Chapter 7 Nicolas Bourbaki: Theory of *Structures*

The widespread identification of contemporary mathematics with the idea of structure has often been associated with the identification of the structural trend in mathematics with the name of Nicolas Bourbaki. Fields medalist René Thom, in a famous polemical article concerning modern trends in mathematical education, asserted that Bourbaki "undertook the monumental task of reorganizing mathematics in terms of basic structural components."[1] Thom further claimed that:

> Contemporary mathematicians, steeped in the ideas of Bourbaki, have had the natural tendency to introduce into secondary and university courses the algebraic theories and structures that have been so useful in their own work and that are uppermost in the mathematical thought of today. (Thom 1971, 695)

The identification of Bourbaki with modern, structural mathematics is not always as explicitly formulated as in Thom's quotation, but it has interestingly been manifest in many other ways. Thus for instance, during the years 1956 and 1957 in Paris the "Association des professeurs de mathématiques de l'enseignement public" organized in Paris two cycles of lectures for its members. The lectures, meant to present high-school teachers with an up-to-date picture of the discipline, were delivered by such leading French mathematicians as Henri Cartan, Jacques Dixmier, Roger Godement, Jean-Pierre Serre, and several others. They were later collected into a volume entitled *Algebraic Structures and Topological Structures*.[2] Thus we have the first component of the two-fold identification: "up-to-date mathematics = structural mathematics." The second component—"Bourbaki's structures = mathematical structures"—was not explicitly articulated therein, yet the editors made this second identification plain and clear by choosing to conclude the book with a motto

1. Thom 1971, 699.
2. Cartan et al., 1958.

quoted from a famous article in which Bourbaki described "The Architecture of Mathematics" in terms of mathematical structures, as follows:

> From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures; and it so happens—without our knowing how—that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation.[3]

Historians of mathematics have also accepted very often this identification of "mathematical structures" with the name of Bourbaki. An explicit instance of this appears in Hans Wussing's well-known account of the rise of the concept of abstract groups. In the introduction to his book Wussing wrote:

> The conscious tendency to think in terms of structures has even produced its own characterization of mathematics. An extreme characterization of this kind, advanced by ... Nicolas Bourbaki ... sees mathematics as a hierarchy of structures. (Wussing 1984, 15)

And of course, the identification of mathematical structures with Bourbaki has had a marked influence *outside* mathematics. The best-known example of this is found in the work of Jean Piaget. In his widely-read general exposition of the central ideas of structuralism, one of the chapters discusses the "new structuralist view of mathematics." In this context Piaget mentions Klein's Erlangen Program, because of its successful use of the concept of group structures, as the first victory of the new approach. "However", he adds:

> ... in the eyes of contemporary structuralist mathematicians, like the Bourbaki, the Erlangen Program amounts to only a partial victory for structuralism, since they want to subordinate all mathematics, not just geometry, to the idea of structure. (Piaget 1971, 28)

In the same place Piaget also pointed out the close correspondence between Bourbaki's so called "mother structures" (i.e. algebraic structures, order structures and topological structures) and the first operations through which the child interacts with the world.[4]

However, as was already seen, the term "mathematical structure" has been used and understood in sharply divergent ways by different authors. The question therefore arises, what is the meaning attributed to the term by those authors identifying it with the work of Bourbaki. Moreover, what did Bour-

3. We quote here from the English version, Bourbaki 1950, 231. This article of Bourbaki is discussed in greater detail below.

baki actually mean by "mathematical structure", and how do mathematical structures appear in Bourbaki's actual mathematical work?

The present chapter discusses the place of the idea of mathematical structure in Bourbaki's work. As already mentioned, Bourbaki's concept of *structure* was meant to provide a unifying framework for all the domains covered by Bourbaki's work. Ore's **structures**—discussed in the previous chapter—had arisen within, and remained focused on, the relatively more limited framework of abstract algebra. Thus Bourbaki's works, and in particular the concept of *structure*, relate to a much larger territory of pure mathematics concerning both motivations and intended scope of application. Bourbaki's concept of *structure* has rarely been explicitly considered as what it really is, namely, *one among several formal attempts* to elucidate the non-formal idea of mathematical structure, and in fact, a rather unsuccessful one at that. It will be seen that there is a wide gap separating the significance of Bourbaki's overall contribution to contemporary mathematics from the significance of this one particular component of Bourbaki's work, namely, the theory of *structures*. Thus, the present chapter examines Bourbaki's concept of *structure*, its relation to Bourbaki's work at large, and the degree of its success in formally elucidating the idea of "mathematical structure."

7.1 The Myth

Nicolas Bourbaki was the pseudonym adopted during the 1930s by a group of young French mathematicians who undertook the collective writing of an up-to-date treatise of mathematical analysis, suitable both as a textbook for students and as reference for researchers, and adapted to the latest advances and the current needs of the discipline. These mathematicians were initially motivated by an increasing dissatisfaction with the texts then traditionally used in their country for courses in analysis,[5] which were based on the university lectures of the older French masters: Jacques Hadamard (1865-1963), Emile Picard (1856-1941), Edouard Goursat (1858-1936), and others.[6]

4. See also Piaget 1973, 84: "From the level of concrete operations—at about 7/8 years—another interesting convergence may be found, that is the elementary equivalence of the three "mother structures" discovered by Bourbaki, and this in itself shows the "natural" character of these structures." See also Gauthier 1969; 1976. The issue of the "mother structures" is further elaborated below. See Aubin 1997, for an illuminating analysis of the close conceptual relationship between Bourbaki, Levi-Strauss and French "potential literature".

5. See Dieudonné 1970, 136; Weil 1992, 99-100.

They also felt that French mathematical research was lagging far behind that of other countries,[7] especially Germany,[8] and they sought to provide a fresh perspective from which to reinvigorate local mathematical activity.

The would-be members of Bourbaki met for the first time to discuss the project in the end of 1934. They stated as the goal of their joint undertaking "to define for 25 years the syllabus for the certificate in differential and integral calculus by writing, collectively, a treatise on analysis. Of course, this treatise will be as modern as possible."[9] The names involved in the project, as well as the details regarding the scope and contents of the treatise were to fluctuate many times in the following decades, but some of the essentials of Bourbaki's self-identity are already condensed in this last quotation: collective work, wide-ranging coverage of the hard-core of mathematics, modern approach.

The first actual Bourbaki congress took place in 1935.[10] The founding members of the group included Henri Cartan (1904-), Claude Chevalley (1909-1984), Jean Coulomb, Jean Delsarte (1903-1968), Jean Dieudonné (1906-1994), Charles Ehresmann (1905-1979), Szolem Mandelbrojt (1899-1983), René de Possel and André Weil (1906-1998).[11] Over the years, many younger, prominent mathematicians joined the group, while the elder members were supposed to quit at the age of fifty. From the second-generation Bourbaki members the following are among the most prominent: Samuel

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6. On the French tradition of *Cours d'analyse*, based on lectures delivered by leading mathematicians see Beaulieu 1993, 29-30.

7. Thus, Weil 1992, 120, described the situation, as he saw it, in the following words: "At that time [1937], scientific life in France was dominated by two or three coteries of academicians, some of whom were visibly driven more by their appetite for power than by a devotion to science. This situation, along with the hecatomb of 1914-1918 which had slaughtered virtually an entire generation, had had a disastrous effect on the level of research in France. During my visits abroad, and particularly in the United States, my contact with many truly distinguished scholars had opened my eyes to the discouraging state of scientific scholarship in France." Weil 1938 contains a general assessment of the state of scientific activity in France during the 1930s. See also Dieudonné 1970, 136.

8. Israel 1977, 42-43 analyses the differences between the German and the French mathematical schools at the time, based on sociological considerations.

9. Quoted from Beaulieu 1993, 28.

10. Cf. Weil 1978, 537-538.

11. Paul Dubreil (1904-1994) and Jean Leray (1906-1998) also attended the early meetings. See Beaulieu 1994, 243 Guedj 1985, 8; Weil 1992, 100 ff. For an account of the participants in the meetings preceding the actual work of the group see Beaulieu 1993, 28-31.

Eilenberg (1913-1998), Alexander Grothendieck (1928-), Pierre Samuel (1921-), Jean Pierre Serre (1926-). All of these mathematicians were pursuing separately their own individual work (usually being among the leading researchers of their respective disciplines), while the activities of Bourbaki absorbed part of their time and effort.

What was initially projected as a modern textbook for a course of analysis eventually evolved into a multi-volume treatise entitled *Eléments de Mathématique*, each volume of which was meant to contain a comprehensive exposition of a different mathematical discipline. Each chapter and each volume of Bourbaki's treatise was the outcome of arduous collective work. Members of the group used to meet from time to time in different places around France. At each meeting, individual members were commissioned to produce drafts of the different chapters. The drafts were then subjected to harsh criticism by the other members, and then reassigned for revision. Only after several drafts had been written and criticized was the final document ready for publication.[12] Minutes of meetings were taken and circulated among members of the group in the form of an internal bulletin called "La Tribu." Although the contents of the issues of "La Tribu" abound with personal jokes, obscure references and slangy expressions which sometimes hinder their understanding, they provide a very useful source for the historian researching the development of Bourbaki's ideas.[13]

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12. Bourbaki's mechanism of collective writing has been documented in several places. See, e.g., Boas 1970, 351; Cartan 1980, 179; Dieudonné 1970, 141; Weil 1992, 105. See also the vivid description of Mac Lane (1988, 337): "Debate at Bourbaki could be vigorous. For example, in one such meeting (about 1952) a text on homological algebra was under consideration. Cartan observed that it repeated three times the phrase 'kernel equal image' and proposed the use there of the exact sequence terminology. A. Weil objected violently, apparently on the grounds that just saying 'exact sequence' did not convey an understanding as to why that kernel was exactly this image." On the introduction of this term, see below the opening passages of § 8.2.

13. According to Weil (1992, 100) since the early meetings of Bourbaki an archive was established, of which Delsarte was first in charge. Later it was kept in Nancy and later on in Paris. Some years ago, the "Association des Collaborateurs de Nicolas Bourbaki" was established at the Ecole Normale Supérieure, in Paris. An archive containing relevant documents, probably including many copies of "La Tribu" was created. Unfortunately, it has yet to be opened to the public. Those issues of "La Tribu" quoted in the present article belong to personal collections. Professor Andrée Ch. Ehresmann kindly allowed me to read and quote from documents belonging to her late husband, Professor Charles Ehresmann. This includes volumes of "La Tribu" from 1948 to 1952. Other quotations here are taken from the personal collections of Chevalley and Szolem Mandelbrojt, as they appear in an appendix to Friedmann 1975.

In the decades following the founding of the group, Bourbaki's books became classic in many areas of pure mathematics in which the concepts and main problems, the nomenclature and the peculiar style introduced by Bourbaki were adopted as standard. Bourbaki's actual influence on the last fifty years of mathematical activity (research, teaching, publishing, resources distribution) has been enormously significant.[14] However, even now that the Bourbaki phenomenon is receding into the past, a fair historical evaluation of Bourbaki's influence on contemporary mathematics remains an arduous task.[15] Such an assessment should take into account, in the first place, the diverse degrees of influence which Bourbaki exerted on mathematical research and on mathematical education during different periods of time and in different countries.[16] Second, it should take into account Bourbaki's varying influence on different branches of mathematics. There are certain branches upon which Bourbaki exerted the deepest influence, like algebra and topology; assessing Bourbaki's influence on them would be tantamount to analyzing the development of considerable portions of these disciplines since the 1940s. Here we can only briefly overview the scope of this influence.

The first chapters of Bourbaki's book on topology were published in 1940,[17] following almost four years of the usual procedure of drafting and criticism. This treatise on topology was meant to provide the conceptual basis needed for discussing convergence and continuity in real and complex analysis. Bourbaki's early debates on topology were gradually dominated by a tendency to define this conceptual basis in the most general framework possible, avoiding whenever possible the need to rely on the traditional, most immediately intuitive concepts such as sequences and their limits. This effort helped understanding, among others, the centrality of compactness in general topol-

14. As Mac Lane 1988, 338, wrote: "A whole generation of graduate students were trained to think like Bourbaki."

15. Beaulieu 1989 contains the most detailed and perhaps only comprehensive historical study of Bourbaki's work written to the present. It concentrates on the first ten years of activity. For more recent works on Bourbaki see: Borel 1998, Cartier 1998, Chouchan 1995, Mashaal 2000.

16. To the best of my knowledge, beyond scattered remarks, there are no detailed studies of Bourbaki's influence on research and teaching of mathematics in individual countries or regions. For Bourbaki's influence on shaping mathematical tastes in American universities, see Lax 1989, 455-456. On the influence of Bourbaki on mathematical education in the USSR see Sobolev 1973. Cf. also Israel 1977, 68.

17. The English version appeared as Bourbaki 1966. For a detailed discussion of Bourbaki's book on topology see § 7.3.3 below.

ogy.[18] It also yielded a thorough analysis of the various alternative ways to define general topological spaces and their central characteristic concepts: open and closed sets, neighborhoods, uniform spaces.[19] Moreover, an important by-product of Bourbaki's discussions was the introduction of filters and ultrafilters as a basis for defining convergence while avoiding reliance on countable sequences. Bourbaki, however, rather than including these latter concepts in the treatise, encouraged Henri Cartan to publish them, while elaborating on their relation to topological concepts, under his own name.[20]

Over the next years alternative approaches to questions of continuity and convergence were developed by other mathematicians, based on concepts such as directed systems and nets. The equivalence of the various alternative systems and those of Bourbaki was proven in the USA by Robert G. Bartle in 1955.[21] Thus, the history of the development of topology, at least from 1935 to 1955, cannot be told without considering in detail the role played in it by both Bourbaki as a group and its individual members.

The significance of Bourbaki's work for the development of algebra has less to do with the redefinition of basic concepts than with the refinement and promotion of the conception of this discipline as a hierarchy of structures.[22] The following quotation of Dieudonné seems to reproduce faithfully Bourbaki's images of algebra during the group's early years of activity:

> The development of "abstract" algebra begins around 1900 when it is recognized that the notion of *algebraic structure* (such as the structure of group, ring, field module, etc.) is the fundamental notion in algebra, putting the nature of the mathematical objects on which the structure is defined in the background, whereas, up to then, the majority of the algebraic theories dealt with calculations principally over the real or complex numbers. (Dieudonné 1985, 64-65)

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18. Cf. Mac Lane 1987a, 166: "The recognition of the importance of compactness and of its description by coverings is a major step in the understanding of topological spaces. It developed only slowly—and was not really codified until Bourbaki, in his influential 1940 volume on topology, insisted." See also Mac Lane 1988, 337.

19. In his own research on topological groups André Weil showed that the metric plays only a secondary role in defining the main topological concepts. He thus developed an alternative approach based on uniform spaces as the adequate conceptual framework for this purpose (Weil 1937). See also Weil 1978, 538.

20. They appeared as Cartan 1937, 1937a. The debates leading to the publication of Bourbaki's book on topology are described in some detail in Beaulieu 1990, 39-41.

21. In Bartle 1955.

22. To a considerable extent what Bourbaki did for topology in terms of a unified presentation, van der Waerden had already done for algebra. See below § 7.3.3.

This image of algebra provided the conceptual framework within which Bourbaki members produced important contributions to the body of knowledge in their own research. This was particularly the case with algebraic geometry which, as Dieudonné said, was "one of the principal beneficiaries" of this conception.[23] In fact, a natural outcome of the above conception was the transformation of the disciplinary aims and scope of algebraic geometry through a redefinition of its classical concepts, by replacing the field of complex numbers by an arbitrary field. In fact, since the mid-forties algebraic geometry underwent a deep transformation which has often been characterized as a reformulation of the discipline as a part of commutative algebra, based on the concepts of "sheaf" and "scheme."[24] Not surprisingly perhaps, among the central figures who brought about this transformation one finds several leading French mathematicians, who were also members of Bourbaki: André Weil, Jean-Pierre Serre and Alexandre Grothendieck.[25] The body of algebraic knowledge, i.e., the concepts, techniques and methods, needed as the common background for understanding these works was precisely that developed in Bourbaki's treatise.[26] And no less important than that: the image of algebra within which their research was produced was the image of algebra first introduced by van der Waerden, and then so eagerly promoted by Bourbaki.

But there was also a second main source for the transformation of algebraic geometry coming from the work of Oscar Zariski (1899-1986) and his followers.[27] Zariski, like his French colleagues (and especially Weil), though independently of them, also set out to redefine the main problems and the conceptual foundation of algebraic geometry in terms of the newly consolidated,

23. See Dieudonné 1985, 59. However, Dieudonné's many accounts of the historical development of mathematics, and in particular of the role played by Bourbaki within it, deserve to be addressed with a critical attitude. (This is done in some detail below, especially in §§ 7.4 and 8.4.) In particular one should point out here, that Bourbaki's own activity did not begin with the structural image of algebra in mind. Rather, this image was absorbed during the first years of the project, as the preparation of the first volumes advanced. See Beaulieu 1994, 247-248.

24. Grothendieck & Dieudonné 1971, 2; Israel 1977, 68-69; Zariski 1950, esp. 77.

25. For their respective seminal works in this area see Grothendieck 1958; Serre 1955; Weil 1946. See also Weil 1954.

26. Cf. Weil 1946, *xiii*.

27. A detailed biography of Zariski, including technical appendices summarizing the significance of his work, appears in Parikh 1991. For Zariski's work on the "algebraization" of the foundations of algebraic geometry see Mumford 1991; Parikh 1991, 87-89.

structural view of algebra (though sometimes laying stress on different points of emphasis).[28] During the sixties several of Zariski's students (e.g., Michael Artin, David Mumford, Heisuke Hironaka) undertook to merge the perspective developed by Zariski with the techniques recently introduced by Grothendieck and Serre.[29] In the introduction to a book on Geometric Invariant Theory, Mumford described the aims of his work in terms that could have been taken from any chapter of Bourbaki, as follows:

> It seems to me that algebraic geometry fulfills only in the language of schemes that essential requirement of all contemporary mathematics: to state its definitions and theorems in their natural abstract and formal settings in which they can be considered independent of geometric intuition. (Mumford 1965, *iv*)[30]

The influence of Bourbaki, both as a group and through the works of its individual members, thus played a central role in shaping the development of algebraic geometry since the mid-1940s, but it is clear as well, that Bourbaki was not alone at that. Moreover, one should also note, that in spite of the strong structural orientation that has been characteristic of this field of research since that time, in the last decades algebraic geometers have returned to old motivations and classical questions.[31]

But if topology and algebra are among the disciplines in which Bourbaki's influence has been felt most strongly, at the other end of the spectrum disciplines like logic and most fields of applied mathematics seem not to have been aware of or influenced by Bourbaki at all. This last generalization, however, must be qualified, because Bourbaki has directly influenced mainstream trends in mathematical economics since the 1960s, through the work of Nobel laureate Gérard Debreu.[32]

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28. For the mathematical divergences and the collaboration between Weil and Zariski (in particular concerning their respective views on algebraic geometry), see Parikh 1991, 84-85 & 90-95. A third author whose contribution to the "algebraization" of the foundations of algebraic geometry must be mentioned is van der Waerden. He has described his own contribution in van der Waerden 1970, esp. 172-176.

29. See Artin 1991; Mumford 1991; Parikh 1991, 147-161.

30. A similar assessment appears in Safarevich 1971, *v*.

31. As Zariski (1972, *xiii*) wrote: "There are signs at the present moment of the pendulum swinging back from 'schemes', 'motives', and so on towards concrete but difficult unsolved questions concerning the old pedestrian concept of a projective variety (and even of algebraic surfaces)." Cf. also Israel 1977, 69.

32. Cf. Ingrao & Israel 1990, 280-288; Weintraub & Mirowski 1994.

Beyond the objective difficulties encountered in analyzing Bourbaki's influence there are additional, subjective problems connected with the mythological status of the group. Ever since the name Bourbaki first appeared in public, the group became the focus of much attention and curiosity among mathematicians, and a full-fledged mythology came to surround it. The essence of this mythology is condensed in the following quotation taken from an article published by Paul Halmos in 1957:

> His name is Greek, his nationality is French and his history is curious. He is one of the most influential mathematicians of the 20th century. The legends about him are many, and they are growing every day. Almost every mathematician knows a few stories about him and is likely to have made up a couple more. His works are read and extensively quoted all over the world. There are young men in Rio de Janeiro almost all of whose mathematical education was obtained from his works, and there are famous mathematicians in Berkeley and in Göttingen who think that his influence is pernicious. He has emotional partisans and vociferous detractors wherever groups of mathematicians congregate. The strangest fact about him, however, is that he doesn't exist. (Halmos 1957, 88)[33]

The legend surrounding the group and the professional stature of the researchers who composed its membership have occasionally impaired the objectivity of appraisals of Bourbaki's scientific contributions. The following is an inspired description of these difficulties:

> Confronted with the task of appraising a book by Nicolas Bourbaki, this reviewer feels as if he were required to climb the Nordwand of the Eiger. The presentation is austere and monolithic. The route is beset by scores of definitions, many of them apparently unmotivated. Always there are hordes of exercises to be worked painfully. One must be prepared to make constant cross references to the author's many other works. When the way grows treacherous and a nasty fall seems evident, we think of the enormous learning and prestige of the author. One feels that Bourbaki *must* be right, and one can only press onward, clinging to whatever minute rugosities the author provides and hoping to avoid a plunge into the abyss. Nevertheless, even a quite ordinary one-headed mortal may have notions of his own, and candor requires that they be set forth. (Hewitt 1956, 507. Italics in the original)

33. Among the many additional articles and books that deal with the myth of Bourbaki the following may be mentioned: Boas 1986; Dieudonné 1970, 1982; Fang 1970; Guedj 1985; Israel 1977; Queneau 1962; Toth 1980. According to Mehrtens 1990, 320, already in the seventies Bourbaki had transformed from myth into history. This assessment may be debated as to its exact dating, but probably not as to its essence.

Among the many laudatory commentaries of Bourbaki's work one also finds articles intended for a popular audience that bolster the myth or borrow from it. One also finds several reviews in which well-known specialists in particular disciplines of mathematics present detailed analyses of specific books within the treatise, pointing out their virtues in approach, clarity of presentation, or in the excellent choice of the exercises. These reviews, however, are sometimes so untypically effusive in extolling the merits of Bourbaki's books that their credibility becomes questionable. Take, for example, the following review by a leading mathematician (himself a member of Bourbaki) of a new edition of one of Bourbaki's books:

> If the preceding editions [of the book] were meant to represent an almost perfect account of the bases for present day mathematics, this is now the perfect basis; the author is sufficiently representative of the mathematical community to make such a claim quite close to the truth. Furthermore, in a time in which indiscriminate use of science and technology threatens the future of the human race, or at least the future of what we now call civilization, it is surely essential that a well integrated report about our mathematical endeavors be written and kept for the use of a later day "Renaissance." As Thucydides said about his "History of the Peloponesian War", this is ... a treasure valuable for all times. (Samuel 1972)

Or as Emil Artin wrote in his review of Bourbaki's book on algebra: "Our time is witnessing the creation of a monumental work."[34]

But obviously not all technical reviews of the *Eléments* are as laudatory as those quoted above. Several reviews, written by specialists in their respective fields, have pointed out the shortcomings (mainly in notation and approach), of this or that specific chapter within Bourbaki's treatise. Thus, for instance, in a review of Bourbaki's book on set theory, B. Jonsson claimed that: "due to the extreme generality, the definitions are cumbersome, and all the results derived are of a very trivial nature."[35] Likewise Paul Halmos, reviewing Bourbaki's book on integration wrote: "I am inclined to doubt whether their point of view will have a lasting influence."[36]

34. Artin 1953. As further instances of reviews praising Bourbaki's work see Gauthier 1972; Mac Lane 1948; Rosenberg 1960. Remarkably, most reviewers of Bourbaki, favorable and critical alike, describe the choice of exercises as excellent; this choice is usually attributed to Jean Dieudonné. Cf. Eilenberg 1942, 1945; Kaplansky 1953, 1960; Kelley 1956; Thom 1971, 698. Examples of praise of Bourbaki's work couched in non-technical language can be found in Fang 1970, Queneau 1962, Toth 1980.

35. Jonsson 1959.

Beyond this kind of criticism, directed at particular issues within Bourbaki's output, one also finds critical attitudes concerning the more general influence of Bourbaki on the overall picture of twentieth-century mathematics. The following quotation is an instance of the latter, expressed by Fields-medalist Michael Atiyah, who, without explicitly mentioning the name of Bourbaki, obviously alludes to it. In assessing the dangers of an unrestricted reliance on the axiomatic approach, Atiyah said:

> Most books nowadays tend to be too formal most of the time. They give too much in the way of formal proofs, and not nearly enough in the way of motivations and ideas. Of course it is difficult to do that—to give motivations and ideas... French mathematics has been dominant and has led to a very formal school. I think it is very unfortunate that most books tend to be written in this overly abstract way and don't try to communicate understanding. (Minio 1984, 17)[37]

The opinions quoted above cover a period of nearly thirty years. This has to be taken in account when evaluating them. The present discussion does not aim at evaluating the overall import and influence of Bourbaki's approach on contemporary mathematics. Neither does it analyze the significance of any of Bourbaki's specific contributions to the mainstream disciplines addressed in Bourbaki's treatise. Rather, the present discussion focuses on a restricted, and usually overlooked, aspect of Bourbaki's work, namely, Bourbaki's reflexive attempt to produce a formal concept, the concept of *structure*, meant to elucidate the non-formal idea of a mathematical structure. This particular aspect of Bourbaki's work, however, is central to the group's images of mathematics and its influence is more clearly manifest at that level. In particular, the concept of *structure* also plays an important role in Bourbaki's own historiography. Moreover, Bourbaki, or at least some of its members had hoped, at a relatively early stage of their work, that *structures* would also play a central role in the body of knowledge considered in the *Eléments* (and thus in mathematics at large). This hope, however, was not fulfilled. Thus, understanding the role that *structures* play in Bourbaki's work provides further insights

36. See Halmos 1953. Further critical reviews appear in Bagemihl 1958; Gandy 1959 ("It is possible, then, that this book may itself soon have only historical interest"); Hewitt 1956, 1966; Mathias 1992; Michael 1963; Munroe 1958.

37. For a much harsher attack, both on the significance of the Bourbaki project for mathematics as a whole and on the motivations behind it, see Mandelbrot 1989, 11-12. Further criticism of the view, that accords such a central place in mathematics to the axiomatic method (by Bourbaki and by others) appear in Browder 1975; Israel 1981; Spohn 1961.

regarding both the overall import and influence of that work and the development of the structural approach in mathematics.

7.2 Structures and Mathematics

The concept of structure has been often associated with Bourbaki's mathematics as well as with Bourbaki's putative philosophy of mathematics. Thus, together with the widespread identification of the structural trend in mathematics with the name of Bourbaki, a "structural philosophy of mathematics" has been attributed to the group. What is meant by this? This issue has rarely been discussed in detail. Despite the centrality that many authors claim for Bourbaki's program in contemporary mathematics, very little has been done to elucidate that purported philosophy. In general, when the term "structuralist" is invoked in connection with Bourbaki it is seldom followed by a detailed explanation of the exact meaning of the term.[38] In fact, it seems unlikely that Bourbaki's views can be accurately captured in either a single formula or a fully articulated philosophical picture of mathematics. What was then Bourbaki's conception of mathematics?

It is not unusual to come across pronouncements of Bourbaki members, who insistently characterize Bourbaki's approach as that of the "working mathematician" whose professional interest focuses variously on problem solving, research and exposition of theorems and theories, and which has no direct interest in philosophical or foundational issues. Thus Bourbaki formulated no explicit philosophy of mathematics and in retrospect individual members of the group even denied any interest whatsoever in philosophy or even in foundational research of any kind.[39]

38. An isolated example of a more articulated attempt to analyze Bourbaki's putative structuralist philosophy of mathematics appears in Fang 1970. Fang's book is a lengthy exegesis of Bourbaki's contribution to the vitality of contemporary mathematics. At the same time it is a harsh attack on all those who would dare criticize Bourbaki's views. His account of Bourbaki's philosophy of mathematics, however, adds up to no more than declaring the philosophical and metamathematical formula "mathematics = logic" to be false. It would be exaggerated to claim that Fang's account is supported by sound philosophical arguments. Instead, Fang claims that Bourbaki's own mathematical work is the ultimate representative of contemporary mathematics and the best example that the essence of mathematical thinking cannot be subsumed under the narrow equation proposed by the logicists. Although one could accept this latter claim, it is far from being an explanation of what Bourbaki's "structuralist" program is. See also Kline 1980, 25, for an unsatisfactory account of Bourbaki's "structuralist" program.

Yet, even if it was true that the group steadfastly avoided pronouncements on issues other than pure theorem-proving and problem-solving, Bourbaki's work, like that of any other scientist or group of scientists, proceeded within a particular framework of images of mathematics. Moreover, like any other scientist's system of images of knowledge, Bourbaki's own system has been subject to criticism, it has evolved through the years, and, occasionally, it has included ideas that are in opposition to the actual work whose setting the images provide. Since Bourbaki gathered together various leading mathematicians, it has also been the case that members of the group professed changing beliefs, often conflicting with one another at the level of the images of knowledge. This point will be developed in what follows.

Therefore, when trying to understand Bourbaki's "structuralist" conception of mathematics, it seems more convenient to speak of Bourbaki's images of mathematics rather than of Bourbaki's philosophy of mathematics.[40] Bourbaki's images of mathematics can be reconstructed by directly examining the mathematical work and the historical accounts of the development of mathematics published by the group, by examining pronouncements of different members of the group, and from several other sources as well. Particular caution must be exercised in this regard concerning the status of pronouncements by different members of the group. Jean Dieudonné has no doubt been the most outspoken Bourbaki member, and—more than anyone else—he has spread Bourbaki's name along with what he saw as Bourbaki's conception. André Weil has been the second most active spokesman in this regard. The views of the majority of the group's members—in particular, those views concerning the structural conception of mathematics and the role of the concept of *structure* in the work of Bourbaki—have been usually much less documented or not documented at all. The present account of Bourbaki's images of mathematics will be based mainly on an analysis of the actual contents of the group's work. At the same time, and without attempting to portray Dieudonné

39. Jean Dieudonné (1982, 619) once summarized Bourbaki's avowed position regarding these kinds of questions "as total indifference. What Bourbaki considers important is *communication* between mathematicians. Personal philosophical conceptions are irrelevant for him." (Italics in the original)

40. Giorgio Israel's articles on Bourbaki have described in detail central elements of the group's images of mathematical knowledge. In Israel 1977 he characterizes Bourbaki's views as an "ideology" rather than a philosophy. The term "ideology", however, is far from unequivocal, and is in need of further clarification.

or Weil as official spokesmen for the group, their pronouncements will help provide a fuller picture of those images.

Bourbaki began its work amidst a multitude of newly obtained results, some of them belonging to as yet unconsolidated branches of mathematics; the early years of Bourbaki's activity witnessed a boom of unprecedented scope in mathematical research.[41] In 1948 Dieudonné (signing with the name of Bourbaki) published a now famous article (already quoted above), that was later translated into several languages and which has ever since come to be considered the group's programmatic manifesto: "The Architecture of Mathematics."[42] According to the picture of mathematics described in that article, the boom in mathematical research at the time of its writing raised the pressing question, whether it could still be legitimate to talk about a single discipline called "mathematics", or:

> ... whether the domain of mathematics is not becoming a tower of Babel, in which autonomous disciplines are being more and more widely separated from one another, not only in their aims, but also in their methods and even in their language. (Bourbaki 1950, 221)

In fact, this same question had occupied Hilbert's thoughts several decades before. Hilbert's 1900 list of twenty-three problems called attention to the diversity of problems facing contemporary mathematics. Apparently Hilbert himself was able to embrace all this variety, but he closed his address by raising the question of the unity of contemporary mathematics in terms very similar to those later used by Dieudonné. Hilbert said:

> The question is urged upon us whether mathematics is doomed to the fate of those other sciences that have split up into separate branches, whose representatives scarcely understand one another and whose connections become ever more

41. In order to illustrate the boom of mathematical knowledge during the twentieth-century, Davis & Hersh 1981, 29, have pointed out that in 1868 the *Jahrbuch über die Fortschritte der Mathematik* divided mathematics into 12 disciplines and 30 sub-disciplines, while in 1979 the *Mathematical Reviews* registered 61 disciplines and 3400 sub-disciplines. As for the initial years of Bourbaki's activity, the index of the *Zentralblatt für Mathematik und ihre Grenzgebiete* of 1934 registers 68 disciplines and 197 sub-disciplines.

42. Mehrtens 1990, 318, writes, following a report of Liliane Beaulieu, that Dieudonné published the article without first discussing it as usual in the framework of the group's meetings. Nevertheless, its contents were never contradicted by other Bourbaki members, at least not publicly. Beaulieu's well-documented account of Bourbaki's first years of activity shows how the views expressed by Dieudonné in this manifesto were a consequence of Bourbaki's early stages of activity, rather than a motivation for it. See Beaulieu 1994.

> loose. I do not believe this nor wish it. Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts. For with all the variety of mathematical knowledge, we are still clearly conscious of the similarity of the logical devices, the relationship of the ideas in mathematical theory and the numerous analogies in its different departments. We also notice that, the farther a mathematical theory is developed, the more harmoniously and uniformly does its construction proceed, and unsuspected relations are disclosed between hitherto separate branches of the science. ... Every real advance [in mathematical science] goes hand in hand with the invention of sharper tools and simpler methods which at the same time assist in understanding earlier theories and cast aside older, more complicated developments.[43]

Obviously Hilbert was a main source of inspiration for Bourbaki, and the further the group developed the original plan, the more pressing became the question of the unity of mathematics. In the "Architecture" manifesto, Dieudonné also echoed Hilbert's belief in the unity of mathematics, based both on its unified methodology and in the discovery of striking analogies between apparently far-removed mathematical disciplines. For Dieudonné it was the axiomatic method that accounted for these two unifying tendencies in mathematics. Dieudonné wrote:

> Today, we believe however that the internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts, so as to create something like a central nucleus that is more coherent than it has ever been. The essential part of this evolution has been the systematic study of the relations existing between different mathematical theories, and which has led to what is generally known as the "axiomatic method." ... Where the superficial observer sees only two, or several, quite distinct theories, lending one another "unexpected support" through the intervention of mathematical genius, the axiomatic method teaches us to look for the deep-lying reasons for such a discovery. (Bourbaki 1950, 222-223)[44]

But here Dieudonné went beyond Hilbert and proposed a further idea, directly connected with the axiomatic method and central to Bourbaki's own

43. Quoted from the English translation: Hilbert 1902, 478-479. The Tower-of-Babel metaphor is also used, and a similar warning against the dangers of the situation described appear also in Klein's lectures on the history of nineteenth century mathematics (Klein 1926-7 Vol. 1, 327.)

44. But Bourbaki was in fact hesitant, in the early meetings of 1935, concerning the wholehearted adoption of the axiomatic approach that has come to characterize the group's work so markedly. These qualms are described in Beaulieu 1994, 246.

unified view of mathematics, namely, the idea of mathematical structures. Dieudonné described the unifying role of the structures as follows:

> Each structure carries with it its own language, freighted with special intuitive references derived from the theories from which the axiomatic analysis ... has derived the structure. And, for the research worker who suddenly discovers this structure in the phenomena which he is studying, it is like a sudden modulation which orients at once the stroke in an unexpected direction in the intuitive course of his thought and which illumines with a new light the mathematical landscape in which he is moving about.... Mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does intuition dominate in the genesis of discoveries. But henceforth, it possesses the powerful tools furnished by the theory of the great types of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method, but which were formerly in a completely chaotic state. (Bourbaki 1950, 227-228)

Thus in the "Architecture" manifesto Dieudonné attributed to the structures—and especially to "the theory of the great types of structures" as well as to the tools provided by it—a central role in the unified picture of mathematics. In order to understand exactly what he meant by this, we must take a closer look at Bourbaki's work.

As the Bourbaki project evolved from writing a course in analysis into an up-to-date, comprehensive, account of the central branches of mathematics, Bourbaki laid greater stress on presenting the whole of that mathematical knowledge in a systematic and unified fashion, and within a standard system of notation. As we saw in Part One, a similar task had been successfully undertaken some years before by van der Waerden, albeit for the more limited context of algebra alone. In fact van der Waerden's book was a main source of inspiration for Bourbaki in the group's early stages of activity.[45] At that time, the images of classical, nineteenth-century algebra were still the dominant ones in France, especially due to the lasting influence of Serret's textbook of algebra (§ 2.1 above). The new methods and the recent achievements of Emmy Noether, Emil Artin and their followers in Germany, were still foreign to the general mathematical audience in France. Commenting on the strong impact

45. Cf. Dieudonné 1970, 136-137. However, at the very initial stages of the project, there was no consensus among members of the group as to the convenience of following van der Waerden's approach. As a matter of fact, in the early meetings there was substantial disagreement as to the extent and specific content of the "abstract package" (Bourbaki's expression) that should be included in the book. See Beaulieu 1993, 30; Beaulieu 1994, 244-246.

caused by the first reading of this book, and its influence on the Bourbaki project Dieudonné wrote: "I had graduated from the Ecole Normale, and I did not know what an ideal was, and barely knew what a group was."[46] In retrospect, the main thrust of Bourbaki's initial motivation may be seen as an attempt to reorient French mathematics away from its traditionally dominant conceptions and into the new perspectives lately developed in Germany. In particular, Bourbaki's treatise, as it gradually came to be conceived and worked out, may actually be seen as an extension of van der Waerden's achievement to the whole of mathematics; that is, much the same as van der Waerden had succeeded in presenting the whole of algebra as a hierarchy of structures, so did Bourbaki present much larger portions of mathematics in a similar way.[47]

There is, however, a noteworthy difference between van der Waerden's systematic exposition of modern algebra and Bourbaki's own, more ambitious attempt. Van der Waerden's unification of algebra consisted in a successful restructuring of a whole discipline, which was attained through a redefinition of the images of knowledge of that particular discipline. While his innovation remained mainly at the level of the images of mathematics, it provided a convenient framework within which much important progress in the body of knowledge was later attained. As was seen in § 1.3 above, van der Waerden felt no need of providing an explicit explanation, either formal or non-formal, of what is to be understood by an "algebraic structure" or by "structural research in algebra." Bourbaki, unlike van der Waerden in this respect, not only attempted on various opportunities to explain what the structural approach is and why it is so novel and important for mathematics, but, moreover, in putting forward the theory of *structures* as part of the treatise, they manifested an eagerness to endorse those explanations, and in fact Bourbaki's whole system of images of mathematics, by means of an allegedly unifying, mathematical theory. This eagerness is to be understood, in the first place, as

46. Dieudonné 1970, 137. But perhaps not only French mathematicians were unaware of recent developments in algebra in Germany. According to Zariski's testimony, for instance, having got most of his mathematical instruction in Italy, it was only while writing his important treatise on Algebraic Surfaces (Zariski 1935) that he began to study the works of Emmy Noether as well as Krull's and van der Waerden's books. See Parikh 1991, 68; Zariski 1972, *xi*. For a more detailed account of algebraic research in Italy in the early twentieth century see Brigaglia 1984.

47. Also the extent of the more modern subjects of mathematics that Bourbaki intended to cover in the treatise changed over the first years of activity. See Beaulieu 1994, 246-247.

an instance of that image of knowledge characteristic of twentieth-century mathematics, according to which meta-issues in mathematics are considered to have been meaningfully elucidated only insofar as they have been articulated in *formal* mathematical theories.

A reflexive, formal-axiomatic elucidation of the idea of mathematical structure could prove useful not only as a general frame of reference but also in addressing some central open questions concerning the role of structures in mathematics. One such central issue was the issue of selection. The issue of selection is a central question in science in general, at the level of the images of knowledge. What an individual scientist selects as his discipline of research, and the particular problems he selects to deal with in that particular discipline will largely determine, or at least condition, the scope and potentialities of his own research. What a community of scientists establishes as main open problems and main active subdisciplines will substantially influence the future development of the discipline as a whole. Clearly, the contents of the body of knowledge directly delimits the potential selections of scientists. On the other hand, these contents *alone* cannot provide clear-cut answers to the issue of selection. Criteria of selection are open to debate and, obviously, there are several possible factors that will determine a particular scientist's choice, when confronted with a given body of knowledge.

Bourbaki was very conscious of the centrality of the issue of selection and, from the very beginning of the group's activities, considerable effort was invested in debating it.[48] In the early meetings, that eventually led to the creation of the core Bourbaki group, an important criterion for the selection of issues to be treated in the projected treatise on analysis was their external applicability and their usefulness for physicists and engineers. Over the first years of activities, however, given the more abstract inclinations of certain members and the way in which the writing of the chapters evolved, gradual changes affected the criteria of selection guiding the group's work.[49]

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48. Evidence for this can be gathered by browsing through the issues of "La Tribu." Below in § 9.4, one particular debate on selection is discussed in detail, namely, the one concerning the possible inclusion of categories and homological algebra in the treatise. See also Beaulieu 1990, 39-41, concerning discussions on the most suitable axiomatic definitions for topology. It should also be stressed here, that, according to Dieudonné's retrospective appraisal (Dieudonné 1970, 22), selection "is the crucial part in Bourbaki's evolution. I think that we can attribute much of the hostility that has been shown towards Bourbaki ... to this strict selection."

49. Cf. Beaulieu 1993, 30-31.

As the axiomatic approach increasingly became a dominant concern for Bourbaki, the problem of selecting, and especially that of justifying, the most interesting theories to be included in the treatise became a pressing one.[50] As Henri Cartan wrote in retrospect, on the face of it the choice of axioms could seem to be completely arbitrary; in practice, however, a very limited number of such systems constitute active mathematical research disciplines, since theories "built upon different axiomatic systems have varying degrees of interest."[51] Moreover, certain axiomatic systems on which mathematicians may decide to invest their research efforts were variously dismissed by others as "axiomatic trash."[52] Thus, Bourbaki's image of mathematics as it became consolidated around the 1940s and as it was expressed in Dieudonné's "Architecture"—an image reportedly centered around the view of mathematics as the science of axiomatic systems—implied the necessity of formulating criteria to explain how the chaff of "axiomatic trash" is winnowed from the grain of the mathematically significant axiomatic systems.

Given the success attained by reflexive theories in mathematical research in the decades preceding Bourbaki's activity, and, more specifically, given the pervasive influence of Hilbert on Bourbaki's image of mathematics, it would be natural, or at least plausible, to expect that an answer to the above-posed question be given by means of a reflexive mathematical theory, within which the correctness of the choice could be endorsed by mathematical proof. Bourbaki's formulation of the theory of *structures* could be seen as an actual response to that expectation.

But on the other hand, considering the intellectual inclinations of the mathematicians involved in the Bourbaki project, one is justified in thinking that each member of the group had strongly conceived opinions of what should be considered as mathematically interesting and what should not, independently of the elaboration of a formal theory meant to "classify the fundamental disciplines of mathematics."[53] The reliance on each mathematician's

50. A general discussion of this issue appears in Spalt 1987. Spalt, following Lakatos, contrasts two different kinds of mathematics: Structure-mathematics vs. informal mathematics (*inhaltlichen Mathematik*). While in the latter, according to Spalt, the problem of justification (*Rechtfertigung*) of concepts and methods is more naturally solved, in the former it appears in all of its acuteness and remains unsolved.

51. Cartan 1980, 177.

52. A term also used by Dieudonné (e.g., in Dieudonné 1982, 620).

intuition in order to decide on this question was asserted by Henri Cartan as follows:

> There is no general rule in mathematics by which one can judge what is interesting and what is not. Only a thorough understanding of existing theories, a critical evaluation of the problems at hand or a sudden, unexpected flash of intuition can enable the researcher to choose the appropriate axiom system. (Cartan 1980, 177)

One might thus expect, that in many of the issues discussed in the Bourbaki meetings, the intuitions of the various members concerning the correct selections often differed and perhaps even clashed, until common agreement was finally reached.[54] Thus, the publication of a formal theory of *structures* and its connection with the issue of selection bring to the fore interesting dualities and tensions, which are characteristic of much of the Bourbaki undertaking: standardized techniques vs. ingenuity in problem-solving, collective work vs. individual genius, formalized generalization vs. intuitive grasping of peculiar mathematical situations.

At any rate, one can see how the thorough adoption of the axiomatic approach as the main tool for the exposition of mathematical theories, together with the images of knowledge associated with that approach, create a direct connection between the issue of selection and Bourbaki's formulation of the theory of *structures*.[55] As it happened, however, and as will be seen in the following sections, this theory did not effectively provide answers to this, or to any other reflexive issue. Nevertheless, Bourbaki's images of mathematics, and in particular the group's actual choices, though obviously not derived

53. As claimed in Cartan 1980, 177: "The concept of *structure* ... allowed a definition of the concept of isomorphism and with it a classification of the fundamental disciplines within mathematics."

54. To quote Cartan once more (1980, 179): "That [a final product] can be obtained at all [in Bourbaki's meetings] is a kind of miracle that none of us can explain."

55. As with other issues, Dieudonné retrospectively attempted to rationalize Bourbaki's choices, explaining them with arguments other than the professional authority of the group's members. Cf. Dieudonné 1982, 620. It is worth stressing, however, that in spite of formulating Bourbaki's alleged objective criteria of selection, Dieudonné wrote that (1982, 623): "No one can understand or criticize the choices made by Bourbaki unless he has a solid and extended background in many mathematical theories, both classical and more recent." For a criticism of the views expressed here by Dieudonné, see Hermann 1986. For a more general criticism of Bourbaki's choices see Benoit Mandelbrot's comments in Albers & Alexanderson (ed.) 1985, 222. In Mandelbrot 1989, 11, he also wrote: "For Bourbaki the fields to encourage were few, and the fields to discourage were many."

from a formal theory, proved to be enormously fruitful in certain quarters of mathematics. Still more interesting, Bourbaki's criteria of selection have very often been accepted *as if* they were actually backed by such a reflexive theory. This point will be further elaborated below.

A second clue to understanding Bourbaki's images of mathematics and the attempt to include a formal theory of *structures* in Bourbaki's treatise is connected with the group's view of mathematics as the science dealing with axiomatic formal systems, as well as its self-proclaimed status as "legitimate heir" to Hilbert's alleged "formalist" doctrine. Whether or not Bourbaki's outlook, or those of its individual members, faithfully reflects Hilbert's own views is debatable, although this is not the place to do so. What is relevant for our discussion here is to recognize that a main reason why Bourbaki adopted a formalist position was to avoid philosophical difficulties. As already stated, members of Bourbaki consistently declared themselves first and foremost to be "working mathematicians", and their actual views concerning philosophical or foundational issues is perhaps most frankly expressed in the following quotation of Dieudonné:

> On foundations we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes, we run to hide behind formalism and say: "Mathematics is just a combination of meaningless symbols" and then we bring out chapters 1 and 2 [of the *Eléments*] on Set Theory. Finally we are left in peace to go back to our mathematics and do it as we have always done, working in something real. (Dieudonné 1970, 145)

This position of "Platonism on weekdays and formalism on Sundays", which is so widespread among working mathematicians, becomes especially worthy of attention in the case of Bourbaki. It has been claimed elsewhere that such a position is untenable as a consistent philosophical account of mathematics, since it involves both logical inconsistency and a distorted description of the actual doings of the mathematician.[56] Nevertheless, it is an accepted image of mathematics, that has at least helped many a twentieth-century mathematician confer some meaning to his own scientific work. This seems to be the case as well for Bourbaki.

Yet since Bourbaki does not represent rank-and-file mathematicians, but rather those who assumed a role of leadership and intended to come forward

56. See Hersh 1971 (1985), 11 ff.

with an innovative, meaningful interpretation of what mathematics essentially is, it may come as a surprise that, while raising the banner of rigor and parsimony in mathematics, Bourbaki was willing to adopt the above-mentioned philosophical position without any reluctance. It is not a criticism of Bourbaki's philosophical sophistication, or lack of it, which concerns us here but rather the question, how is the elaboration of the theory of *structures* connected with Bourbaki's images of mathematics. The above-described mixture of a declared formalist philosophy with a heavy dose of actual Platonic belief is illuminating in this regard. The formalist imperative, derived from that ambiguous position, provides the necessary background against which Bourbaki's drive to define the formal concept of *structure* and to develop some immediate results connected with it can be conceived. The Platonic stand, on the other hand, which reflects Bourbaki's true working habits and beliefs, has led the very members of the group to consider this kind of conventional, formal effort as superfluous. Indeed, of all the apparatus developed in the first book of the treatise following that formalist imperative, only feeble echoes appear in the other volumes, where Bourbaki's real fields of interest are developed. This desire to avoid philosophical issues by adopting an inconsistent position, and the image of mathematics associated with that position, prepare us to understand Bourbaki's need to elaborate a formal theory of *structures* and the futility of this undertaking.

This general picture of Bourbaki's images of mathematics and of their connections with the elaboration of a theory of *structures* allegedly standing at the basis of the building of mathematics provides an adequate background against which to examine in greater detail the contents of Bourbaki's treatise and the actual role played by *structures* within it.

7.3 *Structures* and the Body of Mathematics

The present form of the *Eléments*, composed of ten books, was nearly attained in the early seventies. The ten books are: I. Theory of Sets; II. Algebra; III. General Topology; IV. Functions of a Real Variable; V. Topological Vector Spaces; VI. Integration; Lie Groups and Lie Algebras; Commutative Algebra; Spectral Theories; Differential and Analytic Manifolds.[57] The first six books of the treatise were intended to be more or less self-contained. The other four presuppose knowledge of the first six volumes and, for that reason, Bourbaki gave them no number. The French edition of the *Elé-*

ments bears the subtitle "The Fundamental Structures of Analysis", and the headings of the various volumes reflect one of Bourbaki's most immediate innovations, namely the departure from the classical view according to which the main branches of mathematics were taken to be geometry, arithmetic and algebra, and analysis. Each book in the treatise is composed of chapters that were published successively, though not necessarily in the order indicated by the index. The theory of *structures* appears in Book I, which deals with the set theory. We will now examine the concept of *structure* and the role played by it in this and in the other books of the *Eléments*.

7.3.1 Set Theory

As with other subjects included by Bourbaki among the "Fundamental Structures of Analysis", such as algebra and topology, the group's initial plans did not envisage a systematic, axiomatic elaboration of the theory of sets as an independent subject. Rather, the original idea was to use only elementary set-theoretical notions, introduced from a naive perspective, such as the direct needs of a treatise on analysis would require. This approach reflected a long-standing tradition with respect to set theory in France. During the first three years of actual work on the treatise, however, attention shifted gradually away from the classical issues of analysis and most of the effort was in fact directed toward more basic and "abstract" issues.[58] The chapters of Bourbaki's volume on set- theory were published only during the 1950s, but a summary of results appeared as early as 1939.

Book I of the *Eléments, Theory of Sets*, is composed of four chapters and a "Summary of Results": 1. Description of Formal Mathematics; 2. Theory of Sets (first French edition of both chapters: 1954); 3. Ordered Sets, Cardinals, Integers (1956); 4. Structures (1957). The "summary" was first published in French in 1939.[59] The book is preceded by an introduction on formalized languages and the axiomatic method. No mathematician, it is said, actually works in a fully formalized language, but rather in natural language. However, "his

57. From the first meetings onwards, several different suggestions were put forward as to the desired organization of the treatise into separate volumes. Several subjects that had been contemplated in the plans drafted about 1941, and ranging from algebraic topology to partial differential equations, to numerical analysis, were left out, first because of the impossibility of collective work during the war, and later because of sensible changes in the main interest of the group following the admission of younger members. See Beaulieu 1994, 249-251.

58. See Beaulieu 1994, 246-247.

experience and mathematical flair tell him that translation into formal language would be no more than an exercise of patience (though doubtless a very tedious one).” The first aim of the book then is to present one such formalized language. This language should be general enough to allow the formulation of all the axiomatic systems of mathematics. The existence of such a language is warranted by the fact that:

> ... whereas in the past it was thought that every branch of mathematics depended on its own particular intuitions which provided its concepts and primary truths, nowadays it is known to be possible, logically speaking, to derive practically the whole of mathematics from a single source, the theory of sets. (Bourbaki 1968, 9)

However, since even the complete formalization of set theory alone turns out soon to be impracticable, strings of signs that are meant to appear repeatedly throughout the book are replaced from the beginning by symbolic abbreviations, and condensed deductive criteria are introduced, so that for every proof in the book it will not be necessary to explain every particular application of the inference rules. The final product is a book which, like any other mathematical book, is partially written in natural language and partially in formulae but which, like any partial formalization, is supposed in principle to be completely formalizable. At any rate, the claim is made that the book on set theory lays out the foundations on which the whole treatise may be developed with perfect rigor.[60]

Naturally, when dealing with formal systems the problem of consistency immediately arises. Bourbaki did not attempt to address the problem of consistency through any kind of formalistic device. Rather, Bourbaki deviated here from the professed formalist position and, without further ado, reverted to empiricist considerations. Thus, Bourbaki stated that a contradiction is not expected to appear in set theory because it has not appeared after so many years of fruitful research.[61] But this is not the only sense in which the formalistic apparatus introduced by Bourbaki fulfills no real foundationalist role.

Take for example the discussion on formalized languages, following the introduction. A proof is defined as a series of relations of terms formed according to specified rules. A theorem is defined as any relation appearing in a

59. There have been several reprints (with some minor changes) and translations of this volume into other languages. Unless otherwise stated, all quotations of *Theory of Sets* below, are taken from the English translation (Bourbaki 1968). There are minor differences between the successive editions, but these are referred to and commented separately in this section.

60. Cf. Bourbaki 1968, 11.

proof. All these steps and the stress laid upon the claim that the formal expressions are devoid of meaning are standard for a formalistically-oriented foundation of mathematics. However Bourbaki added a rather unusual commentary:

> This notion [of a theorem] is therefore essentially dependent on the state of the theory under consideration, at the time when it is being described. A relation in a theory τ, becomes a theorem in τ when one succeeds in inserting it into a proof in τ. To say that a relation in τ "is not a theorem in τ" cannot have any meaning without reference to the stage of development of the theory τ. (Bourbaki 1968, 25)

In like manner the concept of a contradictory theory was defined here by Bourbaki as time-dependent: as long as a contradiction has not been proven to exist within the theory, that theory is free from contradiction. Time, however, could prove it to be otherwise. Now the notion of a meta-mathematical concept (and a central one, at that) as being time-dependent, although debatable in itself, could be accepted or overlooked if appearing in an introduction to a book in any standard mathematical discipline. It would likewise not be surprising were it found, e.g., in an intuitionistic book. But it seems strangely out of place in the framework of a book allegedly intended to provide the sound basis and the framework on which the whole picture of the basic branches of mathematics is to be developed from a declared *formalistic* perspective.

Bourbaki's style is often described as one of uncompromising rigor with no heuristic or didactic concessions to the reader.[62] This characterization fits perhaps the bulk of the treatise, but not *Theory of Sets*. In fact, the further one

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61. Cf. Bourbaki 1968, 13. Cf. also one of Bourbaki's earlier publications (1949, p. 3): "... absence of contradiction, in mathematics as a whole or in any given branch of it, thus appears as an empirical fact rather than as a metaphysical principle ... We cannot hope to prove that every definition ... does not bring about the possibility of a contradiction." Imre Lakatos claimed, in an article entitled "A Renaissance of Empiricism in Recent Philosophy of Mathematics?" (Lakatos 1978 Vol. 2, 24-42), that foundationalist philosophers of mathematics, from Russell onwards, when confronted with serious problems in their attempts to prove the consistency of arithmetic, have not hesitated to revert to empirical considerations as the ultimate justification for it. Although Bourbaki is not mentioned among the profusely documented quotations selected by Lakatos to justify his own claim, it seems that these passages of Bourbaki could easily fit into his argument. Bourbaki's empiricist solution to the issue of consistency is also discussed in Israel & Radice 1976, 175-176.

62. As Queneau 1962, 9 puts it: "La seule concession que fasse Bourbaki aux considèrations heuristiques, ce sont ces notes historiques." Of course, there may be utterly different opinions about what constitutes "heuristic considerations" when it comes to the thinking of a layman and that of a mathematician.

advances through the chapters of *Theory of Sets*, encountering ever-new symbols and results, the more one finds additional heuristic explanations of the meaning of the statements, even when they are not especially difficult. Typical is the following example towards the end of the chapter, in the section dealing with quantified theories:

> C36. Let A and R be relations in τ, and let x be a letter. Let τ' be the theory obtained by adjoining A to the axioms of τ. If x is not a constant of τ, and if R is a theorem in τ',then $(\forall_A x)R$ is a theorem in τ. (Bourbaki 1968, 42)

After a two-line proof of C36 is given, the following comment appears:

> In practice we indicate that we are going to use this rule by a phrase such as 'Let x be any element such that A'. In the theory τ', so defined, we seek to prove R. Of course, we cannot assert that R itself is a theorem in τ. (*ibid.*)

In this way, the formal language that was introduced step by step is almost abandoned and quickly replaced by the natural language. The recourse to extra-formalistic considerations in the exposition of results within a text-book is, of course, perfectly legitimate. What should be noticed here, however, is the departure from Bourbaki's behavior in other books of the *Eléments* and the divergence between Bourbaki's pronouncements and what is really done in *Theory of Sets*.

There is, in fact, written evidence of Bourbaki's uncertainty about how to address these problems. The kinds of problems addressed in the first chapters of *Theory of Sets* were not a major concern of the entire group; as a matter of fact, it overlapped the research fields of very few of its members. Nevertheless, those issues had to be addressed if the desired formal coherence of the treatise was to be achieved. Several issues of "La Tribu" record different proposals regarding the desired contents of *Theory of Sets*, as well as several technical problems encountered while developing them in detail. We will consider this point below; yet it is pertinent to quote here a report on the progress in the work on *Theory of Sets* by 1949, in which the above issues of formalized languages and inference rules were dealt with. This report reads:

> Since the first session, Chevalley raised objections concerning the notion of a formalized text, which threaten to hinder the whole publication. After a night of contrition, Chevalley turned to more conciliatory opinions and it was agreed that there are serious difficulties to it, which he was assigned to mask as unhypocritically as possible in the general introduction. A formalized text is in fact an ideal notion, since one has seldom seen any such a text and in any case Bourbaki has

> none. One should therefore speak with discretion about those texts in chapter 1 and indicate clearly in the introduction what separates us from them.[63]

Set theory and the foundational problems of mathematics were close to Chevalley's own research, and this may be the reason why he insisted, perhaps more than any other member of the group, that *Theory of Sets* be published soon.[64] The publication, however, was constantly postponed since problems surrounding the formalized language continued to appear. The book in its final form, rather than being the outcome of a coherently envisaged foundation for mathematics, is a compromise between the desire to bring formalization to its most extreme manifestation, as demanded by Chevalley, and the need to produce a readable book that would fit the overall style of the treatise and the standard reader's interest. This is also probably one of the reasons for the ad-hoc character of *Theory of Sets* when regarded as part of the treatise.

In Chapter 2 the axioms for sets are introduced and some immediate results are proven. Many concepts are treated here using a rather idiosyncratic notation. Bourbaki's overall influence is very often manifest in the extended use, in several fields of mathematics, of new nomenclature and notation introduced in Bourbaki's treatise. But as Paul Halmos has remarked, many of the concepts and notations introduced here are used no more than once and therefore could have been dispensed with. Moreover, as Halmos claimed:

> It is generally admitted that strict adherence to rigorously correct terminology is likely to end in being pedantic and unreadable. This is especially true of Bourbaki, because their terminology and symbolism are frequently at variance with commonly accepted usage. The amusing fact is that often the "abuse of language" which they employ as an informal replacement for a technical name is actually conventional usage: weary of trying to remember their own innovation,

63. "La Tribu" - April 13-25: 1949: "Dès la première séance de discussion, Chevalley soulève des objections relatives à la notion de texte formalisé; celles ci menacent d'empêcher toute publication. Après une nuit de remords, Chevalley revient à des opinions plus conciliantes, et on lui accorde qu'il y a là une sérieuse difficulté qu'on le charge de masquer le moins hypocritement possible dans l'introduction générale. Un texte formalisé est en effet une notion idéale, car on a rarement vu de tels textes et en tous cas Bourbaki n'en est pas un; il faudra donc ne parler dans le chap. I qu'avec beaucoup de discrétion de ces textes, et bien indiquer dans l'introduction ce qui nous en sépare."

64. In fact, in his student years at the Ecole Normale Chevalley developed a strong friendship with the early-deceased Jaques Herbrand, with whom he shared his early interest in logic. Later on Chevalley worked mainly on class field theory, group theory, algebraic geometry and the theory of Lie algebras. See Dieudonné & Tits 1987.

the authors slip comfortably into the terminology of the rest of the mathematical world. (Halmos 1957, 90)[65]

Chapter 3 deals with ordered sets, cardinals and integers. Ordered structures, as will be seen below, are among the so called “mother structures” of mathematics, to which Bourbaki accords a central role in its picture of mathematics; it is surprising, then, that they are considered in no other place in the entire treatise. Lattices, for example, are only briefly mentioned (p. 146), although it should be noticed that a relatively large number of exercises concerning them are included at the end of the chapter.[66]

Finally, Chapter 4 develops the concept of *structure*, Bourbaki’s formalized notion of structure. Before defining *structures* Bourbaki introduced some preliminary concepts. The basic ideas behind those concepts can be formulated as follows: take a finite number of sets $E_1,E_2,...,E_n$, and consider them as the building blocks of an inductive procedure, each step of which consists either of taking the Cartesian product ($E\times F$) of two sets obtained in former steps or of taking their power set $\boldsymbol{B}(E)$. For example, beginning with the sets E, F, G the outcome of one such procedure could be: $\boldsymbol{B}(E)$; $\boldsymbol{B}(E)\times F$; $\boldsymbol{B}(G)$; $\boldsymbol{B}(\boldsymbol{B}(E)\times F)$; $\boldsymbol{B}(\boldsymbol{B}(E)\times F)\times\boldsymbol{B}(G)$ and so forth. Bourbaki introduces a formal device for defining and characterizing every possible construction of the kind described above. The last term obtained through a given construction of this kind for n sets $E_1,E_2,...,E_n$ is called an “echelon construction scheme S on n base sets” and it is denoted by $S(E_1,E_2,...,E_n)$. Given one such scheme and n additional sets E_i', and n mappings $f_i: E_i\rightarrow E_i'$, a further formal straightforward procedure enables one to define a function from $S(E_1,E_2,\ldots,E_n)$ to $S(E_1',E_2',\ldots,E_n')$ (i.e., to the corresponding system built over the sets $E_1',E_2',...,E_n'$ instead of $E_1,E_2,...,E_n$). This function is called the “canonical extension with scheme S of the mappings $f_1,\ldots,f_n$” and it is denoted by $<f_1,...,f_n>^S$. This function is injective (resp. surjective, bijective) when each of the f_i’s are. To define a “species of structure” Σ, one takes:

(1) n sets x_1, x_2, .., x_n, as “principal base sets.”

(2) m sets A_1, A_2, .., A_m, the “auxiliary base sets”, and finally

(3) a specific echelon construction scheme $S(x_1,...,x_n,A_1,...,A_m)$.

65. For a detailed review of Chapters 1-2 of *Theory of Sets*, see Halmos 1955. A technical criticism of Bourbaki’s system of axioms for the theory of sets is developed in Mathias 1992.

66. Only half a page of the 17 pages in Section 1 of Chapter 3 is devoted to lattices (section 1.11). In contrast, out of 24 exercises to this section, 7 deal with lattices. In fact, lattices are treated in some detail in Chapter 7 of Bourbaki 1972, especially pp. 512-529.

This scheme will be called the “typical characterization of the species of structure Σ.” Such a scheme is obviously a set. A *structure* is now defined by characterizing some of the members of this set by means of an axiom of the species of structure. This particular axiom is a relation which the specific member $s \in S(x_1,...,x_n, A_1,...,A_m)$ together with the sets $x_1,...,x_n,A_1,...,A_m$ must satisfy. The relation in question is constrained to satisfy the conditions of what Bourbaki calls a “transportable relation”, which means roughly that the definition of the relation does not depend upon any specific property of S and the sets in themselves but only refers to the way in which they enter in the relation through the axiom. The next example introduced by Bourbaki makes things clearer.

An internal law of composition on a set A is a function from $A \times A$ into A. Accordingly, given any set A, form the scheme $\boldsymbol{B}((A \times A) \times A)$ and then choose from all the sub-sets of $(A \times A) \times A$ those satisfying the conditions of a “functional graph” with domain $A \times A$ and range A. The axiom defining this choice is a special case of what we call algebraic *structures*.[67]

Together with this example Bourbaki also showed, using the previously introduced concepts, how ordered-*structures* or topological-*structures* may be defined. That these are Bourbaki’s first examples is by no means coincidental. These three types of *structures* constitute what Bourbaki calls the *mother structures*, a central part of Bourbaki’s images of mathematics which we shall discuss below.

After defining *structures* Bourbaki introduced further concepts connected with that definition. However, in the remainder of the chapter, recurring reference to n principal base sets and m auxiliary base sets is avoided by giving all definitions and propositions for a single principal base set (and for no auxiliary set) while stating that “the reader will have no difficulty in extending the definitions and results to the general case” (p. 271). This is a further instance of how Bourbaki ignored in *Theory of Sets* their own self-imposed strict rigor of the other books in the treatise.

These concepts deserve closer inspection since they reveal the ad-hoc character of the notions set forth in *Theory of Sets*. Bourbaki’s purported aim in introducing such concepts is to expand the conceptual apparatus upon which the unified development of mathematical theories will be developed. However, all this work turns out to be rather superfluous, since, as will be

67. This example appears in Bourbaki 1968, 263.

seen, these concepts are used in a very limited—and certainly not especially illuminating or unifying—fashion in the remainder of the treatise.

*** Isomorphism**: Let U, U' be two *structures* of the same type Σ on n principal base sets, $E_1,E_2,...,E_n$ and $E_1',E_2',...,E_n'$ respectively, and let n bijections f_i: $E_i \rightarrow E_i'$, be given. If S is the echelon construction scheme of Σ, then $\langle f_1,...,f_n \rangle^S$ is defined as an isomorphism if

$$\langle f_1,...,f_n,Id_1,...,Id_m \rangle^S(U) = U'$$

where Id_i denotes the identity mapping of an auxiliary set A_i into itself. This definition uses the concept of canonical extension introduced above to express in a precise fashion the desirable fact that the isomorphism ‘preserves’ the structure.

*** Deduction of *structures***: Bourbaki defines a formal procedure for deducing a new species of structures from a given one. For instance, if the species of topological group structures is defined on a single set A by a generic *structure* (s_1,s_2), where s_1 is the graph of the composition law and s_2 the set of the open sets of A, then each of the terms s_1 and s_2 is a procedure of deduction and respectively provides the *group* and the *topology* underlying the topological group *structure* (s_1,s_2). Likewise, a commutative group structure can be deduced from either a vector space, or from a ring or from a field.

*** Poorer-Richer *structures***: Among the examples introduced in order to clarify the mechanism of deduction of *structures* defined above, a criterion is defined which enables one to order *structures* with the same base sets and the same typical characterization as *poorer* or *richer*, according to whether the axiom defining the latter can be “deduced” from the former. For example, the species of a commutative group is *richer* than the species of groups.

*** Equivalent species of *structures***: This definition enables one to identify the same *structure* when it is defined in different ways (e.g. commutative groups and $\boldsymbol{Z}$-Modules).

*** Finer-Coarser *structures***: This is a further relation of order defined between *structures* of the same species. Roughly, a given species of *structures* will be finer the more morphisms it contains with E as source and the fewer morphisms it contains with E as target.

These concepts will be discussed below, and their limited generalizing value within the treatise will be examined.

The subsequent sections of *Theory of Sets* are devoted to special constructions which can be made within the framework of the *structures*: inverse image of a *structure,* induced *structure*, product *structure,* direct image and quotient *structure.* The last section of the chapter deals with Universal Mappings. These are defined for an arbitrary *structure,* and the conditions are stated for the existence of a solution to the universal mapping problem in a given *structure.* It is proven that when this case holds its solution is essentially unique. The unwieldiness of the *structure*-related concepts is here perhaps more apparent than in any other place, since, for this specific problem, a fully developed and highly succinct version of the categorical formulation of the Universal Mapping Problem is available.[68] This point will be further developed in § 8.3 below.

After all this painstaking work, the book closes with a "Summary of Results" ("Fascicule de résultats") containing all the results of set theory which will be of some use in the remainder of the treatise. However, the term "Summary" does not accurately describe the contents of this last section. "Fascicule de résultats" seems a more precise name, because what one finds is neither all the results nor a presentation of them exactly as they appeared in the book but rather "all the definitions and all the results needed for the remainder of the series." If the book's stated aim was to show that a sound, formal basis for mathematics can be given, the Fascicule's purpose was to provide the lexicon needed for what follows and to explain the non-formal meaning of the terms within it. This sudden change of approach, from a strict formal style to a completely informal one, is clearly stated and justified by Bourbaki in the opening lines of the Summary:

> As for the notions and terms introduced below without definitions, the reader may safely take them with their usual meanings. This will not cause any difficulties as far as the remainder of the series is concerned, and renders almost trivial the majority of the propositions. (Bourbaki 1968, 347)

68. See also Mac Lane 1971, Chpt. 3.

Thus, for example, the huge effort invested in Chapters 2-3 is reduced to the laconic statement: “A set consists of *elements* which are capable of possessing certain *properties* and of having relations between themselves or with elements of other sets” (p. 347. Italics in the original). A footnote explains further:

> The reader will not fail to observe that the “naive” point of view taken here is in direct opposition to the “formalist” point of view taken in chapters I to IV. Of course, this contrast is deliberate, and corresponds to the different purposes of this Summary and the rest of the volume.

The purpose of the summary, then, is to provide, in completely non-formal terms, the common basis upon which the specific theories will later be developed. It is only in this non-formal fashion that Book I is related to the rest of the treatise and, in particular, that the concept of *structure* appears as a unifying concept.

As for *structures,* the whole formal development is reduced in the Fascicule to a short, intuitive explanation of the concepts (even shorter than the one given in the present account) in which the main ideas are explained. The only important concept associated with *structure* which is mentioned, is that of isomorphism. No mention at all is made of derived-, initial-, quotient-, coarser- and finer-, and other *structures* defined in Chapter IV. This summary of results is essentially different from its counterparts in the other books of the series (for example that of “Topological Vectorial Spaces”),[69] both because of its variance from the actual contents of what it allegedly summarizes and because of the striking and total absence of technicalities.

As already noted, the “Fascicule” first appeared in French in 1939, whereas the first edition of the four chapters of *Theory of Sets* appeared (in French) only between 1954 and 1957. This interval saw many important developments in mathematics and, in particular, the emergence of category theory. It is likely that these developments stimulated Bourbaki’s own thinking and that this contributed to the gap between the contents of the “Fascicule” and that of the book itself. These developments will be discussed again in § 8.5 below.

69. It is important to remark, however, that the kind of Summary appearing in *Topological Vectorial Spaces* is not itself free of problems. One reviewer (Hewitt 1956, 508) wrote: “The ‘Fascicule de Résultats’ is of doubtful value. It would seem difficult to appreciate or use this brief summary without first having studied the main text; and when this has been done, the summary is not needed.”

Yet beyond the gap between the content of the Fascicule and that of the chapters, a shift in Bourbaki's conception of the role of *structures* within the treatise, and therefore within the whole picture of mathematics, is detectable already within the fascicule itself. In fact, a small but notable difference between the first and the third edition of the "Fascicule" exists, namely the addition of a footnote in the third edition. This footnote states that: "The reader may have observed that the indications given here are left rather vague; they are not intended to be other than heuristic, and indeed it seems scarcely possible to state general and precise definitions for structures outside the framework of formal mathematics." (p. 384)

By "outside the framework of formal mathematics", one should understand here "outside the conceptual framework proposed by Bourbaki in *Theory of Sets*." Thus, in spite of declarations to the contrary elsewhere, Bourbaki here implicitly admitted (concealing this confession, as it were, in a footnote) that the link between the formal apparatus introduced in *Theory of Sets* and the activities of the "working mathematician" (Bourbaki's declared main addressee) is tenuous, and, at best, of purely heuristic value.

After this account of the way in which *Theory of Sets* was constructed to enable the final definition of a structure and its related concepts, it is time to inspect more closely the use to which these concepts are put in the different books of the treatise.

7.3.2 Algebra

Bourbaki's book on algebra comprises nine chapters, the first editions of which appeared in print between 1942 and 1959 and which later underwent several re-editions.[70] The image of algebra dominating Bourbaki's book on algebra is essentially the same as that of *Moderne Algebra* in the sense that different algebraic structures are presented in a somewhat hierarchical manner. Thus, for instance, vector spaces are presented as a special case of groups and, therefore, all the results proven for groups hold for vector spaces as well. However, this hierarchy is absolutely non-formal since it is not anchored in terms of the concepts defined in the four chapter of *Theory of Sets*.

Neither commutative groups nor rings are presented as *structures* from which a group can be "deduced", nor is it proven that ***Z***-modules and commutative groups are "equivalent" *structures,* to take but two concepts. Some of

70. All quotations below are taken from the English version Bourbaki 1973.

the *structure*-related concepts do appear in the opening sections of the book, but the rather artificial use to which they are put and their absence from the rest of the book suggests that this initial usage was an *ad-hoc* recourse to demonstrate the alleged subordination of algebraic concepts to the more general ones introduced within the framework of *structures*. For example, readers are told that the definition of an “isomorphism of magmas” (§ 1.1), namely a bijection between two sets endowed with internal laws of composition which “preserves” the laws of composition, conforms to the “general definitions.” However, the formal verification of this trivial fact is actually much more tedious than it may appear at first sight. In fact, according to the definitions, one must first specify the echelon construction scheme of a “magma” (this is done as an example in Bourbaki 1973, § 1.4); then one should show, as explained above, that the defining axiom (namely the relation “$F \in \boldsymbol{B}((A\times A)\times A)$ as a functional graph whose domain is $A\times A$”) is a “transportable relation” for the given scheme, and finally, that

$$<f>^{S}(U) = U'$$

where $<f>^{S}$ is the canonical extension with scheme S and the function f, and U the *structure* in question.

All this exacting verification is neither accomplished not even alluded to in the book, nor is any similar assertion thoroughly verified in what follows. For example, the reader is reminded that the central theorem for a monoid of fractions of a commutative monoid can be expressed in the terminology introduced in *Theory of Sets* by saying that the problem in question “is the solution of the universal mapping problem for E, relative to monoids, monoid homomorphisms and homomorphisms of E into monoids which map the elements of S to invertible elements.” It follows, from a theorem proven in *Theory of Sets* for universal mappings, that the solution given here is essentially unique. This is one of the very few results of *Algebra* which can be pointed out as being obtained as a consequence of the general results obtained in *Theory of Sets*. However, due to the unwieldiness of the concepts, the formal verification of the conditions under which the particular case in question can be treated by using the general one is itself an elaborate process that is not carried out in the book, rendering doubtful, once again, the advantages of having invested so much effort in the general concepts.

The only theorems proven in terms of *structures* are the most immediate ones, such as the first and second theorems of isomorphism (§ 1.6, prop. 6&7),

and even they receive a special reformulation for groups later on in the same book (§ 4.5, Theorem 3). No new theorem is obtained through the *structural* approach and standard theorems are treated in the standard way. The Jordan-Hölder theorem (§ 4.7) aptly illustrates this situation, especially because elsewhere it had been proven within a wider conceptual framework of which group theory is a particular case,[71] while Bourbaki's proof was rather more restricted.

These remarks are not intended to imply that there is one best way to prove this, or any other, theorem. The point is merely to stress the fact that *structure*-related concepts, even within the framework of Bourbaki's own treatise, do not actually stand behind any generalization that is operationally important.

These are the only, feeble connections between algebraic structures and *structures* in Bourbaki's *Algebra*. As the book advances further into the subsequent theories in the hierarchy of algebraic structures, the connection with *structures* is only scarcely mentioned, if at all. Ironically, the need for a stronger unification framework was indeed felt in later sections. Such was the case, for instance, in Chapter 3 where three types of algebras defined over a given commutative ring are successively discussed: tensor-, symmetric- and exterior- algebras. Although a separate treatment is accorded to each type of algebra, this treatment nearly repeats itself in its details three times, one after the other. Thus Bourbaki defines each kind of algebra and then discusses, for each case separately: "functorial properties", "extension of the ring of scalars", "direct limits", "Free modules", "direct sums", etc.[72] This is worth mentioning not only because a unified presentation of the three could have been more economic and direct but especially because all the above mentioned issues lend themselves naturally to a categorical treatment and this possibility is not even mentioned here. The "functorial properties" of the algebras are explained through the use of the standard categorical device of "commutative diagrams",[73] but without mentioning the concepts of functor or category.[74] A further interesting point in this context is that for all the three cases a side

71. See for example Ore 1937 or George 1939. See Birkhoff 1948, 88 for a survey of different proofs of this theorem.

72. Bourbaki 1973, 484-522.

73. See § 8.1 below.

74. A similar situation, where categories and functors could have made the presentation more concise and more general, but their use was avoided, is found in Bourbaki's book on Commutative Algebra. This is discussed in greater detail below in § 7.3.4.

comment was added to the effect that a certain elementary fact, proven for the three cases "is a solution of the *universal mapping problem*",[75] for which the reader is referred to Chapter IV of *Theory of Set.* This result, however, is not formally proven, and what is more significant, it is not used for any purpose in the rest of the section or of the book.

7.3.3 General Topology

Bourbaki's book on general topology comprises ten chapters and a Fascicule de résultats, whose first editions appeared successively between 1940 and 1953, and then underwent several re-editions.[76] In this book one finds the single most outstanding example of a theory presented through Bourbaki's model of the hierarchy of *structures,* starting from one of the "mother structures" and descending to a particular *structure,* namely that of the real numbers.[77] According to the plan in the introduction of the book, the theory of topological spaces is presented in the opposite way to that in which it historically originated. The approach is characterized by the introduction of topological structure independent of any notion of real numbers or any kind of metric.

However, as with *Algebra,* the hierarchy itself is in no sense introduced in terms of the *structure*-related concepts. Thus for instance, topological groups are not characterized as a *structure* from which the *structure* of groups can be "deduced." *Structure*-related concepts appear in this book more than in any other place in the treatise but, instead of reinforcing the purported generality of such concepts, a close inspection of their use immediately reveals their *ad-hoc* character.

As a first example, take the concept of homeomorphism, which is defined (as was "isomorphism" defined in *Algebra)* as a bijection preserving the *structure* of the topology. This definition is claimed to be "in accordance with the general definition."[78] Again, the verification of this simple fact (which is nei-

75. Bourbaki 1973 485, 497, 507.

76. All quotations are taken here from the English version Bourbaki 1966.

77. However, it took considerable work and discussion within Bourbaki to arrive at this conception. Liliane Beaulieu (1990, 39) has described the first report on topology prepared by André Weil and presented for discussion in a Bourbaki congress of 1936 as follows: "One striking feature in Weil's report is that he first introduced the most familiar examples of topological concepts and spaces as a motivation to admit progressively more general ones. This is opposite of what later became Bourbaki's principle 'proceeding from the general to the particular'." Beaulieu also documented the subsequent transformation of the initial report into its definite formulation.

ther done nor suggested in the book) is a long and tedious (though certainly straightforward) formal exercise.

A more elaborate example appears in Chapter 3, dealing with topological groups, and more explicitly, in section 7.1 on "Inverse limits of algebraic structures." Since these notions mix algebraic and topological ideas, one would expect to find *structural* ideas applied here in order to analyze the relationship between these two separate mathematical domains. And in fact, one does find them, only to discover that their mention is somewhat deceptive. Thus Bourbaki relates the idea of "inverse limits" to *structural* ideas in the following words:

> Let Σ be a species of algebraic structures, and let Σ_0 be the *impoverished* structure corresponding to Σ. Whenever we speak of an inverse system of sets $(X_\alpha, f_{\alpha\beta})$ endowed with structures of species Σ, we shall always suppose that the $f_{\alpha\beta}$ are *homomorphisms* for these structures. If we endow $X = lim X_\alpha$ with the internal and external laws of the X_α, then X carries an algebraic structure of *species* Σ_0. Naturally it remains to be seen in each particular case whether or not this structure is of species Σ. (Bourbaki 1966 Vol. 1, 285. Italics in the original)

On the face of it, this could provide a felicitous instance of the application of *structural* concepts in order to elucidate an interesting mathematical situation. Yet, beyond the declaration of what should be done in *structural* terms, nothing of the sort is actually done. Instead, the above quotation is followed by the reformulation of the general setting for the particular cases of groups and rings, in which cases the question in the last sentence of the quotation may be answered in the affirmative.

Thus, the failure of *structures* to play a significant role as a generalizing concept is illustrated in *General Topology* not only by the infrequency of its applications, but *precisely* through the uses to which the concept is actually put. Far from being general concepts used in apparently *different* situations (as claimed by Bourbaki), many *structure*-related concepts appear *only* in a few instances of *Topology.*[79] Such concepts seem, therefore, to have been defined in *Theory of Sets* just to be handy for *General Topology*, but no other use was found for them in the whole treatise. Naturally, this is perfectly legitimate from the formal point of view, but it is much to the detriment of any claim

78. Bourbaki 1966, 18.

79. Such as in section 4.2., where a partial ordering of topologies is defined. The topologies are ordered from *coarser* to *finer.*

about the generalizing value of *structures*. Moreover, it certainly contradicts a leading principle of Bourbaki concerning the axiomatic treatment of concepts, namely that "a general concept is useful only if it is applicable to a number of more special problems and really saves time and effort."[80] Following that principle, Bourbaki did not hesitate to qualify other theories, as "insignificant and uninteresting." By now, it should be clear that Bourbaki's own theory of *structures* does not satisfy that principle.

7.3.4 Commutative Algebra

The remaining books of Bourbaki's treatise rely mainly on concepts taken from *Algebra* and *General Topology* and the concept of *structure* is totally absent from them.[81] In Bourbaki's *Commutative Algebra*, consisting of seven chapters whose first editions appeared between 1961 and 1965,[82] one finds a remarkable departure from the group's self-imposed methodological rules. In this book the limitations of *structures* as a generalizing framework are interestingly manifest and, in fact, they are explicitly acknowledged.

Consider the discussion on "flat modules." As it happens, this is a concept which is better understood in terms of concepts taken from homological algebra, a mathematical discipline which was not dealt with in the treatise until 1980. While it is often the case that when formally introducing concepts in a book of the treatise, Bourbaki illustrates those concepts by referring to an example which had not been yet introduced in that specific book, if the example is not a logical requisite for a full understanding of the concept itself and it appears in another place of the treatise, Bourbaki presents the example between asterisks and gives the corresponding cross-reference. This policy is explained in the "Mode d'emploi" that serves as preface to each of the books in the treatise.

In the case of flat modules, a whole section (§ 4) was included "for the benefit of the readers conversant with homological algebra", in which Bourbaki showed "how the theory of flat modules is related to that of the Tor functors."[83] The concept of functor and the particular case of the Tor functor are

80. Cartan 1980, 180.

81. As a matter of fact, the term "structure" is used once, but with a completely different meaning. See Book X on "Differential and Analytic Manifolds": F. § 6.2.1, p.62.

82. All quotations are taken here from the English version Bourbaki 1972.

83. Bourbaki 1972, 37.

not developed in the treatise, but Bourbaki thought it important to present the parallels between the two approaches: Bourbaki's own approach and the functorial approach to homological algebra. In order to do this, Bourbaki freely used concepts and notations foreign to the treatise. This is one of the few instances in the treatise where, instead of sticking to the usual notation between asterisks, Bourbaki gave a reference to a book or article outside it. Thus, the reader is referred to a forthcoming volume of the treatise where categories, and in particular, Abelian categories were eventually to be developed. Until then, however, one could also consult Cartan & Eilenberg 1956 or Godement 1958.

A cursory examination of issues of "La Tribu" during the fifties uncovers recurring attempts to write chapters on homological algebra and categories for the *Eléments*. Eilenberg himself, who had initiated together with Saunders Mac Lane the study of categories (§ 8.2 below), was commissioned several times to prepare drafts on homology theories and on categories, while a Fascicule de résultats on categories and functors was assigned successively to Grothendieck and Cartier.[84] However, the promised chapter on categories never appeared as part of the treatise. The publication of such a chapter could have proved somewhat problematic when coupled with Bourbaki's insistence on the centrality of *structures*; the task of merging both concepts, i.e. categories and *structures*, in a sensible way, would have been arduous and not very illuminating, and the adoption of categorical ideas would probably have necessitated the rewriting of several chapters of the treatise.[85] In this regard, it is interesting to notice that when the chapter on homological algebra was finally issued (1980) the categorical approach was not adopted. Although the conceptual framework provided by categories had become the standard one for treating homological concepts since the publication of the above mentioned textbook of Cartan and Eilenberg, in Bourbaki's own presentation these concepts are defined within the narrower framework of modules. And naturally, the concept of *structure* was not even mentioned there.

84. Cf. for example "La Tribu" #28 (June 25 - July 8; 1952); # 38 (March, 11-17; 1956); # 39 (June 4-July 7; 1956); #40 (October 7-14; 1956).

85. This point is elaborated in detail below in § 8.4.

7.4 *Structures* and the Structural Image of Mathematics

Although the Bourbaki project started with a relatively limited aim in mind, namely, the writing of an up-to-date treatise in analysis, in the early 1940s, after several years of activity, a much more ambitious program was consolidated. The *Eléments* eventually assumed the form of a unified, comprehensive presentation of *the whole picture* of the essentials of mathematics from a single, best point of view.[86] This conception, however, eventually proved overly sanguine and Bourbaki soon realized that they must limit themselves to only a more reduced, if still highly significant portion of mathematics.[87] Moreover, it became clear that the accelerating pace of developments in research would make it impossible to bring the *Eléments* fully up-to-date. Nevertheless, Bourbaki continued to regard each volume as a definitive survey containing all the basic knowledge needed for understanding and pursuing research in the particular disciplines considered.[88] The *Eléments* was intended to provide the basis for the "classical" component of mathematical knowledge, which it was assumed would remain basically unchanged during the foreseeable future. Thus, it was supposed to provide all the tools needed for developing the second component of mathematics, the living one, as made manifest in current mathematical research.[89]

The evidence presented above suggests that *Theory of Sets*, and particularly the concept of *structure* defined in it, are not essential to the contents of

86. Diuedonné (1970, 145) expressed this conception in retrospect as follows: "Bourbaki sets off from a basic belief, an unprovable metaphysical belief we willingly admit. It is that mathematics is fundamentally simple and that for each mathematical question there is, among all possible ways of dealing with it, a best way, an optimal way." As with other issues, the opinions of other members on this point are less documented, if at all. However, the spirit of the whole project and of the specific discussions on each chapter of the treatise, as documented in the various issues of "La Tribu", indicate in this case that Dieudonné's report expresses faithfully an idea shared by other members of the group.

87. Dieudonné 1970, 136: "Little by little, as we became rather more competent and more aware, we realized the enormity of the job that had been taken on, and that there was no hope of finishing it as quickly as [planned]." See also Fang 1970, 43: "The grand plan notwithstanding ... [Bourbaki's] work will remain unfinished because modern mathematics will never be completed." See also Israel 1977, 67.

88. Cf. Boas 1970, 351.

89. Dieudonné opened his book *A Panorama of Pure Mathematics - As seen by Nicolas Bourbaki* (1982a), with an account of mathematics as composed of two different parts, in the above terms, namely a 'classical' and a 'living' one. What Dieudonné includes under the 'classical' part of mathematics equals the contents of the *Eléments*.

the *Eléments*. One can read and understand any book of Bourbaki's treatise without first learning the theory of *structures*. *Theory of Sets* could in principle be omitted from the series, for it has neither heuristic value nor logical import for any particular theory discussed in the other volumes of the treatise, which form the heart of Bourbaki's real interests. More than any other section of *Theory of Sets*, the theory of *structures* could safely be skipped by any potential reader. Seen from the vantage point of what Bourbaki envisaged for the treatise, namely, to provide the necessary, basic toolkit for the working mathematician, the concept of *structure* seems to be forced and unnatural.

Yet it is not only within Bourbaki's own work that the concept of *structure* plays no mathematically meaningful role. While the various books of the *Eléments* generally turned into widely quoted and even classic references for the topics covered therein, and a considerable portion of the concepts, techniques, notation and nomenclature, introduced by Bourbaki were readily adopted by the practitioners of those branches, this was not the case for *Theory of Sets* and the *structure*-related concepts.[90]

This conclusion can be easily confirmed by examining any scientific review journal. Consider for instance the *Index of Scientific Citations*, during the period 1962 to 1966, the apogee of Bourbaki's influence. The index during these five years includes over 435 quotations of the *Eléments*, but only three of them refer to the chapter on *structures*. Of these three quotations, one appears in a theoretical biology article.[91] In general, the ideas of *Theory of Sets* seem to have inspired organizational schemes for non-mathematical disciplines more than they directly influenced mathematical research.[92]

But if the description of Bourbaki's work presented here is correct, why—it may be asked—have "mathematical structures" come to be generally identified with Bourbaki? The answer is quite simple, and it has to do with the distinction, stressed all along in the present book, among the various formal and non-formal meanings of the term "structure." This distinction has been often

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90. Nevertheless, it should be stressed, that a renewed interest in Bourbaki's concept of *structure* has arisen lately in the framework of current research in model theory, incidentally in connection with the work of José Sebastiao e Silva (mentioned in footnote 3 in the introduction to Part Two above). See, e.g., Da Costa 1987, 144-145; Da Costa & Chuaqui 1988. According to Da Costa (1986, 143 ff.), the limitations of Bourbaki's theory is a consequence of its focusing on syntactic issues. Sebastiao e Silva's ideas, in Da Costa's view, if properly elaborated (using also techniques developed by Alfred Tarski (1901-1983)), could provide the semantical dimension lacking in Bourbaki's concepts, thus providing the conceptual basis for new avenues of research in model theory.

91. Cf. Gillois 1965.

left vague in the historical writings of Bourbaki and of some Bourbaki members. In order to discuss this point properly, it is first necessary to elaborate briefly on the issue of Bourbakian historiography.

Each of the books of Bourbaki's treatise is accompanied by an account of the historical evolution of the discipline considered in it. These accounts were later collected and published in a volume entitled *Eléments d'histoire des mathématiques* (1969), which was widely read among mathematicians and often praised by them.[93] Dieudonné and Weil have been among the Bourbaki members who have expressed a clear and sustained interest in the history of mathematics. Besides having taken active part in the writing of the *Eléments d'histoire*, they independently published abundantly on the history of mathematics.[94]

Bourbaki's historiography, as manifest in the *Eléments d'histoire* as well as in the individual writings of Dieudonné and of Weil, has been strongly connected with their overall conception of mathematics. In particular, they have applied similar criteria to differentiate important from unimportant ideas in both *present* mathematical research and *past* mathematical theories. Naturally enough, this has also been the case regarding the centrality of structures in mathematics.

Bourbakian historiography has been criticized in the past for its "Whiggish" approach. As a matter of fact, this is perhaps the domain in which Bourbaki's writings have been more harshly criticized.[95] Yet most of the criticism

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92. A typical example of this is provided by the so-called "structuralist" trend in contemporary philosophy of science. (For an account of the development of the trend see Diederich 1989). Wolfgang Stegmüller (1979), for instance, described the structuralist approach as "A Possible Analogue to the Bourbaki Programme in Physical Sciences." In attempting to clarify the internal structure of physical theories he applied an axiomatization procedure which allegedly follows "Bourbaki's programme." According to Stegmüller Bourbaki performed a formalization of all mathematics using "set-theoretical rather than metamathematical methods." By following a similar approach in order to formalize physical theories, Stegmüller intends to enable a "realistic" formal treatment of them and of their semantics. Bourbaki's approach, claims Stegmüller, represented an improvement over Russell's "impractical" foundational work in mathematics; thus it is bound as well to represent a parallel improvement over Carnap's "'impractical' foundational work" in the philosophy of empirical science. Stegmüller was pursuing here an approach initially proposed by Suppes 1969. See also Da Costa 1987; Moulines & Sneed 1979.

93. For typical examples of enthusiastic appraisals of Bourbaki's *Eléments d'histoire,* see Mac Lane 1986 and Scriba 1961.

94. The bibliography at the end of this book include some, but not all of the historical writings of Dieudonné and of Weil.

directed towards Bourbaki's historiography has not addressed the problematic status of *structures* within it. This latter issue can be addressed here following the above analysis of the roles played by the formal and by the non-formal concepts of "structure" in Bourbaki's mathematics. In fact, one can observe in Bourbaki's historiography a noteworthy combination of the tendency to apply to historical research criteria of selection initially conceived for refashioning mathematics in terms of structures, on the one hand, and of the alleged centrality of *structures* for mathematics, on the other hand. This combination has brought forward an historical narrative, according to which the idea of structure (and even perhaps the concept of *structure*) not only is essential to the present overall picture of mathematics, but has even been instrumental in bringing about its historical ascendancy. The equivocal use of the term "structure" in its various meanings has only complicated things more.

Consider, for example, the following quotation of Dieudonné that was already cited above:

> Today when we look at the evolution of mathematics for the last two centuries, we cannot help seeing that since about 1840 the study of specific mathematical objects has been replaced more and more by the study of mathematical *structures*. (Dieudonné 1979, 9. Italics in the original)

Taken as a very general statement, this is a seemingly straightforward historical assessment, although its import cannot be understood without knowing the precise meaning of the term "mathematical structure." But as it is, one tends to accept such a general claim, and even more so after Dieudonné adds

95. Spalt 1987 is in fact a book-long criticism of Bourbaki's historiography. See especially pp. 2-4 & 24. See also Spalt 1985: "Strukturalistische Mathematikgeschichtsforschung ist kaum etwas anderes als ein kriminalistisches Aufsuchen jener Begriffe (oder deren struktureller Pendants) in der Vergangenheit, die aus heutiger—sprich: Bourbakis—Sicht die "wahre Natur" der mathematischen Theorien ausmachen. Lehrsätze oder Methoden früherer Zeit sind solcher Historie nur dann und insoweit von Bedeutung, als sie sich als "Spezialfälle" oder "Vorläufer" zeitgenössischer Verallgemeinerungen erfassen lassen."

Grattan-Guinness 1979 harshly criticizes the historiographical approach of Dieudonné (ed.) 1978, especially on grounds of its retrospectively applying present criteria of selection and of its failing to refer to any existing secondary literature. Freudenthal 1981 praises that volume edited by Dieudonné, claiming that "in spite of its shortcomings this *is* a good history of mathematics." However the shortcomings stressed by Freudenthal in some detail are very serious and can hardly be overseen. For further criticism of Bourbaki's historiography see also Israel 1977, 64-65; 1978, 63-69; 1981, 209-211; and Lakatos 1976, 135 & 151.

a dose of caution and states that, indeed, the concept of structure was foreign to mathematicians even as late as 1900. Thus, he further wrote:

> ... this evolution was not noticed at all by contemporary mathematicians until 1900, because not only was the general notion of mathematical structure foreign to them, but the basic notions of specific structures such as group or vector space were emerging very slowly and with a lot of difficulty. (*ibid.)*

The quotation suggests that after 1900, the general notion of "mathematical structure" became known or somewhat clearer to mathematicians. This is a questionable assertion in itself, but not one that requires an elaborate criticism. In fact, Dieudonné's claim is highly problematic because it *cannot* be taken as an assertion merely about the general emergence of structures. To be sure, his claim is followed by a footnote specifying that the term "mathematical structure" is to be taken *in the specific technical sense defined by Bourbaki in the fourth chapter of the first book of the Eléments*! The quotation should, then, read as the claim that since about 1840, and more explicitly after 1900, mathematics has increasingly become the study of *structures*! This statement is quite different from the general one suggested above and, as the preceding chapters clearly show, it can in no sense be accepted as historically correct.

Of course, not all articles by or about Bourbaki assume the identity of the non-formal and formal sense of the term "structure" as explicitly as Dieudonné did in the above-quoted passage.[96] But even when this identification appears in more ambiguous terms,[97] it supports a pervasive assumption that seems to underlie many accepted accounts of the structuralist approach to mathematics and of the central role played by Bourbaki in its consolidation

96. Cf., e.g., Dieudonné 1982, 619: "[T]he connecting link [between the diverse theories within the treatise] was provided by the notion of *structure*." (Italics appear here in the original, but not following the convention adopted in the present book to denote the formal term. It is therefore not clear, in Dieudonné's text, whether he means the formal or the non-formal sense.

97. Cf., for instance, the following account of Bourbaki's early years of activity, in which both senses of the term are subtly intermingled (Weil 1992, 114): "In establishing the tasks to be undertaken by Bourbaki, significant progress was made with the adoption of the concept of structure, and of the related notion of isomorphism. Retrospectively these two concepts seem ordinary and rather short on mathematical content, unless the notions of morphism and category theory are added. At the time of our early work these notions cast light upon subjects which were still shrouded in confusion: even the meaning of the term 'isomorphism' varied from one theory to another. That there were simple structures of group, topological space, etc., and then also more complex structures, from rings to fields, had not to my knowledge been said by anyone before Bourbaki, and it was something that needed to be said." A similar statement appears also in Weil 1978, 537.

and expansion. This is particularly the case when it comes to the putative link between the hierarchy of structures and the alleged centrality of the so-called "mother structures."[98] The mother structures appear in Bourbaki's *Architecture* manifesto as follows:

> At the center of our universe are found the great types of structures, ... they might be called the mother structures ... Beyond this first nucleus, appear the structures which might be called multiple structures. They involve two or more of the great mother-structures not in simple juxtaposition (which would not produce anything new) but combined organically by one or more axioms which set up a connection between them... Farther along we come finally to the theories properly called particular. In these the elements of the sets under consideration, which in the general structures have remained entirely indeterminate, obtain a more definitely characterized individuality. (Bourbaki 1950, 228-29)

It should be stressed again that this description of the mother structures is *not* an integral part of the formal, axiomatic, theory of *structures* developed by Bourbaki. The classification of structures according to this scheme is mentioned several times in *Theory of Sets*, but only as an illustration appearing in scattered examples.[99] Many assertions that were suggested either explicitly or implicitly by Bourbaki or by its individual members—i.e., that all of mathematical research can be understood as research on structures, that there are mother structures bearing a special significance for mathematics, that there are exactly three, and that these three mother structures are precisely the algebraic-, order- and topological-structures (or *structures*)—all this is by no means a logical consequence of the axioms defining a *structure*. The notion of mother structures and the picture of mathematics as a hierarchy of *structures* are not results obtained within a mathematical theory of any kind. Rather, they belong strictly to Bourbaki's non-formal images of mathematics; they appear in non-technical, popular, articles, such as in the above quoted passage,[100] or in the myth that arose around Bourbaki.[101]

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98. Which is incidentally hinted at in Weil's passage quoted in the preceding footnote.

99. For example in *Theory of Sets* pp. 266, 272, 276, 277, 279.

100. And also, in greater detail, in Cartan 1980, 177.

101. It should be stressed, however, that in Bourbaki's "Architecture" manifesto, one also reads (Bourbaki 1950, 229), that the picture of mathematics as a hierarchy of *structures* is nothing but a convenient schematic sketch, since "it is far from true that in all fields of mathematics, the role of each of the structures is clearly recognized and marked off." Furthermore, "the structures are not immutable, neither in number nor in their essential contents."

The equivocal blurring of the formal and non-formal meanings of the term "mathematical structure" and the related issues surrounding the mother structures allows one to identify the influence of Bourbaki's images of mathematics in places where the name Bourbaki is not even mentioned. In fact, the centrality accorded by Bourbaki to the idea of structure, while associating it with the theory of *structures*, has been implicitly taken for granted even by historians who consciously attempt to adopt a historiographical approach opposed to that of Bourbaki. Thus, whenever an author classifies mathematical structures as algebraic, topological and ordered structures, one may assume that he has taken Bourbaki's scheme for granted and has accepted that the mother structures are a meaningful mathematical idea.

A noteworthy instance of this implicit acceptance of Bourbaki schemes appears in Wussing's book on the rise of the abstract concept of group. In this book, contrary to Bourbaki's historiographical tendency, much effort is invested in order to avoid hindsight in the exposition of the development of mathematical ideas. However, in its epilogue, when the author explains the connection between the rise of the abstract concept of group and the rise of the structural trend in mathematics in general, he wrote:

> Within the limits of my study, and to the extent to which these limits bear on the history of the structural concept of "group", the connections between structural thinking and classical mathematics are relatively clear. The very advanced systematization of algebraic structures within contemporary mathematics, that is, the existence of "universal algebra", suggests analogous studies of the genesis of other algebraic structures. But in view of the absence of methodological models and the state of modern mathematics, one is likely to encounter far greater difficulties in the study of ordered and topological structures. (Wussing 1984, 258)

In this quotation, Wussing wholly adopts not only Bourbaki's mother structures scheme, but also the assumption that the possibility of properly elucidating the idea of structure depends on the existence of a formal concept of "structure" for a specific domain. Thus Wussing claims that it was the existence of a concept of "universal algebra" which should encourage the research into the rise of other particular algebraic structures. On the contrary, the absence of a general formal concept of topological- or order-structure has, in his view, hindered the undertaking of historic studies in those areas. But as the present book intends to show, the situation is precisely the opposite. The existence of formal concepts of structures may lead to incorrect historical interpretation, since it induces overlooking the important non-formal aspects of the actual historical process. It should be added, however, that in Wussing's book, as in several other similar works, the issue of the rise of the structural method is just an offshoot of the main argument, an afterthought or a programmatic statement for future works. This is probably the reason why he and other authors are much less critical on this issue than in the general tone of their works.[102]

It is remarkable that in some of Bourbaki's writings one finds an ambiguous attitude towards the validity of the notion of the hierarchy of *structures*. On the one hand, Bourbaki has cautioned that the picture of mathematics as a hierarchy of *structures* is nothing but a convenient scheme since "it is far from true that in all field of mathematics the role of each of the structures is clearly recognized and marked off." Furthermore, "the structures are not immutable, neither in number nor in their essential contents."[103] On the other hand, the inclusion of those examples in *Theory of Sets*, amidst Bourbaki's formal treatment of a theory of structures has had the effect, intentionally or unintentionally, of conferring upon them, metonymically as it were, that special kind of truth-status usually accorded to deductively obtained propositions.

This ambiguous link between the idea of mother structures and a formal mathematical theory has been manifest not only in works on the history of mathematics but also in works that address questions concerning the nature of mathematics and its applicability to other disciplines. Perhaps the best known example of this kind is reflected in Piaget's manifest enthusiasm for Bourbaki's work, already mentioned above. Hans Freudenthal has commented on this wrong-headed view as follows:

> The most spectacular example of organizing mathematics is, of course, Bourbaki. How convincing this organization of mathematics is! So convincing that Piaget could rediscover Bourbaki's system in developmental psychology. ... Piaget is not a mathematician, so he could not know how unreliable mathematical system builders are. (Freudenthal 1973, 46)[104]

102. For additional accounts of the rise of the structural approach in mathematics in which the mother structures are similarly alluded to see Behnke 1956, 29-31; Birkhoff 1974, 336; Novy 1973, 223; Purkert 1971, 23. Even historians who have critically approached Bourbaki's pronouncements on mathematics in general, seem to have admitted Bourbaki's claims on the centrality of mother *structures* at face value. Cf. Mac Lane 1987a, 33 ff. (and § 9.2 below); Israel 1978, 60-61; Mehrtens 1990, 139.

103. Bourbaki 1950, 229. And of course the term "structure" is equivocally used in this passage, so that it is not completely clear whether it should be taken in its formal or in its non-formal meaning.

But Piaget was not alone in failing to realize how unreliable mathematical system builders can be. Some mathematicians seem to have been equally gullible.[105] Bourbaki' members, especially in the first years, hardly saw themselves as "unreliable system builders" nor did they see their formulations as provisional. Doubts about the certainty of Bourbaki's program arose only later on, whereas the image of mathematics as revolving around the concept of *structure* persisted long after that. This change in attitude is shrouded in the ambiguity of claims advanced by Bourbaki members, especially Dieudonné, indicating that "the connecting link [between the diverse theories within the treatise] was provided by the notion of *structure*."[106] If "structure" is taken to mean *structure* then Dieudonné's claim reflects Bourbaki's initial confidence. In that case, however, they are imprecise. If, on the contrary, "structure" is given its non-formal meaning, then Dieudonné's claims may be sound, but they say something different, and indeed significantly less, than they were meant to assert.

The present chapter has analyzed the role played by the concept of *structure* within Bourbaki's *Eléments de mathématique*. The influence of Bourbaki's treatise on twentieth-century mathematics presents significant parallels to that of van der Waerden's book on algebra. As textbooks that compiled much important, previously existing research work, their central contribution consisted in the restructuring of the disciplines studied in them. This restructuring implies a redefinition of the subdisciplines involved, and of their boundaries and interrelations. It also implies a selection of basic concepts, basic tools and basic problems in each subdiscipline. In the case of Bourbaki, given the ambitious scope of the enterprise, this restructuring had implications for mathematics as a whole. One should also bear in mind the fact that both works exerted a strong influence mainly on the images of mathematics, rather

104. For a further criticism of Piaget's reliance on Bourbaki's schemes see Lurcat 1976, esp. 278-280.

105. Consider for example the following assessment of Bourbaki's contributions (Gauthier 1972, 624): "Le Chap. IV sur les 'Structures' est sans doute le plus novateur: on sait que ce thème de structure est proprement bourbakiste. C'est le groupe Bourbaki qui a, en effet, thématisé les structures en mathématiques et les a catégorisées selon les trois grandes espèces de structures-mères." Cartan, on the other hand, explicitly declared that, after twenty years of activity, "there may be some concepts among the fundamentals in Bourbaki's textbook which have already become outdated." Cf. Cartan 1980, 180.

106. Dieudonné 1982, 619. Italics in the original.

than directly through the body of mathematics. This is not intended to belittle the scope or importance of this influence, but rather the opposite. In framing the main open problems, the accepted tools, the aims of mathematical education, etc., in the decades that followed its publication, the *Eléments*, and the picture of mathematics implied by it were to become a central force in the development of mathematical research, and in the growth of mathematical knowledge.

The concept of *structure*, allegedly a central pillar in Bourbaki's building of mathematics, plays no actual role in the presentation of theories within the treatise. Nevertheless this concept and the associated hierarchy based on the three mother structures have been often considered as if they in fact provide a solid, reflexive foundation of Bourbaki's images of mathematics and, in a certain sense, endow them with a kind of justification that is usually absent from alternative schemes.

Bourbaki's work and the way it influenced the subsequent development of mathematics interestingly illuminate the interplay between images and body of knowledge. On the one hand, Bourbaki's creation of a new system of images of knowledge together with the impressive body of knowledge—which was produced under the influence of this image—were to direct and condition a considerable portion of mathematical research for several decades. On the other hand, the inherent force of these images of knowledge was partially bolstered by a reflexive body of knowledge, the theory of *structures*, which in fact did *not* provide the kind of foundational support attributed to it. The mathematical significance of Bourbaki's overall contribution remains the same if one deletes or ignores the existence of the theory of *structures*. Moreover, it is arguable, although by no means certain, that Bourbaki's actual influence on mathematics would have remained the same had not the entire book, *Theory of Sets*, and the theory of *structures* been published at all. Nevertheless, one may wonder how the history of the idea of structure (both in and outside mathematics), as well as its historiography, would have looked like had not Bourbaki formulated the theory of *structures* and had not the distinction between the formal and non-formal meanings of the term "mathematical structure" been blurred in the writings of various individual members of the group.