Noether's Contribution to the Rise of the Structural Approach in Algebra

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Introduction

The centrality of the notion of a mathematical structure was a prominent motif in mathematical discourse throughout the twentieth century. A representative instance of this appears in a well-known article of 1979 by Jean Dieudonné (1906-1922), where he used the following terms to describe "The Difficult Birth of Mathematical Structures" (Dieudonné 1979, p. 8):

> Today when we look at the evolution of mathematics for the last two centuries, we cannot help seeing that since about 1840 the study of specific mathematical objects has been replaced more and more by the study of mathematical *structures*. ... this evolution was not noticed at all by contemporary mathematicians until 1900, because not only was the general notion of mathematical structure foreign to them, but the basic notions of specific structures such as group or vector space were emerging very slowly and with a lot of difficulty.

Similarly, in his well-known *A History of Algebra-From al-Kharizmi to Emmy Noether,* Bartel L. van der Waerden (1985, p. 76) stressed that:

> Modern algebra begins with Evariste Galois. With Galois, the character of algebra changed radically. Before Galois, the efforts of algebraists were mainly directed towards the solution of algebraic equations... After Galois, the efforts of the leading algebraists were mainly directed towards the structure of rings, fields, algebras, and the like.

And in this regard, Emmy Noether has been consistently acknowledged among the most prominent mathematicians who contributed to the consolidation and impact of this structural approach, not only in algebra but also in other mathematical disciplines as well (Rowe and Koreuber 2021). Thus for instance, in his epochal textbook on *Algebraic Surfaces*, Oscar Zariski (1935) explicitly wrote:

> In [*Algebraic Surfaces*] I tried my best to present the underlying ideas of the ingenious geometric methods and proofs with which the Italian geometers were handling these deeper aspects of the whole theory of surfaces ... I began to feel distinctly unhappy about the rigour of the original proofs (without losing in the least my admiration for the imaginative geometric spirit that permeated these proofs); I became convinced that the whole structure must be done over again by purely algebraic methods [such as in the work of Emmy Noether and in the important treatise of van der Waerden].

Paul Alexandroff and Heinz Hopf published that same year the first volume their influential treatise on *Topology* (1935) and they also found it important to make Noether's impact explicit:

> The tendency to strict algebraization of topology on group theoretic foundations, which we follow in our exposition, goes back entirely to Emmy Noether. This tendency seems self-evident today. It was not so eight years ago. It took all the energy and the temperament of Emmy Noether to make it the common property of topologists, and to let it play the role it does today in framing the questions and the methods of topology.

We could go on and on indicating important research articles and textbooks published after 1930 and that bear the mark of the structural approach and of Noether's work and personality. Perhaps the most important ones to be mentioned in this context are Saunders Mac Lane and Garret Birkhoff's *Survey of Modern Algebra* (1941), which influenced generations of mathematicians in the USA, and the multi-volume *Éléments de mathématique*, published along decades starting in 1939 by the members of the Bourbaki group.

In this chapter, I outline some of the main features of Noether's work in algebra and explain the sense in which it represented a synthesis of various threads that preceded it, and that opened new avenues of research by organizing the entire discipline around an innovative approach that came to be known as "structural". The interested reader will find a detailed treatment of the topics discussed here, and a fully annotated body of references to the relevant sources, in my book *Modern Algebra and the Rise of Mathematical Structures* (Corry 2004).

What is that thing called "structural algebra"?

The significance and impact of Noether's work in algebra cannot be properly understood without turning attention to the seminal textbook published in 1930 by the Dutch mathematician Bartel Leendert van der Waerden (1903–1996) under the title of *Moderne Algebra*. This textbook signified a true paradigm-shift in the way that the discipline of algebra, its aims and methods, were conceived. Like many other good textbooks, this one presented a synthesis of a large number of recent works that called for a unified and systematic presentation of the topics it considered. Algebraic knowledge, van der Waerden indicated, had not only grown dramatically over the preceding decades. A fundamental change had also affected the very understanding of the discipline as a whole. He thus wrote (van der Waerden 1930, p. 9, italics in the original):

> The recent expansion of algebra far beyond its former bounds is mainly due to the "abstract", "formal", or "axiomatic" school. This school has created a number of novel concepts, revealed hitherto unknown inter-relations and led to far-reaching results, especially in the theories of *fields* and *ideals*, of *groups* and of *hypercomplex numbers*. The chief purpose of this book is to introduce the reader into this whole world of concepts.

But in his book, van der Waerden did much more than just introducing the reader into a new world of concepts and innovative techniques. The very way in which the material was presented and organized involved an original insight of far-reaching consequences, namely, the realization that a certain family of abstract mathematical notions (groups, rings, fields, etc.), defined via sets of formal axioms, should be best seen as comprising instances of one and the same underlying idea, namely, the general idea of an algebraic structure. Under the new approach pursued in the book, the aim of algebraic research would become now the in-depth elucidation of the individual kinds of structures, based on the recurrent use of several common fundamental concepts, questions and techniques (e.g., isomorphisms, homomorphisms, quotients, residue classes, composition series and direct products, inclusion properties and chain conditions, etc.), and the search after similar kind of mathematical results concerning each of them. This is, in fact, the essence of the "structural approach to algebra." Strange as it may sound nowadays, this fundamental insight had not been definitely achieved, let alone systematically presented in a panoramic textbook, before van der Waerden's.

It is important to emphasize that nowhere in the book did van der Waerden state what is an algebraic structure, either at the general, non-formal level or by means of the introduction of a rigorously defined mathematical concept. Rather, he just worked out in detail, chapter after chapter, the basic concepts and properties relevant to each of the domains he included under the general notion of structure. Neither did he specify a list of main tools to be repeatedly used in the investigation of the individual structures. Rather, he just put to work these tools under a single methodological perspective, thus yielding a unique and innovative view of what algebra is all about.

One fundamental innovation implied by van der Waerden's approach was a redefinition of the conceptual hierarchy underlying the discipline of algebra. The various system of numbers were not considered here, as was the case with previous textbooks, to be the underlying foundation over which the entire edifice of algebra, including the properties of polynomials, was to be erected. Rather, it was the other way round. The real and the rational numbers were now conceived each as particular case of a more general, abstract algebraic construct. The concept of a field of fractions could be defined, for instance, for integral domains in general, and the rational numbers were thus realized as a particular case of this general kind of construction. Real numbers were defined in purely algebraic terms as an infinite, ordered, "real field". Additional, nonalgebraic properties such as continuity or density could simply be ignored as part of the characterization of the real numbers in the newly defined algebraic context.

The task of finding the real and complex roots of an algebraic equation and of understanding their mutual interrelations, which had been the hard core of algebra over the previous centuries, was relegated in van der Waerden's book for the first time to a subsidiary role. Three short sections in his chapter on Galois theory dealt with this specific application of the theory, and in the discussion no previous knowledge of the properties of real numbers is assumed. The new conceptual hierarchical underlying this structural view of algebra was illuminatingly visualized in a diagram (*Leitfaden*) appended to the Table of Contents, which indicated the logical interdependence of the chapters (Fig. 1).

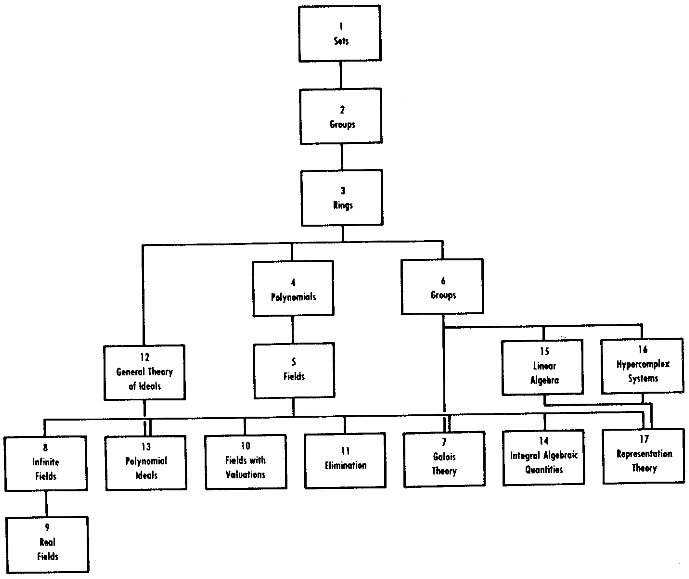

Fig. 1: *Moderne Algebra* - Leitfaden

Van der Waerden wrote his book under the decisive influence of two mathematicians whose lectures he had attended recently: Emmy Noether in Göttingen and Emil Artin (1898–1962) in Hamburg. A considerable part of the contents of the book was directly taken from their lectures. The influence of Artin was significant and it cannot be underplayed but it related to two main topics: (1) his innovative and strikingly structural approach to Galois theory and (2) his work on "real fields". Noether's influence was much more thoroughgoing and much broader, as it transpired through the whole structure of the book as well as in the detailed treatments of the various topics discussed. The forthcoming sections will elaborate on this central point.

What is, then, that thing called „classical algebra":

If the new disciplinary conception of algebra involved in the structural approach is best understood by looking at van der Waerden's *Moderne Algebra*, so we can likewise examine the contents and structure of Heinrich Weber's *Lehrbuch der Algebra* (Weber 1895) as the most adequate manner to understand the disciplinary image that was previously dominant, particularly in the German-speaking world.

When the first volume of the *Lehrbuch* appeared in 1895, Weber (1843–1912) was surely acquainted with the latest advances in various branches of algebra, and in particular with the increasing awareness of the possibility of defining algebraic concepts in purely abstract terms. As a matter of fact, in 1893 he had been the first person to provide, in the framework of one and the same article, abstract axiomatic definitions of both groups and fields, while presenting the latter as a group with an added operation that satisfies some additional axioms. Significantly, the aim of the article was to clarify the "Foundations of Galois Theory" (Weber 1893). Weber introduced all the elements needed for establishing in general terms the isomorphism between the group of permutations of the roots of the equation and the group of automorphisms of the splitting field that leave the elements of the base field invariant. His article implicitly directed attention to the interplay between what we can retrospectively see as the structural properties of both entities, but did not become explicit at that. Nevertheless, it is remarkable that, for all what in retrospect seems innovative in it, this article had minimal direct influence on the algebraic research of other, contemporary mathematicians. Even more strikingly, it did not even influence the perspective adopted by Weber in his own *Lehrbuch*, which embodied the main traits of the classical nineteenth-century image of algebra as the science of polynomial equations and polynomial forms, and soon became the standard textbook of choice in the discipline.

As many other important textbooks, the *Lehrbuch* appeared when its author felt that the development of the discipline over the preceding decades had rendered the existing textbooks obsolete and had brought about the need for a systematic, coherent presentation of many new results and of their main applications. In this case, the state-of-the-art techniques and conceptions related to the problem of finding the roots of polynomial equations, in association with ideas that originated with Galois and were developed by many others, dominated a considerable portion of the book. But like in all previous books in algebra, the theory of polynomials still appeared here as conceptually dependent on, and derivative from, a thorough knowledge of the properties of the various systems of numbers.

The increasingly visibility of techniques associated with Galois theory as the main tool for studying solvability of polynomial equations was duly represented in the book, but at the same time many remnants of ideas that would eventually discarded from algebraic discourse still played a prominent role in the presentation of the discipline. Thus, for instance, "symmetric functions," a concept that had been used by Lagrange in his early research on solvability of polynomial equations and that developments in algebra over the nineteenth century rendered as rather dispensable. Weber included a treatment of them in the *Lehrbuch* as part of a tradition of which his approach to algebra was still part and parcel. In this tradition, a treatment of the theory of polynomial equations should include every particular technique devised to deal with their solvability, as he indeed did here.

Likewise was the case with what is nowadays considered as "analytic" and thus not directly related to an exposition of the basic ideas of algebra. For example, the concept of the root of an equation which is discussed in terms of limits, continuity, ε-δ arguments etc. Weber also discussed Sturm's problem concerning the question of the number of real roots of a given polynomial equation that lie between two given real numbers. This, and further similar problems, are solved with the help of derivatives and other analytical tools. Likewise, Weber discussed well-known approximation techniques, such as interpolation and Newton's method of tangents. Weber also discussed roots of unity, but he did so without mentioning in any way their group-theoretical properties as had already been thoroughly known since the work of Gauss.

Weber's treatment of Galois theory appears in the book after nearly five hundred pages of thorough discussion of various aspects of the resolution of polynomial equations. Although Weber did refer to his own article of 1893, in which he had insisted upon the potential interest involved in studying finite fields, he considered here only infinite fields, or, more specifically, sets of complex numbers closed under the four arithmetical operations of addition, multiplication, subtraction and non-zero division. He briefly mentioned also fields of functions but in no way did he research fields as an autonomous concept with intrinsic interest, even at the relatively elementary level that he did for groups.

Groups are mentioned for the first time in the *Lehrbuch* after more than five hundred pages of discussion of polynomials, and even here one does not find a general treatment of groups. Rather, at this stage Weber considered group of substitutions of one root of a function with another. He defined a group of permutations and the Galois group of a given field, and he explained how the concept can be applied to the theory of polynomial equations. He explicitly stated that the focus of interest does not lie in the study of the properties of the group of permutations as such, but only insofar as it sheds light on the theory of equations. He also proved that the alternating group is simple, which he needed for the proof of the impossibility of solving the general fifth degree equation in radicals.

A thoroughly abstract treatment of groups, similar to that of Weber's own 1893 article appears only in the second volume of the *Lehrbuch*. After the basic concepts of the theory of groups were introduced in the first four chapters of the second volume in a general and abstract way, Weber stated the object of the abstract study of groups. His formulation stresses the need he felt to explain to his contemporaries the meaning of the very use of abstract concepts of this kind (p. 121):

> The general definition of group leaves much in darkness concerning the nature of the concept.... The definition of group contains more than appears at first sight, and the number of possible groups that can be defined given the number of their elements is quite limited. The general laws concerning this question are barely known, and thus every new special group, in particular of a reduced number of elements, offers much interest and invites detailed research.

He also pointed out that the determination of all the possible groups for a given number of elements was still an open question.

The *Lehrbuch* became the standard German textbook on algebra at the turn of the twentieth century and it underwent several reprints. The image of algebra conveyed by Weber's book was to dominate the algebraic scene for almost thirty years. As a matter of fact, this was the textbook from which van der Waerden himself learnt his undergraduate algebra in Amsterdam in 1919-1924 (van der Waerden 1975, p. 32). Its influence can be easily detected, among others, through the widespread adoption of a large portion of the terminology introduced in it. As matter of fact, even after the publication of van der Waerden's introduction of the new, structural image of algebra Weber's influence did not totally disappear. One can notice this by looking at several books published in the 1920s which are closer to the *Lehrbuch* than to *Moderne Algebra*, in terms of their overall view of the discipline. This is the case with *Modern Algebraic Theories* (1926), by Leonard Eugene Dickson (1874-1954), and *Höhere Algebra* (1926) by Helmut Hasse (1898-1979). The clearest sign of the *Lehrbuch*'s longstanding influence on algebraic activity, especially within

Germany, is provided by the publication in 1924 of another textbook by Robert Fricke (1861-1930). Fricke wrote his book upon request of Weber's publisher after the *Lehrbuch* had sold out. In spite of the relatively long time since the original publication, and the many important adavnces in algebraic research since then, Fricke chose to essentially abide by the conception of algebra embodied in Weber's presentation He stressed this very clearly in the name he chose for his own textbook: *Lehrbuch der Algebra - verfasst mit Benutzung vom Heinrich Webers gleichnamigem Buche*.

The impact of Emmy Noether on the rise of the structural approach to algebra may be understood with reference to the thorough-going transformation noticeable in the image of the discipline as it was presented, successively, in Weber's and in van der Waerden's textbooks. An illuminating way to explain how Noether was instrumental in bringing about this transition is by focusing on developments that arose from three main threads of ideas at the turn of the century: (1) work on algebraic integers, (2) the study of polynomials and their factorization properties, and (3) the rise of the idea of abstract fields. Noether's path-breaking synthesis that led to the rise of modern algebra was based on her deep acquaintance with a very broad range of current works. She was able to identify significant trends and to select and improve the most important concept and techniques that her predecessors had started to develop (On this point see McLarty 2017, Merzbach 1983). The forthcoming sections are devoted to discussing these three important threads.

Dedekind's Ideals

The first thread to be mentioned as part of the genealogy of ideas that led to Noether's innovations, and one whose importance can hardly be exaggerated, relates to the work of Richard Dedekind (1831-1916) on the question of factorization of domains of algebraic integers. Noether herself testified to the deep impact that this work had on her own, as she repeatedly used to state: "*Es steht alles schon bei Dedekind*". This statement was meant to refer, above all, to Dedekind's theory of ideals, the basic ideas of which he started to work out as early as 1856, and which he published between 1871 and 1894 as supplements to successive versions of his edition of the *Vorlesungen über den Zahlentheorie* of Gustav Lejeune Dirichlet (1805- 1859).

Dedekind's theory of ideals arose against the background of previous work of Ernst Eduard Kummer (1810–1893) on the question of factorization in domains that generalize the system of Gaussian integers $a+ib$, with a,b integers. Carl Friedrich Gauss (1777-1855) had investigated this domain in his work on biquadratic reciprocity. He succeeded in generalizing the basic ideas of the arithmetic of $\mathbb{Z}$, by identifying counterparts of the prime numbers that would help prove a generalized version of the fundamental theorem of unique factorization. Kummer went on to explore questions related with higher reciprocity and he did so by carrying Gauss's ideas further on into even more general domains of numbers. Specifically, he investigated domains of numbers of the kind $a + \varrho b$ (a,b integers) within $\mathbb{C}$, where ϱ could be either $\sqrt{-t}$ ($t \neq 1$) or $\sqrt[n]{-1}$ ($n \geq 3$).

With the help of the ingenious idea of "ideal complex numbers", he was able to make important progress in these questions, but he also became aware very soon of significant limitations inherent in his theory. Dedekind studied in detail Kummer's work and became strongly impressed by it. Dedekind's main mathematical strenght was on conceptualizing complicated mathematical situations and coming up with the correct settings where the main problem should be successfully analyzed. Accordingly, he realized that the hurdles encountered in Kummer's theory derived from the need to choose specific "ideal numbers" in every particular case to be considered, thus obscuring the more general principles underlying the mathematical situation at stake (Corry 2004, pp. 80-92).

In order to overcome these hurdles, Dedekind started by focusing on those subsets of $\mathbb{C}$ which are closed under the four arithmetic operations, which he called "fields" (*Zahlkörper*). Further, within any given field Ω of complex numbers he defined a specific collection of numbers, *D*, the "algebraic integers" of Ω, comprising all numbers in Ω which are roots of some irreducible monic polynomial with coefficients in $\mathbb{Z}$. This collection *D*, he realized, plays within Ω the same role that—for the purposes of investigating factorization properties— $\mathbb{Z}$ plays within the smallest sub-field of $\mathbb{C}$, namely $\mathbb{Q}$. Thus if we want to investigate for example the factorization properties of generalized Gaussian integers of the kind $G = \{\ a + b\sqrt{-3}\ /\ a,b \text{ in } \mathbb{Z}\ \}$, then we need to place ourselves within the number field $\Omega = \{\ a + b\sqrt{-3}\ /\ a,b \text{ in } \mathbb{Q}\ \}$, and then, within Ω, we need to consider the collection *D* of all its algebraic integers. It is in *D*, rather than in *G*, Dedekind understood, that the correct laws of factorization will surface. By doing so, given that $G \subseteq D$, we will have obtained the factorization laws that apply to *G*.

Thus, Dedekind's first important insight in this regard was to have identified the correct domain of numbers where factorization has to be investigated. Dedekind's second important insight was to define the correct tool with which to do so. This is where his ground-breaking concept of "ideal" appears. Ideals embody one of Dedekind's central methodological principles, with the help of which he was able to introduce many important and useful new concepts, namely, rather than looking at the *individual numbers* and their properties when trying to understand their behavior, it is more important to look at *collections of numbers* and at their properties as collections (i.e., mostly *inclusion* properties). This proved to be a crucial principle in the case of ideals. While studying the factorization properties of an algebraic integer δ in a number field, Dedekind focused on properties of the collection $i(\delta)$ of all multiples of δ. Particularly he pointed out the following two seemingly obvious properties of that collection:

- If α and β both belong to $i(\delta)$, then both $\alpha+\beta$ and $\alpha-\beta$ must also belong to $i(\delta)$;
- If β belongs to $\mathrm{i}(\delta)$ and x is any (algebraic) integer in the domain considered, then βx must also belong to $i(\delta)$.

Dedekind called this collection $i(\delta)$ the "principal ideal generated by δ". The important insight here was that, side by side with these principal ideals, there are other collections of algebraic integers which, while not themselves also principal ideals, they do satisfy the same two properties. Moreover, Dedekind understood that such collections will be of help for understanding factorization in its broadest setting. Thus, Dedekind defined an ideal *M* to be *any* collection of algebraic integers satisfying the following two properties:

(1) If $\alpha, \beta \in M$, then $\alpha + \beta \in M$ and $\alpha - \beta \in M$;
(2) If $\beta \in \mathrm{M}$ and x is any (algebraic) integer in *D*, then $\beta x \in M$.

Working out the details of a theory of ideals meant, above all, defining operations among the ideals, identifying properties of specific kinds of ideals (such as "prime ideals") and, of course, formulating generalized versions of the fundamental theorem of unique factorization as known to hold in the basic case of $\mathbb{Z}$. The various, successive versions of Dedekind's theory were meant to make the theorems *less and less dependent* on the need to choose specific representatives in the collection of numbers considered for the sake of defining the properties or of proving theorems related to them. Gradually, Dedekind set to himself a more radical aim, namely, coming up with proofs which would ideally be formulated *solely in terms of the collections themselves and of their inclusion properties*. This was a quest for increasingly "structural", and less "computational" understanding of the mathematics involved here, and one that Noether would later on continue to pursue with increased sharpness (McLarty 2006).

One issue in which the gradual changes along the successful versions of the theory become particularly noticeable is that of prime ideals and their multiplicities. In the first published version of the theory, dating from 1871, Dedekind formulated the main factorization theorem as follows (1930-32, Vol. 3, p. 258):

- Every ideal is the *l.c.m.* of all the powers of prime ideals that divide it. (*)

This should be read as follows: an ideal A is a multiple of another ideal B (or "is divisible by B") whenever $A \subseteq B$. An ideal P is prime if its only divisors are itself and the ideal U, U being $i(1)$ (or the collection of all algebraic integers in the field considered). Notice that in present-day terms, Dedekind's prime ideals are called "maximal ideals", while an ideal is called nowadays prime if for any product of ideals contained in it, at least one of the factors is also contained in it. Domains in which every ideal can be uniquely written as a product of prime ideals are called nowadays "Dedekind domains". It was Noether, as will be seen below, who indicated the specific importance of such domains.

Now, the main problem here is how to define the "power" of an ideal. In order to do so, Dedekind introduced the following equivalent definition of a prime ideal:

An ideal P is prime if for every product $\alpha\beta \in P$, either $\alpha \in P$ or $\beta \in P$.

This definition allowed him to add a new concept, "simple ideal", whose formulation, like the latter, alternative one, requires choosing specific elements in the ideal and checking some of the properties of their powers, so as to define the power of the ideal (For details, see Corry 2004, pp. 92-120). This definition in turn, provides a rather workable tool for proving useful results. For example:

every principal ideal $i(\mu)$ is the l.c.m. of all powers of simple ideals containing μ.

Further:

every prime ideal is indeed a prime simple one.

Hence, one can speak only of prime ideals, while forgetting about the simple ones. Further:

if all the powers of the prime ideals that divide a given ideal M, divide also the principal ideal i(δ), then M divides the principal ideal $i(\delta)$.

Finally, with this results at hand he was able to prove the fundamental factorization theorem (*).

Dedekind's definition and use of simple ideals in order to deal with multiplicity of factors in the first version of his theory epitomizes the inherent reasons for attempting improved versions. Simple ideals were not defined through an abstract property, as a special class of prime, or of other kind of ideals, but rather as collections of integers satisfying a specific kind of congruence which required selecting specific numbers. Dedekind was able to couch his new factorization theorem in ideal-theoretical language, but in an important sense he did not yet abandon the traditional outlook which he wanted to overcome. Dedekind worked hard in the subsequent versions in order to omit this concept as well as some other similar ones.

In 1879, for example, Dedekind was able to reformulate the main concepts and theorems by developing in detail a full arithmetic of ideals, including operations such as division and negative powers, without relying on specific choices. This allowed him to avoid defintions and proofs based on choices of specific numbers and to develop a theory with a distinct structurally-oriented flavor. An emblematic results found in this version is the following:

> Given a chain of modules $A_1, A_2, \ldots A_n, \ldots$ contained in a given finitely-generated module N, and such that A_i is contained in A_{n+1} for all indexes i, then there is an index k such that $A_i = A_k$, for all $k > i$.

This is already quite close to the kind of structural formulations that one may find later on in the work of Noether, particularly because of what appears here as an early version of the "ascending chain condition" (*a.c.c.*), which will become the hallmark of her approach. In the proof, however, Dedekind constructed a certain determinant whose rows contain the coordinates of a specific set of numbers, with respect to a given basis. Dedekind stated (1930-32, Vol. 3, p. 527) that "this proof is not satisfactory because it depends upon specific choices of numbers, and, moreover, because the theory of determinants is alien to the proper content of the theorem."

It is important to stress that Dedekind's early use of this kind of "chain condition" was essentially different from that which we will later find in Noether's work, and this difference is indeed crucial. Dedekind's proof relied on a lemma which is based on properties of complex numbers and algebraic integers, and which, moreover, requires choosing specific elements in the module. He did not singled out this chain condition as an abstract kind of properties that we might postulate as an abstract axiom to characterize all domains for which unique factorization theorems hold but rather used it as a specific property of the numbers he was working with.

This is, then, the important point to stress here: Dedekind's fields were the *subject matter* of his theory, whereas ideals were a *tool* for handling the main question arising in the study of this subject matter, namely, unique factorization. Fields and ideals were not, one may say, mathematical entities belonging to one and the same family, or different manifestations of a common underlying notion. Similar was the case with Dedekind's quite idiosyncratic approach to Galois theory: he focused on sub-fields of the complex numbers, and the (extensions) containing the roots of given polynomial equations. This was the *subject matter* to be investigated. The group of a given equation was a *tool* with the help of which, the question of solvability could be analyzed. Thus, sub-fields of $\mathbb{C}$ and their algebraic integers were for Dedekind the subject-matter of "higher arithmetic", whereas polynomial equations and polynomial forms was the subject-matter of "algebra"; ideals and groups, one the other hand, were the right tools to deal with the main problems arising in each of these domains. In both cases, the main properties on the basis of which the corresponding mathematical theory is built are the properties of the systems of numbers, which are assumed to be well-established in their own.

Polynomials and Factorization

The second thread in the genealogy of ideas leading to Noether's innovations is the one that concerns the issue of factorization in systems of polynomials, specifically ideas associasted with the work of Emmanuel Lasker (1868–1941) Francis Sowerby Macaulay (1862–1937) (Lasker 1905, Macaulay 1913). The starting point of their works go back to ideas developed by Hilbert early in his career. He had famously proved that for any module M of polynomials in n variables $x_1,..., x_n$, there exists a finite sub-collection $F_1, ...,F_k$, such that any member of M may be written as $X_1F_1 + ... + X_kF_k$, where X_i represent arbitrary polynomials. This collection is called a basis. Hilbert also proved his famous *Nullstellensatz*, according to which given a finite set of polynomials $F_1, ...,F_h$, and another polynomial F, such that F vanishes in all the common roots of $F_1, ...,F_h$, then there exists a natural number k, and h polynomials $A_1, ...,A_h$,such that $F^k = A_1F_1 + ... + A_hF_h$.

Lasker, better known for being World Chess Champion for 27 years, had studied in Göttingen, where he got to know Hilbert, and later completed his PhD in Erlangen under Max Noether (1844-1921), Emmy Noether's father. While studying factorization properties for systems of polynomials, Lasker inotrduced the idea of a primary ideal, which, in modern terms and considering a general ring R, can be stated as follows:

> An ideal A of an arbitrary ring R is called a primary ideal if given two elements a,b of R, such that their product ab belongs to A but a does not belong to A, then there exists an integer k such that b^k belongs to A.

The connection of this idea with Hilbert's *Nullstellensatz* is evident. The infleunce of Dedekind's work on ideals is also evident, but Lasker never became invovled with the idea of possibly formulating a general theory of rings and ideals valid both for numbers and polynomials. Rather he just trasnferred some of Dedekind's main concepts from the realm of numbers into the realm of polynomials, which he considered to be a completely separate one, with its own properties that are independent, in many important senses, from those of numbers.

Lasker proved two main results:

1. every module is representable as an intersection of a finite number of primary modules and a module such that at no point $a_1, ...,a_n$ all the polynomials belonging to the module vanish.
2. for every polynomial form f belonging to the prime ideal P associated with a primary ideal M, there exists an integer h such that f^h is an element of M.

The important point is that for proving the latter result, Lasker relied on the following lemma:

> given an infinite ascending sequence of ideals $M_1, M_2,...$ there exists a number n so that for $N > n$, the module M_N is contained in $(M_1, M_2,..., M_n)$, the *l.c.m.* of the n modules $M_1, M_2,... M_n$.

While this lemma can be seen as a version of the *a.c.c.* specifically formulated for polynomial forms, Lasker proved it using Hilbert's basis theorem, and in this sense it is a direct consequence of specific properties of polynomails, rather than a general principle valid in an abstract ring. Remerkably, Lasker did not mention the reciprocal theorem, namely, that the said chain condition implies the existence of a finite basis. In the framework of polynomials, this statement would have been superfluous.

Lasker's results were complemented in 1913 by Macaulay. Macualay proved the *uniqueness* of the decomposition and provided an algorithm to actually perform it. The Lasker-Macaulay theorem thus states that every ideal of polynomials can be uniquely decomposed as an intersection of finitely many primary ideals. Noether would eventually prove the more general theorem, nowadays known as the Lasker-Noether theorem, namely that "every ideal in a ring with *a.c.c.* can be uniquely decomposed as an intersection of finitely many primary ideals".

From *p*-adic numbers to abstract fields

The third thread to be considered here, and one that became a crucial turning point in this story, is the one associated with the seminal work on abstract fields published in 1910 by Ernst Steinitz (1871–1928). The opening section of Steinitz's article, "Algebraische Theorie der Körper", included an abstract defintion of fields as presented in Weber's article of 1893, but the focus of his inquiry diverged significantly from that of Weber. In Steinitz's own words (1910, p. 5):

> Whereas Weber's aim was a general treatment of Galois theory, independent of the numerical meaning of the elements, for us it is the concept of field that represents the focus of interest. . . . The aim of the present work is to advance an overview of all the possible types of fields and to establish the basic elements of their interrelations.

Of fundamental importance in his quest was the idea of the charcateristic of the field. Steinitz showed for the first time that any given field contains a "prime field" which is isomorphic, according to the characteristic of the original field, either to the field of rational numbers or to the quotient field of the integers modulo p (p prime). Then, after thoroughly studying the properties of these prime fields, he proceeded to classify all possible extensions of a given field and to analyze which properties are passed over from any field to its various possible extensions. Since every field contains a prime field, by studying prime fields and the way in which properties are passed over to extensions, Steinitz would attain a full picture of the structure of all possible fields.

Besides the characteritic, Steinitz also worked out in some details additional importat ideas such as seprabale elements in the field, and the degree of trascendence of an extension. He also proved that every field has an algebraically closed extension. Remarkable is also the way in which Steinitz devoted specific attention to his use of set-theoretical ideas. At the time, this was still quite a novel issue which needed to yet to find a proper place in mathematical discourse and Steinitz was fully aware of the advantages but also of the dangers involved in it. This is manifest in his remark about the use of the Axiom of Choice and the need to do this is with great care.

The main source of inspiration for Steinitz's work came from the original research that Kurt Hensel (1861–1941) had recently conducted on the thoery of p-adic numbers. What is special about these p-adic numbers is that they embodied a truly new kind of entity, that was full of mathematical meaning and interest, and which, unlike what is found in Dedekind, "counts neither as the field functions nor as the field of numbers in the usual sense of the word." Moreover, the p-adic numbers do not comprise a field of numbers located somewhere between $\mathbb{Q}$ and $\mathbb{C}$ as had been all the number fields theretofore investigated. The importance of the characteristic as a main notion in the theory arose no doubt from Hensel's ideas.

It is interesting that this same work of Hensel gave rise to the definition of abstract rings, but not in itself to an abstract investigation of rings in parallel to that of Steinitz. Hensel had presented his work in two main books, *Theorie der algebraischen Zahlen* (1908) and *Zahlentehorie* (1913). In both books he was assisted by the then graduate student Abraham Halevy Franekel (1891-1965). Fraenkel raised the question what would happen with Hensle's numbers if, instead of a prime number p, a composite number g would be used as the basis for representing numbers as series of powers. The mian issue here is that when doing so, one obtains a system of numbers where divisors of zero do appear. Hensel thought that such systems are devoid of interest, whereas his student Fraenkel undertook the exercise of formulating general axioms that define an abstract system with two operations, one of which does not necessarily comprise inverse elements, and in which divisors of zero may appear (Fraenkel). Fraenkel attempted to adapt to the case of ring the kind of ideas developed by Steinitz in his article on fields. On the other hand, he did not connect, in any way, the idea of a ring with the question of unique factorization as presented in the works of Dedekind on algebraic numbers and of Lasker on polynomials, as Noether would do about ten years later. Indeed his articles on this topic remained at the most elementary level and did not open ways to use the general notion of ring as a tool for actual research (Corry 2000).

Noether's Abstract Rings

Noether's first publication dealing with the question of unique factorization in abstract rings, "*Idealtheorie in Ringbereichen*", appeared in 1921. She stated very clearly that the aim of the paper was:

> … to translate the factorization theorems of the rational integer numbers and of the ideals in algebraic number fields into ideals of arbitrary integral domains and domains of general rings. (Noether 1921, p. 25)

In order to do so, she started by focusing on the unique representation of an integer as a product of powers of prime numbers written in the following alternative way:

$$a = p_1^{r_1} p_2^{r_2} \dots p_s^{r_s} = q_1 q_2 \dots q_s$$

In this representation, some elementary properties are immediately noticeable:

> (1) Any two different factors q_i, q_j are relatively prime and they cannot be further decomposed into factors having this property. Moreover, since the factors are relatively prime, the product of the q_i's represents also their *l.c.m.*
>
> (2) Given q_i, q_k and an integer b such that q_k divides $b^2 \cdot q_i$, then, it follows that q_k necessarily divides $b \cdot q_i$;
>
> (3) Given that q divides $b^2 \cdot c$, but q does not divide b, then q necessarily divides a power of c. However, the product of two factors q_i, q_k does not satisfy this property anymore;
>
> (4) Each factor q_i in the above decomposition cannot be represented as the *l.c.m.* of two proper divisors of q_i.

The theorem of unique factorization of integer numbers could now be formulated in terms of the primary components q_i. To each q_i there is a uniquely associated prime number p_i and a natural number σ, such that $p_i^{\sigma} = q_i$, and the theorem is stated as follows:

> In any two different factorizations of a rational integer numbers into irreducible, greatest primary factors q, the number of factors, the corresponding prime numbers (up to their sign), and the exponents coincide. (Noether 1921, p. 25)

Then, substituting the word "number" by "ideal generated by a number", she obtained the theorem for ideals in algebraic number fields, as Dedekind had formulated it.

From here, she moved into the realm of an abstract rings Σ. By this time, the general idea of an abstract ring was still totally unfamiliar to her audience so that she started by defining a ring and stating some of its most basic properties, following the work of Fraenkel. Echoing the core of Hilber's work on polynomials she also required that the rings satisfy the "finitness condition" (*Endlichkeitbedingung*), namely, that every ideal in the ring should have a finite basis. She then took the fundamental step of proving the equivalence between the finiteness condition and the *a.c.c.*, now formulated in purely abstract terms.

Within this spirit of ideas, Noether proceeded to define new concepts that were all formulated strictly in terms of chain conditions and other inclusion properties, and to prove several factorization properties in the same terms. Thus, for instance, Noether defined primary and prime ideals as follows: D is a primary ideal in Σ if from $AB \subseteq D$ and $A \nsubseteq D$, it follows that there exists an integer n such that $B^n \subseteq D$. If, in addition, n=1 for every ideal B, then D is a prime ideal. For any primary ideal D, there exists a unique prime ideal P containing D, such that a power of P is contained in D. The lowest such power is called the exponent of D. The main factorization theorem is the following:

> In a ring satisfying the *a.c.c.*, every ideal is representable as the reduced intersection of a finite number of indecomposable ideals (which are also primary); the number of such ideals and the collection of associated prime ideals is invariant for every given ideal, though possibly the specific primary ideals used in the factorization are not.

In her 1921 paper, Noether proved four factorization theorems of this kind. All four were already known for the domains of polynomials, but the previously existing proofs relied, as Noether herself remarked, on the fact that every polynomial is uniquely representable as a product of irreducible polynomials. This result, in turn, depended on properties of polynomials that are directly derived from those of the systems of real and complex numbers. Noether's main insight was to understand that the factorization of ideals may indeed be formulated independently of this property of polynomials, and that what determines it is, in fact, what may be properly called a "structural property", namely, the *a.c.c.* Moreover, Noether clearly stated the uniqueness conditions of each of the different factorizations. In the introduction she mentioned a series of earlier articles in which uniqueness had been discussed, but always in a partial, non-systematic fashion. The decomposition theorems of this paper are all "multiplicative" ones: namely, they present the ideals as products or *l.c.m.* of other ideals. However, Noether mentioned the existence of other, "additive" decomposition theorems and pointed out that a translation of the latter kind of decompositions into the former would be possible using the concepts introduced in her article, together with some additional ones. These additional concepts were meant to define properties of the systems of residue classes defined by the ideals in the ring. The latter happen to constitute themselves a ring satisfying the same general properties as the original one. In fact, the original ring might itself have been considered as a special case of a residue system.

Noether's structural concerns appear clearly conveyed by these kinds of remarks. Moreover, they condition the direction of her research, since they suggest both the kinds of questions that should be addressed and the preferred way to answer them. Noether chose here decomposition questions as her main focus of interest, but these are addressed from a peculiar perspective. Not only are the subjects of her research abstractly defined by means of axioms, but, moreover, contrary to the classical approach in which the systems of numbers have a privileged status and the properties of other algebraic constructs may be derived from those of the number-systems, under this approach there is no essential difference between a given algebraic system and the building blocks that constitute it. Faced with these systems the question arises, how their properties (in this case, decomposition properties) are passed over back and forth, from the original structure into its building blocks (the quotient systems). These kinds of questions had already been worked out by Steinitz in the case of fields, but in extending Steinitz's point of view to rings, Noether was in fact establishing a guideline for research with much more general implications, namely, that these questions are relevant for a more general class of interesting, abstractly defined algebraic systems.

In her next important paper (Noether 1926), Noether approached the problem of factorization within a much more maturely conceived axiomatic framework. She proved a series of decomposition theorems that hold in a commutative ring *R*, to which five additional axioms are successively added. Taken together, these five axioms add up to define what is known nowadays as a "Dedekind ring", namely a ring in which every primary ideal is a power of a prime ideal. The five axioms are formulated at the beginning of the paper as follows:

I. *R* satisfies the *a.c.c.*

II. Every proper descending chain of ideals in *R*, each of which contains a given non-zero ideal, is a finite chain.

III. There exists a unit for the multiplication in *R*.

IV. There are no divisors of zero in *R*.

V. The field of fractions of the ring *R* is integrally closed (i.e., each element of the field of fractions, which is an integer with respect to *R*, belongs in fact to *R*).

Noether considered abstract rings here, not only as an appropriate framework for formulating general theorems of factorization, but increasingly as an object of intrinsic interest that raised structural questions worthy of detailed study. Given a ring *T* with identity and having no divisors of zero, and given a fixed sub-ring *R* of *T* containing the identity, Noether defined an *R*-Module as a sub-set of *T* satisfying the usual conditions for the addition and for the product by an element of *R*. Consequently, An ideal is a module all of whose elements belong to *R*. In these terms, one illustrative example of the kinds of theorem proved here by Noether deals with finite modules, namely, modules for which a finite basis exists. The theorem is formulated as follows:

> If *R* is a commutative ring with unit satisfying the *a.c.c.*, and *M* is a finite *R*-module, then also *M* satisfies the *a.c.c.* (Noether 1926, p. 34)

The structural spirit of the theorem is manifest in that it considers an abstractly defined domain and establishes the conditions under which certain properties of the basic domain are passed over to a new domain which is derivative from it. And what is of special significance is that the *proof* relies on a correspondence between the lattice of sub-modules of *M* and that of the ideals of *R*. This lattice expresses in a succinct fashion the inclusion relations of the sub-domains of the two domains in question.

This structural approach is also explicitly worked out when Noether explains how the properties expressed as axioms I to V carry over from certain basic rings to their finite extensions. In particular, if axioms I-V hold for a given ring *R*, they hold for the domain *S* of all integers in *T* relative to *R*. At this stage it is clear how to proceed:

> It is enough to prove for subordinate fields of numbers and of functions that the basic domains satisfy axioms I-V: integer numbers, single-variable polynomials, functional domains of many-variables polynomials. (Noether 1926, 37)

Thus, while in the case of fields, Steinitz had established that the most basic fields are the prime fields associated with any field, turning these fields into the building blocks of a structural theory of fields, Noether's factorization theorems embodied a natural step in the process of determining the building blocks of an euqally structural, abstract theory of rings.

In the main result of the paper, Noether considered the consequences of adding axioms II-V to the *a.c.c.* Assuming all the five axioms allows proving that primary ideals are powers of their associated prime ideals. As a consequence, the main decomposition theorem is the following:

> If a ring *R* satisfies axioms I-V, then every ideal in *R* is uniquely representable as intersection of a finite number of powers prime ideals. (Noether 1926, 53)

Finally, in the closing section of the article Noether proved the equivalence of the double-chain condition (i.e. the simultaneous occurrence of the ascending and descending chain conditions) with the existence of a composition series (A sequence $E \leq H_n \leq ...H_1 \leq G$ of sub-groups of a given group constitute a composition series when each subgroups is normal relative to its immediate predecessor and when the factor groups H_i / H_{i+1} are all simple groups). She formulated all of her arguments in terms of modules in general. She did not mention operations among elements of the module, and relied purely on arguments dealing with the interrelation of the ideals and the sub-modules involved. Moreover, as she explained in a footnote, since the modules are in fact Abelian groups under addition, the composition series are indeed principal series (In a composition series, it might be the case that H_{i+1} is normal in H_i, but not in G. If, however, all the H_i's are normal in G, then the series is called a principal series. For Abelian groups, every composition series is also a principal series). However, since all the theorems are formulated in terms of composition series, they remain valid also for the case of non-Abelian groups. The converse is also true: the existence of a composition series implies the double chain condition. As a by-product of this result, Noether also proved a general version of the Jordan-Hölder theorem using induction on the length of the series in the standard way.

In the following years, Noether became increasingly active in research in other topics of algebra such as representation theory and hypercomplex numbers. He contributed important results in these fields as well, always conducting her research within the same structural spirit that was visible in her two articles on rings. In particular, Noether focused her interest in the non-commutative cases and in the search after the "structural invariants" that arise in such cases. In the non-commutative case it is somewhat limitative to rely on the properties of the operations defined on the individual elements of the abstract ring. Decomposition theorems in this case are best proved purely in terms of inclusion properties of sub-domains. This perspective afforded, in her view, fundamental insights to understanding the commutative case as well.

Noether's abstractly conceived concepts provided a natural framework in which conceptual priority may be given to the axiomatic definitions over the numerical systems considered as concrete mathematical entities. Still, her axiomatic conception, perhaps because of her own deep acquaintance with the classical aims of concrete algebraic research, was not one that aimed at formal games with symbols, devoided of a more concrete matematical meaning. For Noether, the axiomatic analysis of concepts is only one of two complementary aspects, rather than the exclusive essence of mathematical research. Thus she was quoted as saying:

> In mathematics, as in knowledge of the world, both aspects are equally valuable: the accumulation of facts and concrete constructions and the establishment of general principles which overcome the isolation of each fact and bring the factual knowledge to a new stage of axiomatic understanding. (Quoted in Corry 2004, p. 249)

Other works in Algebra in the early 20th Century

In spite of her decisive influence, it is obvious that the rise of the structural approach in algebra was the outcome of the combined impact of works by various mathematicians who worked out their ideas since the turn of the twentieth century. This impact was felt not only in terms of innovative techniques and results, but also in terms of a thoroughgoing reconceptualization of the discipline, its internal organization and its place in the overall economy of mathematics (Corry 2007). Besides Noether, the most significant influence over van der Waerden in writing his book was that of Emil Artin. Van der Waerden attended in 1926 a course on algebra in Hamburg taught by Artin in collaboration with Otto Schreier (1901-1929). They presented an updated overview of recent work by Steinitz on fields and Noether on rings, as well as a presentation on Galois theory that would essentially be the one that van der Waerden followed in his textbook. But of especial interest for van der Waerden was the recent work on real fields, by Artin and Schreier, which he would reproduce in chapter X of his book.

The problem which Artin addressed and which led to his definition of real fields, together with Schreier, is the following:

> Let F *be* an algebraically closed field. Find all proper subfields K of F, such that F is an algebraic extension of finite degree of K.

In 1924 Artin found a solution for a field F which is an algebraic closure of $\mathbb{Q}$. Two years later, together with Schreier, he found a more general solution, valid for any algebraically closed field of characteristic 0: F has to be an extension of K of degree 2, and K has the property that -1 cannot be expressed as a sum of squares (Artin and Schreier 1926, 1927). Besides the intrinsic interest of this result, the definition of real fields in purely algebraic terms was of the utmost importance for the consolidation of van der Waerden's view of the entire discipline of algebra as a hierarchy of structures, since it lent definitive support to the idea that the field of real numbers has no essential conceptual priority, but rather it is a very specific case of combined ideas that relate to algebraic structures. This idea is emblematically represented in the fact that, in van der Waerden's *Leitfaden*, real fields appeared at the deep-bottom of the hierarchy of concepts, as a particular case of infinite fields.

Throughout his career, Artin distinguished himself as a brilliant expositor who published various, highly influential textbooks on *Galois theory* (1942), *Rings with minimum condition* (1948), *Geometric algebra* (1957), and *Class field theory* (1961). All these texts bear the strong structural character that in van der Waerden's book implied a groundbreaking innovation, and that, in turn, had been highly influenced by Artin.

In the German context, we should also mention here the works of Berlin mathematicians such as Issai Schur (1875-1941) and Georg Ferdinand Frobenius (1849-1917), on matrices, algebras and hypercomplex numbers (Curtis 1999; Hawkins 2013). Taken individually all these works contributed with important results that were in the background to Noether's work, but they did not came up with a new, coherent overall picture of the discipline that would challenge the existing one as embodied, above all, in Weber's *Lehrbuch*. Similar is the case with the work of American mathematicians such as Leonard Dickson (1874-1954) and Joseph Wedderburn (1882-

1948), and their predecesors Benjamin Peirce (1809-1880) and Josiah Willard Gibbs (1839-1903), who had worked on vector calculus, linear groups, Galois fields, and hypercomplex numbers (Artin 1950, Fenster 1998, Parshall 1985).

A last point of particular interest to be mentioned here, which helps understand the crucial role of Noether, as the main figure in this story, concerns the work of Hilbert. Hilbert's far-reaching influence on all fields of activity in mathematics at the turn of the century—including those related to algebra—cannot be overemphasized, of course, and precisely because of this, it allows us stressing the particularly novel character of Noether's work. I already mentioned above the work of Lasker, and the way in which it was influenced by Hilbert. Indeed, there are other aspects Hilbert's work which, on the face of it, are directly related to the most important developments in early-twentieth century mathematics and which we tend to associate with the rise of the structural approach in algebra. In the first place is his work on the theory algebraic invariants. This was the first field of research in which Hilbert distinguished himself as an upcoming figure of mathematics by the end of the nineteenth century. His achievements derived, among other things, from a rejection of the algorithmic style that dominated the discipline and by focusing several conceptual central ideas that he adopted from number theory. Then, working in the theory of algebraic number fields, he adopted the more conceptual perceptive developed by Dedekind, while preferring it over that of Leopold Kronecker (1823-1891) which was more algorithmic (Corry 2001). Hilbert's famous account of 1897, *Zahlbericht*, which definitely embodied this preference, turned into one of the most influential ones in the theory of algebraic number fields, thus helping promote Dedekind's point of view. Finally, the central role of the modern axiomatic method, a main component of the structural approach to algebra, is justly identified with the work of Hilbert, beginning with his 1899 book on the foundations of geometry (Hilbert 1899).

In all of these respects it may indeed make sense to speak about Hilbert as the harbinger of a new conception of algebra, and yet, against the description provided above, several important considerations shed light on the gap that still was necessary to overcome in order to reach the new conception embodied in the work of Noether. Take as an example of this the famous list of twenty-three problems put forward by Hilbert in 1900. Five of them are somehow related to algebra in the "classical" sense, but none to algebra in a more "modern" sense. There is no problem related to the theory of groups, algebras or systems of hypercomplex numbers, and not even to Galois theory. In his Göttingen lectures, where he typically presented the ideas that were occupying his mind over the years, we find none of the topics related to questions of factorization in abstract rings, group theory, abstract fields, or the like. Nor did any of his outstanding doctoral students (no less than 68, including the likes of Hermann Weyl) wrote dissertations on such topics.

As already mentioned, Noether proved Lasker's theorem using the "finitness condition", as Hilbert had done before her. But she derived this property from the *a.c.c.* alone, whereas Hilbert had derived it from specific properties of polynomials over $\mathbb{C}$. Hilbert, as far as we are aware, never commented on this point, or indicated its importance. More importantly, there is no know evidence that Hilbert ever commented on the idea of structure as a possible, organizing principle for algebra, let alone mathematics at large. In all of his lectures and publications, he consistency abode by the classical conceptual hierarchy whereby the systems of numbers, either genetically or axiomatically defined, are the basis on which the entire edifice of mathematics is built. Moreover, there seems to be no evidence that he ever commented positively on van der Waerden's book.

Concluding Remarks

Beyond the intrinsically mathematical virtues of Noether's work, it also seems clear that the great influence she was able to exert needs also to be explained by the quantity and the quality of her Göttingen students. It seems unlikely that such a circumstance could have come about in an institutional environment other than the truly unique one which developed in Göttingen between the years 1895, with Hilbert's arrival, and 1933, after Hitler's rise to power (Rowe 2018). Obviously, regardless of their own personal abilities to create a stable group of students around them and to communicate to them their own ideas, neither Dedekind, nor Steinitz, Fraenkel, Lasker or Macaulay enjoyed the opportunity to do in their respective intitutions. The peculiarity of the *Noether Schule* as a cultural phenomenon is discussed in detail in the chapter by Mechtchild Koreuber in the present collection, and its importance in the process of spreading her ideas cannot be overlooked (see also Koreuber 2015). In the words of Hermann Weyl (Cited in Corry 2004, p. 222):

> In my Göttingen years, 1930-1933, she was without doubt the strongest center of mathematical activity there, considering both the fertility of her scientific research program and her influence upon a large circle of pupils.

Among Emmy Noether's direct students we can mention leading algebraists such as Max Deuring (1907-1984), Hans Fitting (1906-1938), Friedrich Grell (1903-1974), Jakob Levitzki (1904-1956), Kenjuri Shoda (1902-1977), Otto Shilling (1911-1973) and Ernst Witt (1911-1991). Several other leading mathematicians co-worked with her and her influence was decisive on their own works. Besides those already mentioned, one should also point out here Helmuth Hasse (1898-1979) and Wolfgang Krull (1891-1971) whose contributions to algebra were of the highest quality and impact (Fenster & Schwermer 2007). As Stefan Müller-Stach explains in his chapter in this collection, one can trace a direct line leading from the contributions of Noether and her successors to many important topics that are central to current algebraic research, such as homological algebra, the application of categorical methods in group theory and topology, algebraic and arithmetic geometry, and algebraic K-theory.

Noether's influence on van der Waerden's *Moderne Algebra* and—through the impact of the latter—on modern algebraic research at large can be characterized as follows: van der Waerden adopted many results of Noether and presented them in a systematic way. Noether—original as her thought was—was not the only important algebraist from whom van der Waerden took his ideas. He also included methods and results of Artin, Krull and other important mathematicians who were working on the same issues and were influenced by (and probably also influenced) Emmy Noether. In van der Waerden's presentation, different mathematical domains were presented as individual instances of algebraic structures, that need therefore to undergo similar treatments: they are abstractly defined, they are investigated by recurrently using a well-defined collection of key concepts, and a series of questions and standard techniques is applied to all of them. Some of these features had already appeared in Steinitz's and in Fraenkel's work, but Noether's research on ideals definitely contributed to give legitimation and interest to the possibility of applying them systematically. Van der Waerden also included issues that had not been developed by Noether but which found a natural place among the other algebraic domains. The most outstanding example of this was group theory, which appeared in van der Waerden's book for the first time as an algebraic theory of parallel status to field theory, ring theory, etc.

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