LEO CORRY

NICOLAS BOURBAKI AND THE CONCEPT OF MATHEMATICAL STRUCTURE*

ABSTRACT. In the present article two possible meanings of the term "mathematical structure" are discussed: a formal and a nonformal one. It is claimed that contemporary mathematics is structural only in the nonformal sense of the term. Bourbaki's definition of *structure* is presented as one among several attempts to elucidate the meaning of that nonformal idea by developing a formal theory which allegedly accounts for it. It is shown that Bourbaki's concept of *structure* was, from a mathematical point of view, a superfluous undertaking. This is done by analyzing the role played by the concept, in the first place, within Bourbaki's own mathematical output. Likewise, the interaction between Bourbaki's work and the first stages of category theory is analyzed, on the basis of both published texts and personal documents.

1. INTRODUCTION

It is commonplace for mathematicians and nonmathematicians alike to refer to the structural character of mathematics in the twentieth century. In structuralist texts, mathematics is described as the paradigm of a structural science.[1] Historians of mathematics usually emphasize the centrality of the concept of "structure" in contemporary mathematical research.[2] "Mathematical structures" appear in contemporary philosophy of mathematics as well. Several philosophers of mathematics have suggested that the concept of structure may provide a solution to many of the most fundamental questions in their discipline.[3]

Together with the widespread identification of contemporary mathematics with the idea of structure, it is also common to associate the structural trend in mathematics with the name of Nicolas Bourbaki. For instance, this identification is explicitly made by the 'structualist' Jean Piaget. Piaget even established a clear correspondence between Bourbaki's so-called "mother-structures" (i.e., algebraic structures, order structures and topological structures) with the first operations through which a child interacts with the world.[4]

However, although nowadays there exists indeed a high degree of agreement that mathematics is 'structural' in character, even a cursory examination of the meaning of the term "structure" in the diverse places where it appears will reveal that the term "mathematical structure" is

Synthese **92**: 315–348, 1992.

used and understood in diverging ways. One may then ask: Can a more precise definition of the supposed 'structural character' of twentieth-century mathematics be formulated? Is the identification of the term with the name of Bourbaki justified on any grounds? If it is not, why was it so identified in the first place?

I will address these and other questions in the present article. I will claim that the "structural character of contemporary mathematics" denotes a particular, clearly identifiable way of doing mathematics, which can however only be characterized in *nonformal* terms. After that specific way of doing mathematics was crystallized and became accepted in the 1930s, diverse attempts were made to provide a *formal* theory within the framework of which the *nonformal* idea of a "mathematical structure" might be *mathematically* elucidated. Many confusions connected to the "structural character of mathematics" arise when the distinction between the formal and the nonformal senses of the word is blurred. When the distinction is kept in mind, it soon becomes clear that Bourbaki's real influence on contemporary mathematics has nothing to do with the concept of *structure*.[5] In what follows, we will discuss the rise, the development and the eclipse of Bourbaki's concept of *structure*.

2. FORMAL AND NONFORMAL CONCEPTIONS OF MATHEMATICAL STRUCTURE

Algebra is the discipline in which the structural approach to mathematics first crystallized. Algebra is presently seen as the study of 'algebraic structures', but throughout the eighteenth and nineteenth centuries, the aim of algebra was the study of polynomial equations and the problem of their solvability. Many new mathematical concepts and theorems were introduced during the nineteenth century which are presently considered part of the hardcore of algebra. However, the deep change undergone by algebra in that period was not just an impressive quantitative growth in the body of knowledge but, rather, a change in the overall conception of the aims, the methods, the interesting questions and the possible answers to be worked out in algebra. In other words, the rise of the structural approach in algebra signified a change in the "images of mathematics".[6]

The first book to present a comprehensive exposition of algebra from the structural point of view was *Moderne Algebra* (1930) by B. L. van der Waerden. Roughly stated, the aim and contents of this book may be characterized by a formulation which seems rather obvious nowadays: to define the diverse algebraic domains and to attempt to fully elucidate their structures. Although this formulation became increasingly accepted and understood among mathematicians and it actually came to dominate algebraic research, it is remarkable that van der Waerden's book did not contain any explicit statement explaining what a structure is or what is meant by "elucidating the structure" of an algebraic domain.

The view of algebra advanced in *Moderne Algebra* brought about a unified perspective in the research of several branches: similar questions were asked about them, similar methodological tools were applied and similar answers were expected. This unified, structural perspective does not necessitate an explicit elucidation of the idea of structure. In fact, this idea belongs to the corpus of tacit knowledge shared by mathematicians in their day-to-day work without, however, being part of any specific formal mathematical theory. One can attempt to render explicit the meaning of this nonformal idea by examining what is actually done in structural mathematical research. One can also examine the introductions to textbooks or other expository papers where mathematicians involved in structural research explain their activities.[7] Such an examination reveals that, like many other tacitly shared images of knowledge, the nonformal idea of mathematical structure is interpreted differently by different mathematicians and that, moreover, the meaning of the term has also changed from 1930 to present-day. For lack of space, we will not carry out such an examination here.

The answer to the question "What is a mathematical structure?", then, may be taken for granted and remain implicit, and it may also be explicitly formulated at the nonformal level. But it is also possible to conceive a formal mathematical theory as an answer to that question. At variance with other exact sciences, mathematics affords 'reflexive' means of formally elucidating its images of knowledge within mathematics itself. Proof theory, for instance, examines, from a formal mathematical point of view, the second-order idea of mathematical proof. This theory has produced significant mathematical information about a central tool of mathematics.[8] In a similar vein, formal mathematical theories have been conceived that attempt to elucidate, in strictly mathematical terms, the second-order idea of structure. Examples of this are Oystein Ore's foundation of abstract algebra in lattice-theoretical terms,[9] the development of universal algebra, category theory and Bourbaki's theory of *structures*.

A full account of the development of the idea of mathematical structure since 1930 should take into account, then, the changing nonformal images of knowledge surrounding that idea, the various formal attempts at elucidating it, and the interrelations among all these. In the present article I deal only with a small portion of that full picture: the evolution of Bourbaki's reflexive attempt and its relation to the rise of category theory.

3. THE MYTH OF BOURBAKI

Nicolas Bourbaki is the pseudonym adopted during the 1930s by a group of young French mathematicians who were dissatisfied with the state of contemporary French mathematics and considered the methods of the old masters (Hadamard, Picard, Borel, Goursat, etc.) inadequate for modern research. The group undertook the task of bringing French mathematics up-to-date through the publication of a comprehensive treatise entitled *Éléments de Mathématique*, each volume of which was to deal with a different field of mathematics.

The idea was conceived in 1933 and the first congress of Bourbaki took place in July 1935. The founding members of the group were Jean Dieudonné, Henry Cartan, André Weil, Paul Delsarte, Claude Chevalley, Szolem Mandelbrojt, Jean Coulomb, Charles Ehresmann and René de Possel.[10] Over the years, many other prominent mathematicians joined the group, including Samuel Eilenberg, Jean Pierre Serre, Alexander Grothendieck and others.

Bourbaki's treatise is the outcome of arduous collective work. Members of the group met from time to time in different places around France and in those meetings individual members were commissioned to produce drafts of the different chapters. The drafts were then subjected to the harsh criticism of the other members and reassigned for revision. Only after several drafts had been written and criticized was the final document ready for publication. Minutes of the meetings were taken and circulated among members of the group in the form of an internal bulletin called *La Tribu*. The contents of the issues of *La Tribu* are sometimes hard to understand because they abound with personal jokes, obscure references and slangy expressions. However, they are very useful for the historian researching the development of Bourbaki's ideas.[11]

In the decades following the founding of the group, Bourbaki's books became classics in several areas of pure mathematics. Bourbaki's style and a considerable part of his[12] innovations in nomenclature and symbolism soon became standard. Moreover, ever since the name of Bourbaki first appeared in public, it became the focus of much attention and curiosity among mathematicians, and a full-fledged mythology developed around the group.[13] The legend of Bourbaki has, more often than not, impaired the objectivity of appraisals of Bourbaki's scientific output.[14] Several reviews of Bourbaki's writings are so uncharacteristically effusive in their extolling of his merits that their credibility becomes questionable.[15] There are, however, also less laudatory technical reviews of the *Éléments*.[16]

Assessing Bourbaki's overall influence on contemporary mathematics is an arduous task. Such an assessment must take into account the diverse degrees of influence which Bourbaki exerted in different periods of time, in different countries and, of course, on different branches of mathematics; algebra and topology were probably the branches on which Bourbaki exerted his most profound influence, while logic and most fields of applied mathematics seem not to have been aware of or influenced by Bourbaki at all. In the present article I will focus my comments on a rather limited aspect of Bourbaki's work, the concept of *structure*. This is a concept, however, very central to Bourbaki's conception of mathematics and, therefore, understanding the role that *structures* play in Bourbaki's work will provide insight into the overall import of Bourbaki.

4. THE STRUCTURES OF BOURBAKI

Bourbaki began his work in the late 1930s facing an unprecedented multitude of newly obtained results, some of them belonging to as yet unconsolidated branches of mathematics. The question arose more than ever whether it could still be legitimate to talk about a single discipline called "mathematics" or

> . . . whether the domain of mathematics is not becoming a tower of Babel, in which autonomous disciplines are being more and more widely separated from one another, not only in their aims, but also in their methods and even in their language. (Bourbaki 1950, p. 221)

To Bourbaki, this divergence was more apparent than real. Like van der Waerden had done for thitherto disparate disciplines, which since then were included under the heading of "modern algebra", Bourbaki undertook the task of presenting the whole picture of mathematical knowledge in a systematic and unified fashion, within a standard system of notation, addressing similar questions, and using similar conceptual tools and methods in the different branches. But Bourbaki went a step beyond van der Waerden and attempted to provide a formal theory of *structures* affording a common foundation for all the other theories considered in his treatise. Bourbaki's work was originally motivated by the idea that *the whole* of mathematics may be presented in a comprehensive treatise from a unified, single best point of view, and the concept of *structure* was to play a pivotal role within it. This initial conception, however, proved overconfident and Bourbaki soon realized that he must limit himself to include in his treatise only a portion of mathematics. In particular, the concept of *structure* was gradually relegated to an ancillary plane. In the present section we will examine the actual role played by the *structures* in Bourbaki's treatise.

In the early 1970s the *Éléments* had nearly attained its present form, composed of ten books: I. *Theory of Sets*; II. *Algebra*; III. *General Topology*; IV. *Functions of a Real Variable*; V. *Topological Vector Spaces*; VI. *Integration*; *Lie Groups and Lie Algebras*; *Commutative Algebra*; *Spectral Theories*; and *Differential and Analytic Manifolds*.[17] Each book is composed of chapters that were published successively, though not necessarily in the order indicated by the index. Book I, on the theory of sets, which will be examined here in detail, is composed of four chapters and a "Summary of Results": 1. "Description of Formal Mathematics"; 2. "Theory of Sets" (first French edition of both chapters is 1954); 3. "Ordered Sets, Cardinals, Integers" (1956) and 4. "Structures" (1957). The "Summary" was first published in French in 1939.[18]

The gap of more than ten years between the publication of the summary and that of the four chapters bears witness to Bourbaki's hesitations about the contents of *Theory of Sets*. The summary reflects the initial conception: *Theory of Sets* was meant to provide a formally rigorous basis for the whole of the treatise, and the concept of *structure* represented the ultimate stage of this undertaking. The result, however, was different: *Theory of Sets* appears as an ad-hoc piece of mathematics imposed upon Bourbaki by his own declared positions about mathematics, rather than as a rich and fruitful source of ideas and mathematical tools. I will now provide support for this claim by examining the book in some detail.

The book is preceded by an introduction on formalized languages and the axiomatic method. Set theory is the theory to which all mathematics is to be reduced and the one which the formalized language is supposed to describe. However, since even the complete formalization of set theory alone is impracticable, strings of signs that are meant to appear repeatedly throughout the book are replaced from the beginning by abbreviating symbols, and condensed deductive criteria are introduced, so that for every proof in the book it will not be necessary to explain every particular application of the inference rules. In the end, we obtain a book which, like any other mathematical book, is partially written in natural language and partially in formulae but which, like any partial formalization, is supposed to be in principle completely formalizable. At any rate, the claim is made that the book on set theory lays out the foundations on which the whole treatise may be developed with perfect rigor.

Bourbaki's style is usually described as one of uncompromising rigor with no heuristic or didactic concessions to the reader. This characterization fits perhaps the bulk of the treatise, but not *Theory of Sets*. In fact, the further one advances through the chapters of *Theory of Sets*, encountering ever new symbols and results, the more Bourbaki assists himself with *heuristic* explanations of the meaning of the statements, even when they are not especially difficult.[19] The formal language that was introduced step by step in Chapter I is almost abandoned and quickly replaced by the natural language. The recourse to extraformalistic considerations in the exposition of results within a textbook is, of course, perfectly legitimate. *What I want to point out, however, is the divergence between this and the other books of the Éléments and between Bourbaki's pronouncements and what is really done in Theory of Sets.*

There is written evidence of Bourbaki's uncertainty about how to address the foundational problems encountered in *Theory of Sets*. Although such problems were not a major concern of the entire group, they had to be addressed if the desired formal coherence of the treatise was to be achieved. Several issues of *La Tribu* record different proposals for the contents of *Theory of Sets* and some technical problems encountered while developing it in detail. We will examine this point below in Section 5. However, it is pertinent to quote here a report on the progress in the work on *Theory of Sets* made by 1949, in which the above issues of formalized languages and inference rules is dealt with:

Dès la première séance de discussion, Chevalley soulève des objections relatives à la notion de texte formalisé; celles ci menacent d'empêcher toute publication. Après une nuit de remords, Chevalley revient à des opinions plus conciliantes, et on lui accorde qu'il y a là une sérieuse difficulté qu'on le charge de masquer le moins hyprocritement possible dans l'introduction générale. Un texte formalisé est en effet une notion idéale, car on a rarement vu de tels textes et en tous cas Bourbaki n'en est pas un; il faudra donc ne parler dans le chap. I qu'avec beaucoup de discrétion de ces textes, et bien indiquer dans l'introduction ce qui nous en sépare. Dans l'introduction des arguments typiques un canular [sic] a été caché, qu'il faudra mettre en évidence: si on veut faire une démonstration raisonnable, il faut, en introduisant un argument typique dans une démonstration (p. ex. "soit U un ouvert qui . . . , que . . ."), ou bien s'assurer que le type en question n'est pas vide (ceci étant une consequence des autres hypothéses) – ou bien dire qu'on pose en axiome le fait qu'il n'est pas vide, soit en vue de raisonner par l'absurde (cf. descente infinie), soit en vue de déduire des conséquences d'une conjecture non démontrée (postulat d'Euclide au 19–ème siècle, hypothèse de Riemann). Pour mieux éclairer sa lanterne on dissèquera une ou deux démonstrations de la suite du traité (p. ex. du chap. I de Top. Géné. où les choses sont assez simples). (*La Tribu*, 13–25 April 1949)

The discussion on formalized languages, then, appears more as an imposition on Bourbaki's treatise than as an urgent mathematical problem worthy of the attention of the group.

In Chapter II Bourbaki introduces his axioms for sets, which constitute a variant of ZF. Bourbaki's system treats ordering as an irreducible mathematical notion and, therefore, the existence of ordered pairs is asserted separately. Many concepts are treated here using a rather idiosyncratic notation which is hard to justify. Bourbaki's overall influence is sometimes reflected in the widespread adoption of new nomenclature and notation introduced in Bourbaki's treatise. Dieudonné has noted that not all of Bourbaki's innovations were adopted. He has found "no apparent reason [for this fact] except the obdurate conservatism ingrained in so many mathematicians".[20] However, if we were to judge by the notation introduced by Bourbaki in Chapter II of *Theory of Sets*, we could find additional reasons. Many of the concepts and notations introduced here are used no more than once. After reminding us that it is possible to use shorter formulations by "abuse of language", Bourbaki returns to the normal usage, swiftly abandoning many of the innovations.[21] Chapter III deals with ordered set, cardinals, and integers.

The concept of *structure* is developed in Chapter IV. Before defining *structures*, Bourbaki introduces some preliminary concepts. The basic ideas behind those concepts can be formulated as follows: take a finite number of sets $E_1, E_2, \ldots, E_n$. Suppose we take those sets as the building blocks of an inductive procedure, each step of which consists either in taking the Cartesian product of sets obtained in the former step or in taking their power set **P**(E). For example, if we begin with the sets E, F, G, the outcome of one such procedure could be: **P**(E)XF; **P**(G); **P**(**P**(E)XF); **P**(**P**(E)XF)X**P**(G); and so forth. Bourbaki introduces a formal device that enables us to define and characterize every possible construction of the kind described above. The last term obtained through a given construction of this kind for n sets $E_1, E_2, \ldots, E_n$ is called an "echelon construction scheme S on n base sets" and it is denoted by $S(E_1, E_2, \ldots, E_n)$. If we are given one such scheme and n additional sets E_i' and n mappings f_i: $E_i \rightarrow E_i'$, it is easy to define an additional straightforward, formal procedure which enables us to define a function from $S(E_1, E_2, \ldots, E_n)$ to $S(E_1', E_2', \ldots, E_n')$ (the corresponding system when built over the sets $E_1', E_2', \ldots, E_n'$ instead of $E_1, E_2, \ldots, E_n$). This function is called the "canonical extension with scheme S of the mappings $f_1, \ldots, f_n$" and it is denoted by $\langle f_1, \ldots, f_n\rangle^S$, and it is injective (resp. surjective, bijective) when each of the f_i's are so. Now to define a 'species of structure' Σ we take:

(1) n sets $x_1, x_2, \ldots, x_n$; the 'principal base sets';
(2) m sets $A_1, A_2, \ldots, A_m$; the 'auxiliary base sets' and
(3) a specific echelon construction scheme:

$$S(x_1, \ldots, x_n, A_1, \ldots, A_m).$$

This scheme will be called the 'typical characterization of the species of structure Σ'. The scheme is obviously a set and the *structure* is now defined by characterizing some of the members of this set by means of an axiom of the species of structure. This axiom is a relation which the specific member $s \in S(x_1, \ldots, x_n, A_1, \ldots, A_m)$ together with the sets $x_1, \ldots, x_n, A_1, \ldots, A_m$ must satisfy. The relation in question is constrained to satisfy the conditions of what Bourbaki calls a 'transportable relation', which means roughly that the definition of the relation does not depend upon any specific property of s and the sets in themselves, but only refers to the way in which they enter in the relation through the axiom. The following example introduced by Bourbaki makes things clearer:

> An internal law of composition on a set A is a function from AXA into A. Accordingly, given any set A, we form the scheme **P**((AXA)XA) and then we choose from all the subsets of (AXA)XA, those satisfying the conditions of a 'functional graph' with domain AXA and range A. The axiom defining this choice is a special case of what we call algebraic *structures*. (Cf. Bourbaki 1968, p. 263)

Together with that example, Bourbaki also shows, using the concepts that were introduced above, how ordered-*structures* or topological-*structures* may be defined. That these are Bourbaki's first three examples is by no means coincidental. These three types of *structures* constitute what Bourbaki calls the *mother-structures*, a central concept of Bourbaki's images of mathematics which we will discuss below.

After the definition of *structure* is given, Bourbaki introduces further concepts connected with that definition. However, in the remainder of the chapter, continuous reference to n principal base sets and m auxiliary base sets is avoided by giving all definitions and propositions for a single principal base set (and for no auxiliary set) while stating that "[t]he reader will have no difficulty in extending the definitions and results to the general case" (Bourbaki 1968, p. 271). This is a further instance of Bourbaki ignoring in *Theory of Sets* the self-imposed strict rigor set out in the other books in the treatise.

I will now consider these concepts in some detail since they reveal the ad-hoc character of all the effort invested in *Theory of Sets*. Bourbaki's purported aim in introducing such concepts is expanding the conceptual apparatus upon which the unified development of mathematical theories will rest later on. However, all this work turns out to be rather redundant since, as we shall see, these concepts are used in a very limited – and certainly not highly illuminating or unifying – fashion in the remainder of the treatise. Let me, then, consider some examples.

* *Isomorphism:* Let U, U′ be two structures of the same type Σ on n principal base sets, $E_1, E_2, \ldots, E_n$ and $E'_1, E'_2, \ldots, E'_n$, respectively, and let n bijections f_i: $E_i \rightarrow E'_i$, be given. If S is the echelon construction scheme of Σ, then $f_1, \ldots, f_n\rangle^S$ is defined as an isomorphism if

$$\langle f_1, \ldots, f_n, \mathrm{Id}_1, \ldots, \mathrm{Id}_m\rangle^S(U) = U'$$

where Id_i denotes the identity mapping of an auxiliary set A_i into itself.

This definition uses the concept of canonical extension introduced above to express in a precise fashion the desirable fact that the isomorphism 'preserves' the structure.

* *Deduction of Structures:* Bourbaki defines a formal procedure for deducing a new species of structures from a given one. For instance, if the species of topological group structures is defined on a single set A by a generic structure (s_1, s_2), where s_1 is the graph of the composition law and s_2 the set of the open sets of A, then each of the terms s_1 and s_2 is a procedure of deduction and, respectively, provides the *group* and the *topology* underlying the topological group structure (s_1, s_2). Likewise, we can deduce a commutative group structure from either a vector space, or from a ring or from a field.

* *Poorer-Richer Structures:* Among the examples introduced in order to clarify the mechanism of deduction of structures defined above, we find a criterion to order structures with the same base sets and the same typical characterization as *poorer* or *richer*, according to whether the axiom defining the latter can be 'deduced' from the former. For example, the species of a commutative group is *richer* than the species of groups.

* *Equivalent Species of Structures:* This definition enables us to identify the same structure when it is defined in different ways (e.g., commutative groups and Z-Modules).

* *Finer-Coarser Structures:* This is a further relation of order defined between structures of the same species. Roughly, the finer a given species of structures, the more morphisms it contains with E as source and the fewer morphisms it contains with E as target.

The final sections of *Theory of Sets* are devoted to special constructions which can be made within the framework of the *structures*: inverse image of a *structure*, induced *structure*, product *structure*, direct image and quotient *structure*. The very last section of the chapter deals with universal mappings. These are defined for an arbitrary *structure*, and the conditions are stated for the existence of a solution to the universal mapping problem in a given *structure*. It is proven that for this case this solution is essentially unique. The unwieldiness of the *structural* concepts is here perhaps more apparent than in any other place, since, for this specific problem, a fully developed and highly succinct version of the categorical formulation of the universal mapping problem is available.[22] I will further comment on this point below.

After the painstaking elaboration of the four chapters, one encounters the "Summary of Results" ("Fascicule de Résultats") containing all the results of set theory which will be of some use in the remainder of the treatise. However, the term 'summary' does not accurately describe the contents of this last section. The original "Fascicule de Résultats" seems a more precise name, because the "Fascicule" indeed contains 'résultats' – not all the results and not results exactly as they appear in the book but, rather, 'all the definitions and all the results needed for the remainder of the series'. If the book's stated aim was to show that we can formally establish a sound basis for mathematics, the fascicule's purpose is to inform us of the lexicon we will use in what follows and of the *informal* meaning of the terms within it. This sudden change of approach, from a strictly formal to a completely informal style, is clearly admitted and openly justified by Bourbaki in the opening lines of the "Summary":

As for the notions and terms introduced below without definitions, the reader may safely take them with their usual meanings. This will not cause any difficulties as far as the remainder of the series is concerned, and renders almost trivial the majority of the propositions. (Bourbaki 1968, p. 347)

For example, the huge effort invested in Chapters I and II is reduced to the laconic statement: "A set consists of *elements* which are capable of possessing certain *properties* and of having relations between themselves or with elements of other sets" (ibid., p. 347; italics in the original). A note explains further:

The reader will not fail to observe that the "naive" point of view taken here is in direct opposition to the "formalist" point of view taken in Chapters I and IV. Of course, this contrast is deliberate, and corresponds to the different purposes of this Summary and the rest of the volume. (Ibid.)

The purpose of the summary, then, is to provide, in a completely informal fashion, the common basis upon which the specific theories will later be developed. It is only in this informal fashion that Book I is related to the rest of the treatise and, in particular, that the concept of *structure* appears as a unifying concept. As we shall see below, this concept has no real mathematical use in the rest of Bourbaki's work.

The whole formal discussion of Chapter IV is reduced in the "Fascicule" to a short, intuitive explanation of the *structural* concepts (even shorter than the one given in the present article). The only important *structural* concept mentioned in the "Fascicule" is that of isomorphism. No mention at all is made of derived-, initial-, quotient-, coarser- and finer-, and other *structures* defined in Chapter IV. This summary of results is essentially different from its counterparts in the other books of the series (for example, that of *Topological Vectorial Spaces*),[23] both because of its variance from the actual contents of what it allegedly summarizes and because of the striking and total absence of technicalities. As I said above, while the "Fascicule" first appeared in French in 1939, the first edition of the four chapters of *Theory of Sets* appeared only between 1954 and 1957. This interval saw the emergence of category theory and it is clear that many ideas developed within that theory brought Bourbaki to rethink his conceptions, thus creating the gap between the contents of the "Fascicule" and that of the book itself. I will return to this point below.

There is a small but notable difference between the first and the third editions of the "Fascicule", namely, the addition of a footnote to the third edition stating that:

> The reader may have observed that the indications given here are left rather vague; they are not intended to be other than heuristic, and indeed it seems scarcely possible to state general and precise definitions for structures outside the framework of formal mathematics. (Bourbaki 1968, p. 384)

The expression "outside the framework of formal mathematics", should be taken to mean here 'outside the conceptual framework proposed by Bourbaki in *Theory of Sets*'. It follows that the concept of *structure* has no working significance outside the discussion of Chapter IV in *Theory of Sets*. In spite of Bourbaki's declarations in many other places, he admitted here that the link between the formal apparatus introduced in *Theory of Sets* and the activities of the 'working mathematician', which was supposedly Bourbaki's real concern, is tenuous and intuitive. This remark openly contradicts the alleged centrality of the concept of *structure* for the whole of mathematics, and it seems to have been banished to the footnotes as to conceal its real, if undesired, significance.

We have seen enough of how *Theory of Sets* is written to enable the final definition of *structures* and the gap between the book and the

"Fascicule". We proceed now to inspect more closely the use that these concepts are put to through the different books of the treatise, in order to complete my line of argumentation.

4.1. *Algebra* (1973)

The image of algebra in Bourbaki's book is essentially the same as that of *Moderne Algebra*, in the sense that different algebraic structures are presented in a somewhat hierarchical way. Thus, for instance, vector spaces are a special case of groups and, therefore, all the results proven for groups hold for vector spaces as well. However, this hierarchy is absolutely informal and it is in no way presented in terms of the concepts defined in *Theory of Sets*, Chapter IV.

Neither commutative groups nor rings are presented as *structures* from which a group can be 'deduced', nor is it proven that Z-modules and commutative groups are 'equivalent' *structures*, to take but two concepts. We do find some of the *structural* concepts in the initial section of the book, but these appear merely as lip-service intended to demonstrate the alleged subordination of algebraic concepts to the more general ones introduced within the framework of *structures*. For example, readers are told that the definition of an "isomorphism of magmas" (Section 1.1), namely a 'composition-preserving' bijection between two sets endowed with an internal law of composition, conforms to the 'general definitions' (i.e., those of *Theory of Sets*, Chapter IV). Yet the formal verification of this trivial fact is much more tedious than it may appear at first sight, since, according to the definitions of that section, one should first specify the echelon construction scheme of a 'magma' (this is done as an example in Section 1.4), then one should show that the defining axiom (namely, the relation "$F \in \mathbf{P}((AX A)XA)$ is a functional graph whose domain is AXA") is a 'transportable relation' for the given scheme, and, finally, that

$$\langle f \rangle^{S}(U) = U'$$

where $\langle f \rangle^{S}$ is the canonical extension with scheme S and the function *f*, and U is the *structure* in question. All this exacting verification is neither done nor suggested in the book, nor is any similar assertion thoroughly verified in what follows. For example, the reader is reminded that the main theorem for a monoid of fractions of a commutative monoid can be expressed in the terminology introduced in *Theory* *of Sets* by saying that the problem in question "is the solution of the universal mapping problem for E, relative to monoids, monoid homomorphisms and homomorphisms of E into monoids which map the elements of S to invertible elements" (Bourbaki 1973, p. 20). It follows, from a theorem proven in *Theory of Sets* for universal mappings, that the solution given here is essentially unique. This is one of the very few results of *Algebra* which can be pointed out as being obtained as a consequence of the general results obtained in *Theory of Sets*. However – I hasten to add – due to the unwieldiness of the concepts, the formal verification of the conditions under which the particular case in question can be treated by using the general one is itself an exacting process that is not carried out in the book, thus rendering doubtful, once more, the advantages of having invested so much effort in the general concepts.

The only theorems proven in terms of *structures* are the most immediate ones, such as the first and second theorems of isomorphism (Section 1.6, Propositions 6 and 7), but even they receive a special reformulation for groups (Section 4.5, Theorem 3). No new theorem is obtained through the *structural* approach and standard theorems are treated in the standard way. The Jordan–Hölder chain theorem (Section 4.7) aptly illustrates this situation, especially since elsewhere it had been proven within a wider conceptual framework of which group theory is a particular case,[24] while Bourbaki's proof is rather more restricted. It is not my intention here to decide which is the best way to prove this, or any other, theorem, but to dispute the generality allegedly attained in the books of Bourbaki and to insist that *structural* concepts do not actually stand behind any generalization that is operatively important.

Needless to say, as the book progresses into the subsequent theories in the hierarchy, the connection to *structures* is not even mentioned. Moreover, the need for a stronger unification framework is pronounced in various sections of the book, particulary in Chapter III on algebras where similar properties of tensor-, symmetric- and exterior-algebras are mentioned as three *different* instances.

4.2. *General Topology* (1966)

Bourbaki's book on *General Topology* is often mentioned as his most influential and innovative. It is also the book containing the most outstanding example of a theory presented through Bourbaki's model of the hierarchy of *structures*, starting from one of the '*mother-structures*' and descending to a particular *structure*, namely, that of the real numbers. According to the plan in the introduction of the book, the theory of topological spaces is presented in the opposite way to that in which it historically originated. The approach is characterized by the introduction of topological structure independent of any notion of real numbers or of any kind of metrics.

However, as with *Algebra*, the hierarchy itself is not in any sense introduced in terms of the *structural* concepts. Thus, for instance, topological groups are not characterized as a *structure* from which the *structure* of groups can be 'deduced'. *Structural* concepts appear in this book more than in any other place in the treatise but, instead of reinforcing Bourbaki's claims of the generality of such concepts, a close inspection of their use immediately reveals their ad-hoc character.

As a first example, I take again the concept of homeomorphism, which is defined (as 'isomorphism' was in *Algebra*) as a bijection preserving the *structure* of the topology. This definition is claimed to be "in accordance with the general definition" (Bourbaki 1968, p. 18). Again, the verification of this simple fact (which is neither done nor suggested in the book) is a long and tedious (though certainly straightforward) formal excerise.

That *structures* fail to play a significant role as a generalizing concept is illustrated in *General Topology* not only because it is so scarcely applied therein, but *precisely* by the uses to which the concept is actually put. Far from being general concepts used in apparently *different* situations (as claimed by Bourbaki), many *structural* concepts appear *only* in a few instances of the *Topology*.[25] Such concepts seem, therefore, to have been defined in *Theory of Sets* just to be handy for *General Topology*, but no other use was found for them in the whole treatise. Naturally, this is perfectly legitimate from the formal point of view, but it is much to the detriment of Bourbaki's claims concerning the unifying value of *structures*. Moreover, it certainly contradicts a leading principle of Bourbaki concerning the axiomatic treatment of concepts, namely, that "a general formulation can justify its existence only when it can be applied to several special problems and when it can really be an aid in saving time and thought".[26] Bourbaki has not hesitated to qualify theories that do not abide by that principle as "insignificant and uninteresting". By now, I think, it is clear that *structures* do not themselves satisfy that principle.

4.3. *Commutative Algebra* (1972)

Other books in Bourbaki's treatise rely mainly on concepts taken from *Algebra* and from *General Topology*, and the concept of *structure* is totally absent from them.[27] In Bourbaki's *Commutative Algebra*, on the other hand, we find a remarkable departure from his self-imposed rules, in which the limitations of *structures* as a generalizing framework are quite obvious and, in fact, explicitly acknowledged.

In Section 1.2, flat modules are defined as part of the theories under inspection. As it happens, this is a concept which is better understood in terms of concepts taken from homological algebra, a mathematical discipline which was not dealt with in the treatise until 1980. Now, it is often the case that, while formally introducing concepts in a book of the treatise, Bourbaki feels the need to illustrate those concepts by referring to an example which had not yet been introduced in that specific book. If the example is not a logical requisite for a full understanding of the concept itself and it appears in another place in the treatise, Bourbaki presents the example written between asterisks and gives the corresponding cross-reference. This policy is explained in the "Mode d'emploi" that serves as a preface to each of the books.

In the case of flat modules, we find a whole section (Section 4) in which "for the benefit of the readers conversant with Homological Algebra", Bourbaki shows "how the theory of Flat Modules is related to that of the Tor functors" (Bourbaki 1968, p. 37). The concept of Tor functor is not developed in the treatise, but Bourbaki thinks it is important to present the parallels between the two approaches: the Bourbaki approach and the functorial approach to homological algebra. In order to do that, Bourbaki freely uses concepts and notations foreign to the treatise. Instead of sticking to the usual prescription of writing it between asterisks, we find on the same page one of the very few examples in the treatise where a reference is given to a book or article outside it. Thus, the reader is referred to a forthcoming volume of the treatise where categories were eventually to be developed. Until then, however, one could also consult Cartan and Eilenberg (1956) or Godement (1958).

A cursory examination of issues of *La Tribu* during the 1950s uncovers recurring attempts to write chapters on homological algebra and categories for the *Éléments*. Eilenberg himself was commissioned several times with the preparation of drafts on homologies and on categories, while a fascicule de résultats on categories and functors was assigned successively to Grothendieck and Cartier.[28]

However, the promised chapter on categories never appeared as part of the treatise. As we shall see in greater detail in the next section, the publication of such a chapter could have proved somewhat problematic when coupled with Bourbaki's insistence on the centrality of *structures*. The task of merging both concepts, i.e., categories and structures, in a sensible way, would have been arduous and unilluminating, and the adoption of categorical ideas would have probably necessitated the rewriting of several chapters of the treatise. This claim is further corroborated by the interesting fact that when the chapter on homological algebra was finally issued (1980), the categorical approach was not adopted therein. Although since the publication of the above mentioned textbook of Cartan and Eilenberg the broad framework provided by categories became the standard framework for treating homological concepts, in Bourbaki's own presentation these concepts are defined within the narrower framework of modules. And, naturally, the concept of *structure* was not even mentioned there. I would like, then, to now more closely consider the connection between categories and *structures*.

5. STRUCTURES AND CATEGORIES

The central ideas of category theory were first outlined by Samuel Eilenberg and Saunders Mac Lane in 1942. Later, in 1945, the two mathematicians published the first systematic and comprehensive exposition of the theory. They defined the concepts of category theory to provide a framework within which to study certain recurring mathematical properties generally denoted as 'natural'. For technical reasons, the attempted conceptual framework was to enable 'all' individuals within a certain mathematical branch to be considered simultaneously. In this sense, the aims of category theory partially overlapped those of Bourbaki's *structure* theory.

Structures were first mentioned in the "Fascicule" of *Theory of Sets*, which was published in 1939. The first full version of Chapter IV, however, appeared only in 1957. By that time category theory had developed considerably and had reached the status of an independent discipline that enabled generalized formulations of several recurring mathematical situations. Mac Lane had further developed some central ideas in his article on 'duality' (1950); Eilenberg (who was himself then a member of Bourbaki) had co-authored two important books – Eilenberg and Steenrod (1952) and Cartan and Eilenberg (1956) – that exhibited the actual usefulness of categories and functors in exposing fully elaborated theories; and Buchsbaum (1955) and Grothendieck (1957) (the latter who was himself a member of the younger generation of Bourbaki) published independently two important articles which brought the significance of Abelian categories to the attention of the mathematical community.

Obviously, Bourbaki must have addressed the question of whether or not to adopt some or all of the language of category theory for use for its own purposes. As noted above, Bourbaki promised as early as 1961, in the first edition of *Commutative Algebra*, to publish a volume about Abelian categories. That volume never appeared, however. The promise and the failure to fulfill it together suggest Bourbaki's ambivalence about the value of the language of category theory to their project. In fact, Bourbaki's internal debates about the worth of adopting the categorical point of view are explicitly documented in several issues of *La Tribu*; in particular, one finds them encapsulated in a brief commentary to the effect that in one of the meetings of Bourbaki:

> L'on a remarqué: une violente offensive de virue [sic] functoriel. (*La Tribu*, 7–14 October 1956)

But before we proceed to quote and comment further on remarks appearing in *La Tribu*, let us observe how the existence of diverging points of view comes to the fore in certain articles published in standard mathematical journals.

As I said above, the final section of *Theory of Sets* addresses the problems of universal mappings and the central theorem stating the uniqueness of the solution to this problem. This whole section was apparently added to the book without any evident and natural connection to the previous sections. It includes a list of examples of the universal mapping problem, which individually appear scattered throughout other books of the treatise. However, neither that section nor the corresponding sections of the books in which the particular theories are developed include even a cursory verification that the examples satisfy the conditions of the theorem formulated in terms of *structures*.

The ideas contained in that section were first formulated by Samuel (in his 1948 article). Particular cases of universal constructions were known well before that, but Samuel was the first to seek a single generalized formulation of a problem that arose in diverse branches of mathematics. Consequently, this was a real test for the usefulness of *structures* and the concepts related to them.

Although Samuel does not mention this in his article, many of his formulations seem to be an attempt to exploit within the Bourbakian conceptual scheme the emphasis of category theory on morphisms. This attempt, however, highlights the limitations of *structures* rather than any advantages provided by them.

Samuel begins his article by mentioning the classical examples of universal constructions: fraction fields of a domain of integrity, free products, completion of uniform spaces, Čech compactification, etc. All of these examples appear again in the corresponding section of *Theory of Sets*. Samuel claimed that it is possible to provide an exact axiomatic formulation of the problem in general terms, and that such a formulation would appear in forthcoming books of Bourbaki's treatise.

Predictably, Bourbaki's concepts did not play a significant role in Samuel's article, despite Samuel's constant reference to various books of the *Éléments*, in which precise definitions for many of the concepts mentioned within the article could presumably be found. These references included chapters of the treatise which were yet to be published. In hindsight, it is clear that the concepts which he referred to did not play a significant role in the chapters of the *Éléments* that Samuel cites. Interestingly, the section of *Theory of Sets* concerning universal constructions is one of the few places in which Bourbaki refers to a work outside the treatise; in that section the reader is referred to Samuel's article to verify that free topological groups constitute an instance of a universal construction.

Free topological groups – the central focus of Samuel's attention in this article – had been considered previously by several mathematicians. Samuel believed that previous treatments had been cumbersome, and that he could simplify the study of free topological groups by considering their "universal" properties.[29] Since he wanted to show the applicability of the concept of universal construction to similar situations in other fields of mathematics, Samuel tried to connect these ideas to Bourbaki's *structures*. Notably, even though Bourbaki's concepts are formally applicable to the problems that Samuel considers, they do not positively contribute to their generalization and solution.

In fact, many categorical ideas pervade the article, as properties of morphisms are emphasized throughout. Samuel defines a T-mapping as a mapping between two *structures* of the same type T which must also satisfy a number of conditions. These conditions are the conditions imposed on morphisms in category theory. An ST-mapping is a mapping sending an S-structure into a T-structure and which satisfies some further conditions, which, incidentally, resemble the defining axioms for functors in category theory. As we have seen, this emphasis on morphisms was in no way a feature of the Bourbakian approach. Rather, it seems to have been taken from the first articles on category theory.

Samuel's article was reviewed by Mac Lane (1948b), who was at that time working on closely related ideas which were to appear in Mac Lane (1950) and which were outlined in an early announcement in Mac Lane (1948a). In his review Mac Lane mentioned some additional conditions which must be imposed upon the ST-mappings in order for Samuel's proof of the universal mapping theorem to be correct. Mac Lane showed that the main concern of his own article, duality phenomena, was closely connected to the problem of universal mappings, and that categorical concepts advanced the understanding of both problems. *Structures*, on the other hand, made no clear contribution to the elucidation of the problem, even though it was the only problem that was treated in a more or less generalized formulation within the *Éléments*.

It is worth pointing out that Mac Lane himself, while analyzing the development of categorical ideas (though being unaware of the problematic relationship between Bourbakian and categorical ideas as analyzed in this section) remarked that the huge influence of Bourbaki had impeded the recognition of certain central categorical concepts. Indeed, Mac Lane often mentioned the fact that, while many individual examples of adjoint functors were well-known in the 1930s, it was only in the 1950s that the concept was actually formulated, in the work of Daniel Kan (1958). The reason for this delay is worthy of independent analysis, but in the present context it is pertinent to see what Mac Lane believes to be the explanation:

Ideas about Hilbert spaces or universal constructions in general topology might have suggested adjoints, but they did not; perhaps the 1939–1945 war interrupted this development. During the next decade 1945–55 there were very few studies of categories, category theory was just a language, and possible workers may have been discouraged by the widespread pragmatic distrust of "general abstract nonsense" (category theory). Bourbaki just missed. (Mac Lane 1971, p. 103)

In this remark, Mac Lane implies that Bourbaki's generalizing concepts, especially those concerning universal constructions, were too cumbersome to allow the identification of the central mathematical ideas at the core of the problems they considered. In Mac Lane's opinion, it was precisely Bourbaki's special language that impeded the discovery of what was already there. In his words:

[Bourbaki's] definition of universal construction was clumsy, because it avoided categorical language .... Bourbaki's idea of universal construction was devised to be so general as to include more – and in particular to include the ideas of multilinear algebra which were important to French Mathematical traditions. In retrospect, this added generality seems mistaken; Bourbaki's construction problem ... missed a basic discovery; this discovery was left to a younger man, perhaps one less beholden to tradition or to fashion. Put differently, good general theory does not search for the maximum generality, but for the right generality. (Ibid.)

The example of universal constructions clearly shows, then, that when Bourbaki published Chapter IV, including results and ideas announced back in 1939, categorical ideas appeared repeatedly throughout Bourbaki's own work, despite the group's refusal to explicity recognize this fact. In the end, as we know, *structures* appeared within the *Éléments* while categories did not. It is clear that the early developments of the categorical formulation, more flexible and effective than the one provided by *structures* rendered questionable Bourbaki's initial hopes of finding *the* single best formulation for each mathematical idea and cast doubt on the initially intended universality of Bourbaki's enterprise. Let us now see how these questions and doubts manifested themselves in Bourbaki's meetings.

As was mentioned earlier, *La Tribu* indicates that members were assigned to prepare drafts about categories and functors on several different occasions, although no chapter dealing with these concepts was ever published. We can also find in *La Tribu* indications of diverse technical problems arising in connection with the detailed treatment of many *structural* concepts. One learns that the definite publication of the results announced in 1939 concerning *structures* was delayed because of questions suggested by category theory. This is indicated by the following report on the tentative contents of *Theory of Sets*:

Chap. IV (Structures) – Un papier de Cartier montre que les résultats de Samuel sur les limites inductives sont des cas particuliers de fourbis ultra-généraux sur la commutation des problèmes universels. Ces fourbis ne s'énoncent bien que dans le cadre des catégories et foncteurs. Cartier propose une méthode métamathématique d'introduire ces dernières sans modifier notre système logique. Mais ce système est vomi car il tourne résolument le dos au point de vue de l'extension . . . . On décide donc qu'il vaut mieux élargir le système pour y faire rentrer les catégories; à première vue le système Gödel semble convenir. Afin de n'être pas le cul entre deux chaises, et aussi afin de ne pas retarder la publication d'un chapitre sur lequel on a beaucoup travaillé, on décide (malgré le véto de Dixmier, retiré in extremis) d'envoyer le chap. IV à l'impression sans modifier les limites inductives, et en ajoutant les petites modifications relatives aux solutions strictes des problèmes universeles. Quant aux catégories et foncteurs, on est finalement convaincus que c'est très important. D'où:

Chap. V (Catégories et foncteurs) – Pour commencer Grothendieck rédigera une espèce de Fascicule de Résultats en style naïf, afin que Bourbaki se rende compte de ce qu'il est utile de pouvoir faire. On formalisera ensuite. (*La Tribu*, 39, 4 June–7 July 1956)

Further evidence of the interplay between *structural* and categorical concepts is provided by Weil's letter to Chevalley, dated 15 October 1950, which was distributed to the members of Bourbaki as an appendix to one of the issues of *La Tribu*. Thus Weil wrote:

Je viens de recevoir les chap. II–III des Ensembles . . . . faut-il reserver le mot "fonction" à une application d'un *ensemble* dans l'univers, comme tu as fait (auquel cas, avec tes axiomes, les valeurs de la fonction forment aussi un ensemble, bien entendu)? ou bien convient-il de nommer "fonction" tout ce à quoi on attache un symbole fonctionnel, e.g. $P(E)$ $A \times B$, $A \cdot B$ (prod. tens.) etc.? Evidemment "fonction" dans le second sens ne serait pas un objet mathématique, mais un vocable métamathématique; c'est sans doute pourquoi il existe (je ne veux nommer personne . . .) des gens qui disent "foncteur" devons-nous accepter ce terme? . . . Quant à [la theorie des structures], ton chapitre débrouille bien la question; mais nous ne pouvons guère ne pas aller plus loin que tu n'as fait, et chercher s'il est possible de donner quelques généralité aux notions de structure induite, structure produit, homomorphisme. Comme tu sais, mon honorable collègue MacLane soutient que toute notion de structure comporte necessairement une notion d'homomorphisme, consistant à indiquer, pour chaqune des données constituant la structure, celles qui se comportent de manière covariante et celles qui se comportent de manière contravariante . . . . Que penses-tu qu'il y ait à tirer de ce genre de considerations?

The above quotations, and many other statements scattered throughout several issues of *La Tribu*, confirm the impression created by direct reading of published material, namely that Chapter IV of *Theory of Sets*, in which Bourbaki's theory of *structures* is developed, was published at a stage when it was already clear that the concept of *structure* could not fulfill the expectations initially attached to it, and that there existed an alternative generalizing mathematical concept at least as comprehensive as that of *structure* and probably more satisfactory.[30]

It is therefore somewhat surprising to read what Dieudonné had to say in 1982 about Bourbaki and categories:

One often hears people wondering why Bourbaki has not undertaken to publish a chapter on categories and functors. I think one of the reasons is the following: the parts of mathematics where those concepts are extremely useful, such as algebraic geometry and algebraic and differential topology, are among those which Bourbaki cannot contemplate including in the treatise . . . . For many other parts of mathematics, it is certainly possible to use the language of categories and functors, but they do not bring any simplification to the proofs, and even in homological algebra, one can entirely do without their use, which would amount to introducing extra terminology. (Dieudonné 1982, p. 622)[31]

This comment seems strange for several reasons. First, as we have seen and contrary to Dieudonné's claim above, in his book *Commutative Algebra* Bourbaki explicitly acknowledged the gains that may be expected from the use of categorical ideas in homological algebra. Bourbaki made this point even at the cost of deviating from a restriction jealously preserved throughout most of the treatise. Second, since 1956, when Cartan and Eilenberg published their classic *Homological Algebra*, the functorial approach was widely adopted in the field, mainly because of its simplicity. Third, like the above analysis of the various books of Bourbaki's treatise shows, it is the *structural* approach that fails to simplify proofs and that ultimately does little more than introduce extraneous terminology.

Having examined the historical and conceptual relationship between categories and *structures*, we can proceed to the final section of this article, in which Bourbaki's influence on the historical account of the rise of the structural approach is briefly examined.

6. BOURBAKI AND THE HISTORY OF STRUCTURES

The evidence presented above suggests that *Theory of Sets* and, particularly, the concept of *structure* are not essential to the *Éléments*. Didactically *and* mathematically, *Theory of Sets* can be totally skipped over, for it has neither heuristic value nor logical import for Bourbaki's treatise. Moreover, while some books within the *Éléments* turned into widely quoted and even classic references for the topics covered therein – and many of the concepts, notation and nomenclature introduced by Bourbaki were readily adopted – that was not the case for *Theory of Sets* and the *structural* concepts.

This conclusion can be confirmed by examining any review journal.

Consider, for instance, the *Index of Scientific Citations*, between 1962 and 1966. The index includes over four hundred and thirty-five quotations of the *Éléments*. Only three of them refer to the chapter on *structures*. One of these three quotations appears not within a mathematical research paper, but in a theoretical biology article (Gillois 1965). In general, the ideas of *Theory of Sets* inspired organizational schemes for nonmathematical disciplines more than they were used in mathematical research.

If this is indeed the case – it might be asked – why have "mathematical structures" come to be generally identified with Bourbaki? The main reason for this is that the above-mentioned distinction between the nonformal concept of mathematical structure and the formal one of *structure* has been often left vague in historical accounts and, in particular, in the writings of Bourbaki and of some Bourbaki members. Consider, for example, the following historical account of Dieudonné:

> Today when we look at the evolution of mathematics for the last two centuries, we cannot help seeing that since about 1840 the study of specific mathematical objects has been replaced more and more by the study of mathematical structures. (Dieudonné 1979, p. 9)

Taken as a very general statement, this is a seemingly inoffensive historical truism. However, it *cannot* be taken as a very general statement, for it is followed by a footnote specifying that *the term 'mathematical structure' is to be taken in the specific technical sense defined by Bourbaki in the fourth chapter of the first book of the Éléments*. The quotation, then, translates to the claim that since 1840 mathematics has increasingly become the study of *structures*. This later claim, even if followed by the qualification that "this evolution was not noticed at all by contemporary mathematicians until 1900" cannot by any means be accepted as historically true.

Not all articles by or about Bourbaki explicitly assume the identity of the nonformal and formal senses of the term 'structure', as Dieudonné did in the above quotation. Most present the relationship between the nonformal and the formal in more ambiguous terms.[32] This ambiguity has produced a highly misleading historical account of the structuralist approach to mathematics and, in particular, of Bourbaki's influence on its consolidation and expansion. A case in point is the centrality of the so-called '*mother-structures*'.

In the above discussion on the contents of *Theory of Sets*, I mentioned that the *mother-structures* (algebraic-, topological-, and order-structures) constitute a central issue in Bourbaki's images of mathematics. In one of Bourbaki's most popular nontechnical articles explicating his conception of mathematics, this point is explained as follows:

> At the center of our universe are found the great types of structures . . . they might be called the mother structures . . . . Beyond this first nucleus, appear the structures which might be called multiple structures. They involve two or more of the great mother-structures not in simple juxtaposition (which would not produce anything new) but combined organically by one or more axioms which set up a connection between them . . . . Farther along we come finally to the theories properly called particular. In these elements of the sets under consideration, which in the general structures have remained entirely indeterminate, obtain a more definitely characterized individuality. (Bourbaki 1950, pp. 228–29)

This description of the *mother-structures*, I hasten to repeat, is not an integral part of the axiomatic theory of *structures* developed by Bourbaki; it appears several times in *Theory of Sets*, but only in scattered examples.[33] That all of mathematical research is a research of *structures*, that there are *mother-structures* bearing a special significance for mathematics, that they are exactly three, and that the three *mother-structures* are precisely algebraic-, order- and topological-structures (or *structures*), all this is by no means a logical consequence of the axioms defining a *structure*. The concept of *mother-structures* and the picture of mathematics as a hierarchy of *structures* are not results obtained within a mathematical theory of any kind. Rather, they belong strictly to Bourbaki's images of mathematics; they appear only in nontechnical, popular articles such as the one quoted above, or in the myths that arose around Bourbaki.

Bourbaki himself had an equivocal attitude about the validity of its own notion of the hierarchy of *structures*. On the one hand, he warns us that the picture of mathematics as a hierarchy of *structures* is nothing but a convenient schematic sketch since "it is far from true that in all fields of mathematics, the role of each of the structures is clearly recognized and marked off". Furthermore, "the structures are not immutable, neither in number nor in their essential contents".[34] But, on the other hand, the inclusion of those examples in *Theory of Sets* amidst Bourbaki's formal treatment of a theory of *structures* is bound to confer them, metonimically as it were, that special kind of truth-status usually accorded to deductively obtained propositions. Thus, Bourbaki's images of mathematics aspire to remain beyond all debate.

Such debate, however, is an integral part of the development of mathematics and it cannot be actually avoided.

The circumstantial link between the idea of *mother-structures* and a formal mathematical theory has been the source of serious misunderstanding, especially outside mathematics. Probably the best-known misunderstanding is reflected in Piaget's manifest enthusiasm for Bourbaki's work. Hans Freudenthal has commented on that misguided view as follows:

> The most spectacular example of organizing mathematics is, of course, Bourbaki. How convincing this organization of mathematics is! So convincing that Piaget could rediscover Bourbaki's system in developmental psychology . . . . Piaget is not a mathematician, so he could not know how unreliable mathematical system builders are. (Freudenthal 1973, p. 46)

But not only Piaget fails to realize how unreliable mathematical system builders are; some mathematicians seem equally ignorant of this fact.[35] Bourbaki himself, especially in his first years, hardly considered himself an 'unreliable system builder' and his formulations to be provisional. Doubts about the certainty of Bourbaki's program arose only later on, but the image of mathematics as revolving around the concept of *structure* remained long after that. This change in attitude is shrouded in the ambiguity of claims such as "the connecting link [between the diverse theories within the treatise] was provided by the notion of *structure*".[36] If 'structure' is taken to mean *structure*, then Dieudonné's claim reflects Bourbaki's initial faith, but, alas, it is completely mistaken. If, on the contrary, we give 'structure' its nonformal sense, then the claim is true, but it says something different, and indeed significantly less than it was meant to say.

Does all this mean that Bourbaki had no significant influence on contemporary methematics? Hardly. What I have shown here is only that Bourbaki's *structures* could have had no influence at all. But I have also shown why, if we want to describe Bourbaki's influence properly, we must then concentrate on what I have called the "images of mathematics". This in no way belittles the extent or importance of Bourbaki's influence. On the contrary, the images of mathematics play a decisive role in shaping the path of development of this discipline.[37] We must only keep in mind that images of knowledge do not behave like deductive systems: unlike formal mathematical theories, images of mathematical knowledge are subjected to continuous debate and change. Bourbaki certainly influenced the images of mathematics through several important aspects afforded by his work, such as clarity of presentation, and economy of means and unity of language. But that influence was enhanced by other factors that are often overlooked: sheer authority, reputation, and even some degree of misconception and arbitrariness. It is in terms of the combination of all these factors that the important role of Bourbaki's *Éléments* in shaping the course of research in many central branches of mathematics during several decades of the present century must be explained. Likewise, it should be clear now that the rise of the structural approach to mathematics should not be conceived in terms of this or that formal concept of structure. Rather, in order to account for this development, the evolution of the nonformal aspects of the structural image of mathematics must be described and explained.

NOTES

* Several persons have read and criticized earlier drafts of this paper. I thank them all for forcing me to formulate my ideas more simply and convincingly. Special thanks are due to Professors Pierre Cartier (Bures-sur-Yvette), Giorgio Israel (Rome), and Andrée C. Ehresmann (Amiens) for stimulating and illuminating conversations.

[1] See, e.g., Gandillac, Goldmann and Piaget 1965, p. 143; and Lane 1970, p. 20.

[2] See, e.g., Bell 1945, Chap. X; Dieudonné 1979, p. 9; and Wussing 1984, p. 15.

[3] See, e.g., Maddy 1980; Resnik 1981, 1982; and Shapiro 1983. See also the survey appearing in the introduction to Aspray and Kitcher (1988, esp. p. 4).

[4] Piaget 1968, p. 21. See also Gauthier 1969, 1976.

[5] In what follows, Bourbaki's formal concept is referred to by *structure* (underlined), while the term without further specifications is used for the nonformal concept.

[6] "Images of mathematics" denote any system of beliefs *about* the body of mathematical knowledge. This includes conceptions about the aims, scope, correct methodology, rigor, history and philosophy of mathematics, etc. (see Corry 1990).

[7] For instance, Hasse 1930; Macaulay 1933; Mac Lane 1939, 1963; and Ore 1931. In Corry (1991) the changes in the images of algebra (and in particular the rise of the structural image of algebra) are analyzed, as they manifest themselves in the textbooks of the late nineteenth and early twentieth century.

[8] A detailed discussion of this reflexive capacity of mathematics and its philosophical relevance appears in Corry (1990).

[9] See Ore 1935, 1936. In a forthcoming article I intend to give an account of this short-lived and now forgotten program, which has some interesting historical implications.

[10] There are some minor disagreements in the various accounts about who precisely were the first members of the group. Cf., e.g., the interview with Chevalley in Guedj (1985, p. 8); he does not mention Coulomb or Ehresmann among the founders.

[11] Some years ago, the "Association des Collaborateurs de Nicolas Bourbaki" was established at the École Normale Supérieure, Paris. An archive containing relevant documents,

probably including many copies of *La Tribu*, was created, but, unfortunately, it has yet to be opened to the public. In the present article I quote some issues of *La Tribu* belonging to personal collections. Professor Andrée C. Ehresmann kindly allowed me to read and quote from documents belonging to her late husband, Professor Charles Ehresmann. This includes volumes of *La Tribu*, from 1948 to 1952. Other quotations here are taken from personal collections of Chevalley and Mandelbrojt, as they appear in an appendix to Friedman (1975), which can be consulted at the library of the Centre Alexandre Koyré in Paris.

[12] I will adhere to the convention of referring to Bourbaki in the third person singular.

[13] See Section I of the bibliography for an extensive listing of secondary literature on Bourbaki. Among the many articles dealing with the myth of Bourbaki we may mention Dieudonné (1970, 1982), Fang (1970), Guedj (1985), Halmos (1957), Israel (1977), Queneau (1962), and Toth (1980).

[14] See the inspired description of Hewitt (1956, p. 507): "Confronted with the task of appraising a book by Nicolas Bourbaki, this reviewer feels as if he were required to climb the Nordwand of the Eiger . . . . Nevertheless, even a quite ordinary one-headed mortal may have notions of his own, and candor requires that they be set forth".

[15] See, e.g., Samuel (1972, p. 1): "As Thucydides said about his 'History of the Peloponnesian War', this is . . . a treasure valuable for all times". See also Artin (1953, p. 474): "Our time is witnessing the creation of a monumental work". Other favorable reviews of Bourbaki's work appear in Eilenberg (1942), Gauthier (1972), Kaplansky (1953), Kelley (1956), and Rosenberg (1960). Examples of praise of Bourbaki's work couched in nontechnical language can be found in the cited articles of Fang (1970), Queneau (1962), and Toth (1980).

[16] Cf. Bagemihl (1958); Gandy (1959, p. 73: "It is possible, then, that this book may itself soon have only historical interest"); Halmos (1953, p. 254: "I am inclined to doubt whether their point of view [on integration] will have a lasting influence"); Hewitt (1956); Jónsson (1959, p. 629: "Due to the extreme generality, the definitions are cumbersome, and all the results derived are of a very trivial nature"); Michael (1963); and Mostowski (1967). It is noteworthy that almost all reviewers of Bourbaki, favorable and critical alike, characterize the choice of exercises as excellent; this choice is usually attributed to Jean Dieudonné.

[17] The first six books of the treatise are supposed to be more or less self-contained. The final four presuppose knowledge of the first six volumes and, for that reason, Bourbaki gave them no numbers. *Differential and Analytic Manifolds* consists only of a fascicule de résultats.

[18] There have been several printings (with some minor changes) and translations into other languages. All quotations of *Theory of Sets*, below, are taken from the English translation of Bourbaki (1968).

[19] Cf., e.g., Theorem C36, in Bourbaki 1968, p. 42.

[20] Dieudonné 1982, p. 620.

[21] Paul Halmos wrote several interesting reviews of Bourbaki's books. See his critical reviews of Chapters I and II in Halmos (1955, 1956).

[22] See Mac Lane 1971, Chap. III.

[23] It is important to remark, however, that the kind of summary appearing in *Topological Vectorial Spaces* was not universally praised. One reviewer, Hewitt (1956, p. 508), said that, "[t]he 'Fascicule de Résultats' is of doubtful value. It would seem difficult to appreciate or use this brief summary without first having studied the main text; and when this has been done, the summary is not needed".

[24] See, for example, Ore 1937; or George 1939. See Birkhoff (1948, p. 88) for a survey of different proofs of this theorem.
[25] Such as in Section 4.2, where a partial ordering of topologies is defined. The topologies are ordered from *coarser* to *finer*.
[26] Cartan, p. 15, quoted in Fang 1970, p. 54.
[27] "Structure" is used once, but with a completely different meaning. See Book 10, *Differential and Analytic Manifolds* in Bourbaki 1968, Section 6.2.1, p. 62.
[28]. Cf., for example, *La Tribu* 28 (25 June–8 July 1952); 38 (11–17 March 1956); 39 (4 June–7 July 1956); and 40 (7–14 October 1956).
[29] Samuel notes the connection between his article and ideas appearing in Kakutani (1944), Markoff (1942), and Nakayama (1943).
[30] Mention should be made of the work of Charles Ehresmann, one of the founding members of Bourbaki, who combined *structural* and categorical concepts in a fascinating and idiosyncratic way. Most of his works in this field appear in Volume II, part 1, of Ehresmann (1981). See especially his 1957 article. Elements of his theory also appear in his 1965 book. An analysis of Ehresmann's work is far beyond the scope of the present article.
[31] See also Samuel's review of Bourbaki's *Algebra* (1972, p. 1):

> [O]ne may guess that Bourbaki is not planning, for the time being, to write about categories or functors: one gets the feeling that, for him, functoriality is more a way of thinking than a way of writing.

[32] This is also the case in some more general historical accounts. When speaking of the development of the structural trend in mathematics, Bourbaki's hierarchy is often taken for granted, and algebraic structures are often mentioned only in conjunction with topological- and order-structures. Cf., for example, Novy 1973, p. 223; Purkert 1971, p. 23; and Wussing 1984, p. 256.
[33] See, for example, in *Theory of Sets*, pp. 266, 272, 276, 277, 279.
[34] Bourbaki 1950, p. 229.
[35] A similar assessment of Bourbaki's contributions appears in Gauthier (1972, p. 624).
[36] Dieudonné 1982, p. 619; italics in the original.
[37] For a detailed argument on this point, see Corry (1990).

REFERENCES

I. Primary Works and Book Reviews:

Artin, E.: 1953, Review of Bourbaki (1939–), Book II, Chaps. 1–7, (1942–52), *Bull. AMS* **59**, 474–79.

Bagemihl, R.: 1958, Review of Bourbaki (1939–), Book I, chap. 3, *Bull. AMS* **64**, 390.

Bourbaki, N.: (1939–), *Éléments de Mathématique*, 10 vols., Hermann, Paris.

Bourbaki, N.: 1949, 'The Foundations of Mathematics', *J. of Symbolic Logic* **14**, 1–8.

Bourbaki, N.: 1950, 'The Architecture of Mathematics', *Amer. Math. Monthly* **57**, 221–32.

Bourbaki, N.: 1960, *Éléments d'Histoire des Mathématiques*, Hermann, Paris.

Bourbaki, N.: 1966, *General Topology*, 2 vols., Hermann, Paris.

Bourbaki, N.: 1968, *Theory of Sets*, Hermann, Paris.

Bourbaki, N.: 1972, *Commutative Algebra*, Hermann, Paris.

Bourbaki, N.: 1972a, 'Univers', in M. Artin, A. Grothendieck, and J. L. Verdier (eds.), *Théorie des Topos et Cohomologie Etale des Schémas*, (SGA 4), Springer Verlag, Heidelberg, pp. 23–72.

Bourbaki, N.: 1973, *Algebra*, Hermann, Paris.

Bourbaki, N.: 1980, *Homological Algebra*, Hermann, Paris.

Cartan, H.: 1959, 'Nicolas Bourbaki und die heutige Mathematik', *Arbeitsgemeinschaft für Forsch. des Landes Nord Westf.*, Heft 76, pp. 5–18.

Dieudonné, J.: 1970, 'The Work of Nicolas Bourbaki', *Amer. Math. Monthly* **77**, 134–45.

Dieudonné, J.: 1979, 'The Difficult Birth of Mathematical Structures, (1840–1940)', in U. Mathieu and P. Rossi (eds.), *Scientific Culture in Contemporary World*, Scientia, Milan, pp. 7–23.

Dieudonné, J.: 1982, 'The Work of Bourbaki in the Last Thirty Years', *Notices AMS* **29** 618–23.

Eilenberg, S.: 1942, Review of Bourbaki (1939–), Book I, Fascicule de Résultats, 1st ed. (1939), *MR* **3**, 55–56.

Eilenberg, S.: 1945, Review of Bourbaki (1939–), Book II, Chap. I, *MR* **6**, #113.

Fang, J.: 1970, *Bourbaki: Towards a Philosophy of Modern Mathematics – I*, Paideia Press, New York.

Friedman, J.: 1975, *L'Origine et le Développement de Bourbaki*, unpublished Ph.D. Dissertation, Centre Alexandre Koyré, École Practique des hautes Études en Sciences Sociales, Paris.

Frink, O.: 1950, Review of Bourbaki (1949), *MR* **11**, 73.

Gandy, R. O.: 1959, Review of Bourbaki (1949), *J. Symbolic Logic* **24**, 71–73.

Gauthier, Y.: 1969, 'La Notion Theoretique de Structure', *Dialectica* **23**, 217–27.

Gauthier, Y.: 1972, Review of Bourbaki (1939–), Book I, *Canadian Mathematical Bulletin* **15**, 623–26.

Gillman, L.: 1964, Review of Bourbaki (1939–), Book II, Fascicule de Résultats, 3rd ed. (1958), *MR* **27**, #4757.

Guedj, D.: 1985, 'Nicolas Bourbaki, Collective Mathematician, An Interview with Claude Chevalley', *Math. Intelligencer* **7**, 18–22 (originally appeared in *Dédales*, Nov. 1981).

Halmos, P.: 1953, Review of Bourbaki (1939–), Book VI, Chaps. 1–4 (1952), *Bull. AMS* **59**, 249–55.

Halmos, P.: 1955, Review of Bourbaki (1939–), Book I, Chaps. 1–2 (1954), *MR* **16**, #454.

Halmos, P.: 1956, Review of Bourbaki (1939–), Book I, Chaps. 3 (1956), *MR* **17**, #1062.

Halmos, P.: 1957, 'Nicolas Bourbaki', *Scientific American* **196**(5), 88–99.

Halmos, P.: 1970, Review of Fang (1970), *MR* **40**, #7066.

Hermann, R.: 1986, 'Mathematics and Bourbaki', *Math. Intelligencer* **8**, 32–33.

Hewitt, E.: 1956, Review of Bourbaki (1939–), Book IV (1953–55), *Bull. AMS* **62**, 507–08.

Hewitt, E.: 1966, Review of Bourbaki (1939–), Book VI, Chaps. 7–8, *MR* **31**, #3539.

Israel, G.: 1977, 'Un Aspetto Ideologico della Mathematica Contemporanea: Il "Bourbakismo"', in E. Donini et al. (eds.), *Mathematica e Fisica: Cultura e Ideologia*, De Donato Editore, Bari, pp. 35–70.

Jónsson, B.: 1959, Review of Bourbaki (1939–), Book I (1957), *MR*, **20**, #3804.

Kaplanski, I.: 1953, Review of Bourbaki (1939–), Book II, Chaps. 6–7, *MR* **14**, #237.
Kelley, J. L.: 1956, Review of Bourbaki (1939–), Book IV, *MR* **17**, #1109.
Kolchin, E. R.: 1951, Review of Bourbaki (1939–), Book II, Chaps. 4–5, *MR* **12**, #6.
Mac Lane, S.: 1948, Review of Bourbaki (1939–), Book II, Chap. 2, *MR* **9**, #406.
Mac Lane, S.: 1986, Review of Bourbaki (1960/1984 English ed.), *MR* **86h**, 01:005.
Michael, E.: 1963, Review of Bourbaki (1939–), Book III, Chaps. 1–2, *MR* **25**, #4480.
Mostowski, A.: 1967, Review of Bourbaki (1939–), Book I, Chap. 4, *MR* **34**, #2425.
Munroe, H. E.: 1958, Review of Bourbaki (1939–), Book V, Chap. 5 (1956), *Bull. AMS* **64**, 105–06.
Queneau, R.: 1962, 'Bourbaki et les Mathématiques de Demain', *Critique* **18**, 3–18.
Rosenberg, A.: 1960, Review of Bourbaki (1939–), Book II, Chap. 8, *Bull. AMS* **66**, 16–19.
Samuel, P.: 1948, 'On Universal Mappings and Free Topological Groups', *Bull. AMS* **54**, 591–98.
Samuel, P.: 1972, Review of Bourbaki (1939–), Book II, Chaps. 1–3 (1970), *MR* **43**, #2.
Stegmüller, W.: 1979, *The Structuralist View of Theories: A Possible Analogue to the Bourbaki Programme in Physical Sciences*, Springer Verlag, Berlin-New York.
Toth, I.: 1980, 'Nicolas Bourbaki. S.A.', Princeton (Preprint).

II. General Works:

Aspray, W. and P. Kitcher (eds.): 1988, *History and Philosophy of Modern Mathematics*, Minnesota Studies in the Philosophy of Science, Vol. XI, University of Minnesota Press, Minneapolis.
Bell, E. T.: 1945, *The Development of Mathematics*, 2nd ed., McGraw-Hill, New York.
Birkhoff, G. H.: 1948, *Lattice Theory*, 2nd ed., AMS Colloquium Publication, Providence.
Buchsbaum, D.: 1955, 'Exact Categories and Duality', *Trans. AMS* **80**, 1–34.
Cartan, H. and S. Eilenberg: 1956, *Homological Algebra*, Princeton University Press, Princeton.
Corry, L.: 1990, 'Linearity and Reflexivity in the Growth of Mathematical Knowledge', *Science in Context* **3**, 409–40.
Corry, L.: 1991, 'Libros de Texto e Imágenes del Algebra en el Siglo XIX', *Llull* **14**, 7–30.
Ehresmann, C.: 1957, 'Gattungen von lokalen Strukturen', *Jber. DMV* **60**, 49–77 (also in Ehresmann (1981), II-1, pp. 125–53).
Ehresmann, C.: 1965, *Categories et Structures*, Dunod, Paris.
Ehresmann, C.: 1981, *Ouevres Complètes et Comentées*, ed. by Andrée Charles Ehresmann, Amiens.
Eilenberg, S. and S. Mac Lane: 1942, 'Natural Isomorphisms in Group Theory', *Proc. Acad. Sci.* **28**, 537–43.
Eilenberg, S. and S. Mac Lane: 1945, 'General Theory of Natural Equivalences', *Trans. AMS* **28**, 231–94.
Eilenberg, S. and N. Steenrod: 1952, *Foundations of Algebraic Topology*, Princeton University Press, Princeton.
Freudenthal, H.: 1973, *Mathematics as an Educational Task*, D. Reidel, Dordrecht.
Gandillac, H., L. Goldmann and J. Piaget (eds.): 1965, *Entretiens sur les Notions de Genèse et de Structure*, Mouton & Co., Paris.

Gauthier, Y.: 1976, *Fondements des Mathématiques*, Les Presses de l'Université de Montréal, Montreal.
George, E.: 1939, 'Über den Satz von Jordan-Hölder', *J. reine und ang. Math.* **180**, 110–20.
Gillois, M.: 1965, 'Relation d'identité en Génétique', *Annales de l'Institut H. Poincaré* **2**, 1–94.
Godement, R.: 1958, *Théorie de Faisceaux*, Hermann, Paris.
Grothendieck, A.: 1957, 'Sur quelques Points d'Algebre Homologique', *Tôhoku Math. Journal* **9**, 119–221.
Hasse, H.: 1930, 'Die moderne algebraische Methode', *Jber. DMV* **39**, 22–34 (reprinted in: 1986, *Math. Intellingencer* **8**, 18–23, English trans. by Abe Shenitzer).
Kakutani, N.: 1944, 'Free Topological Groups and Infinite Direct Product of Topological Groups', *Proc. Imp. Acad. Tokyo* **20**, 595–98.
Kan, D.: 1958, 'Adjoint Functors', *Trans. AMS* **87**, 294–329.
Lane, M. (ed.): 1970, *Introduction to Structuralism*, Basic Books, New York.
Macaulay, F. S.: 1933, 'Modern Algebra and Polynomial Ideals', *Proc. Cambridge Phil. Soc.* **30**, 27–46.
Mac Lane, S.: 1939, 'Some Recent Advances in Algebra', *Amer. Math. Monthly* **46**, 3–19.
Mac Lane, S.: 1948a, 'Groups, Categories and Duality', *Proc. Nat. Acad. Sci.* **34**, 263–67.
Mac Lane, S.: 1948b, Review of Samuel (1948), *MR* **9**, 605.
Mac Lane, S.: 1950, 'Duality for Groups', *Bull. AMS* **56**, 485–516.
Mac Lane, S.: 1963, 'Some Additional Advances in Algebra', in A. A. Albert (ed.), *Studies in Modern Algebra*, Prentice-Hall, Englewood Cliffs, pp. 35–58.
Mac Lane, S.: 1971, *Categories for the Working Mathematician*, Springer Verlag, New York.
Maddy, P.: 1980, 'Perception and Mathematical Induction', *The Philosophical Review* **89**, 163–96.
Markoff, A.: 1942, 'On Free Topological Groups', *Bull. Acad. Sci. USSR* (Ser Math.) **9**, 3–64.
Nakayama, T.: 1943, 'Note on Free Topological Groups', *Proc. Imp. Acad. Tokio* **19**, 471–75.
Novỳ, L.: 1973, *Origins of Modern Algebra*, English trans. by J. Taver, Noordhof International Publications, Leyden.
Ore, O.: 1931, 'Some Recent Developments in Algebra', *Bull. AMS* **39**, 728–74.
Ore, O.: 1935, 'On the Foundations of Abstract Algebra, I', *Annals of Maths.* **36**, 406–37.
Ore, O.: 1936, 'On the Foundations of Abstract Algebra', II, *Annals of Maths.* **37**, 265–92.
Ore, O.: 1937, 'On the Theorem of Jordan–Hölder', *Trans. AMS* **41**, 247–69.
Piaget, J.: 1968, *Le Structuralisme*, Presses Universitaire de France, Paris.
Purkert, W.: 1971, 'Zur Genesis des abstrakten Körperbegriffs (1)', *Schriftenreihe für Geschichte der Naturwissenschaft, Technik, und Medizin* **8**, 23–37.
Resnik, M.: 1981, 'Mathematics as a Science of Patterns: Ontology and Reference', *Noûs* **15**, 529–50.
Resnik, M.: 1982, 'Mathematics as a Science of Patterns: Epistemology', *Noûs* **16**, 95–105.

Shapiro, S.: 1983, 'Mathematics and Reality', *Philosophy of Science* **50**, 523–48.
Waerden, B. L. van der: 1930, *Moderne Algebra*, 2 vols., Springer Verlag, Berlin (English trans. of 2nd ed. – Vol. I by Fred Blum (1949), Vol. II by T. J. Benac (1950) – Frederic Ungar Publishing, New York).
Wussing, H.: 1984, *The Genesis of the Abstract Group Concept. A Contribution to the History of the Origin of Abstract Group Theory*, MIT Press, Cambridge, MA (English trans. by Abe Shenitzer of: 1969, *Die Genesis des abstrakten Gruppenbegriffs*, Berlin).

The Cohn Institute of the History and Philosophy of Science
Tel-Aviv University
Ramat Aviv 69978
Israel