise was here to stay. And in saying so, reference was made not only to the image of mathematical knowledge put forward in the book; implicitly, and often also explicitly, Dieudonné asserted that mathematics had come now to be the discipline dealing not only with structures, but more specifically with Bourbaki's *structures* [Dieudonné 1979]. As a matter of fact, Bourbaki's work did imply many important contributions to twentieth century mathematics, but the concept of *structure* is certainly not among them. Bourbaki's structural image of mathematics, on the contrary, was a main force in shaping mathematical activity all over the world for several decades after its emergence.

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