STEHT ES ALLES WIRKLICH SCHON BEI DEDEKIND? IDEALS AND FACTORIZATION BETWEEN DEDEKIND AND NOETHER

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Abstract. This article analyzes similarities and divergences between the approaches of Richard Dedekind and Emmy Noether to the problem of factorization in fields of algebraic numbers. Dedekind's approach was highly idiosyncratic when seen in the context of late nineteenth-century algebra. In many important senses it can be seen as the harbinger of the central ideas of the structural approach to abstract algebra of which, beginning in the 1920s, Emmy Noether and her school were the main promoters. Still, several decades of intense mathematical research in the hands of important figures separate the works of these two great masters. This raises the question that is addressed in this article, namely, what are the differences between their approaches and what was the process that led from that of the first to that of the second.

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1. Introduction

Mathematical lore has it that Emmy Noether (1882-1935), whenever praised for her brilliant algebraic work, typically reacted by modestly stating: "*Es steht alles schon bei Dedekind*" (All of this is already found in the work of Dedekind). And indeed, whoever glances at Dedekind's texts on the theory of fields of algebraic numbers will not fail to realize, and to be surprised by, the extent to which ideas found there can easily be associated with the kind of reasoning typical of twentieth-century algebra, and that is commonly attributed to Noether's innovations. This is particularly the case concerning the theory of ideals in abstract rings, and the approach to the question of unique factorization in such domains. Still, the plain historical fact is that Richard Dedekind (1831-1916) started to work out his ideas as early as 1856, and published them between 1871 and 1894, in successive versions of his supplements to Dirichlet's *Vorlesungen über den Zahlentheorie*, whereas Noether's first publication on this topic dates from 1921. Long decades of intense mathematical activity separates between them.

The question thus naturally arises whether Noether's generous statement concerning Dedekind's putative priority is historically and mathematically precise, and if so, what is it that one can define as her specific contribution to the rise of modern algebra. After all, over the years that separate their periods of involvement with these matters, many prolific and prominent mathematicians were at work in related questions. Think, to take just one prominent example, of the work done by David Hilbert (1862-1943) in number theory in the 1890s and in particular of his *Zahlbericht* of 1897,[14] which is generally considered to be a milestone of the discipline at the turn of the century. If all of Noether's ideas were already found in Dedekind's work, why did it take so long for colleagues such as Hilbert, to become aware of them, and what was then the value of her specific contributions?

This article is devoted to discuss these questions and to do so by focusing on how questions of factorization were addressed with the help of the notion of ideal. The development of the theory of ideals has attracted the attention of historians of mathematics for a while now. Back in 1980 Harold Edwards published a first in a series of articles of seminal importance for understanding this crucial issue in the history of late nineteenth-century mathematics.[10] I also devoted a central part of my book on the rise of modern algebra to a detailed discussion of this topic.[4] The present article is mostly based on material discussed in the book, and the interested reader will find there further details. (See also [21]

2. Dedekind's Ideals

Dedekind's theory of ideals arose against the background of the work of Ernst Eduard Kummer (1810–1893) on the question of factorization in domains that generalize the system of Gaussian integers $a+ib$. Carl Friedrich Gauss (1777-1855) had investigated this domain in his work on biquadratic reciprocity. He succeeded in generalizing the basic ideas of the arithmetic of $\mathbb{Z}$, by identifying counterparts of the prime numbers that would help prove a generalized version of the fundamental theorem of unique factorization. Kummer went on to explore questions related with higher reciprocity and he did so by carrying Gauss's ideas further on into even more general domains of numbers. Specifically, he investigated domains of numbers of the kind $a + \rho b$ (a and b integers) within $\mathbb{C}$, where ρ could be either $\sqrt{-t}$ (t $\neq 1$) or $\sqrt[n]{-1}$ ($n \geq 3$).

This is not the place to delve into the details of Kummer's work (see [4, pp. 81–92]). Suffice it to say that he achieved important progress with the help of an innovative concept, namely, "ideal complex numbers". In spite of this, he soon became aware of some limitations inherent in his theory. Kummer had been an indefatigable calculator who painfully examined a large number of cases when working out his theory. Dedekind studied in detail this work and became strongly impressed by it. Dedekind's main mathematical strenght was on conceptualizing complicated mathematical situations and coming up with the correct settings where the problem should be analyzed. He realized that the hurdles encountered in Kummer's theory derived from the need to choose specific "ideal numbers" in every particular case considered, thus obscuring the more general principles underlying the situation.

In order to overcome these hurdles, Dedekind started by focusing on those subsets of $\mathbb{C}$ which are closed under the four arithmetic operations, which he called "fields" (*Zahlkörper*). Further, within any given field Ω of complex numbers he defined a specific collection of numbers, $\boldsymbol{D}$, the "algebraic integers" of Ω, comprising all numbers in Ω which are roots of some irreducible monic polynomial with coefficients in $\mathbb{Z}$. In this way he could focus on a collection $\boldsymbol{D}$ which plays within Ω the same role that—for the purposes of investigating factorization properties—$\mathbb{Z}$ plays within the smallest sub-field of $\mathbb{C}$, namely $\mathbb{Q}$.

Thus if we want to investigate for example the factorization properties of generalized Gaussian integers of the kind

$$G = \{\ a + b\sqrt{-3}\ /\ a, b\ \text{ in } \mathbb{Z}\ \},$$

then we need to place ourselves within the number field

$$\Omega = \{\ a + b\sqrt{-3}\ /\ a, b \text{ in } \mathbb{Q}\ \}.$$

Within Ω we need to consider the collection $\boldsymbol{D}$ of all its the algebraic integers and it is here, not in G, that the correct laws of factorization will appear. Given that $G \subseteq \mathbf{D}$, we will have obtained the factorization laws will apply to G.

Thus, Dedekind's first important insight is to have identified the correct *domain* where factorization has to be investigated. Dedekind's second important insight was to define the correct *tool* with which to do so. This is where the ideals make appearance. Dedekind based his entire approach on handling collections of numbers and their properties as collections (mostly inclusion properties) rather than looking at the individual numbers and their properties. This proved to be a crucial decision. In the case of ideals, he proceeded by considering the properties of the collection $i(\delta)$ of multiples of an algebraic integer δ in a number field. He pointed out the following two seemingly obvious ones:

- If α and β both belong to $i(\delta)$, then both $\alpha + \beta$ and $\alpha - \beta$ must also belong to $i(\delta)$;
- If β belongs to $i(\delta)$ and x is any integer in the domain considered, then β x must also belong to $i(\delta)$.

Dedekind called this collection $i(\delta)$ the principal ideal generated by δ. It is obvious that by studying the properties of such collections we will gain insight into the factorization properties. What is less obvious is that, side by side with the principal ideals, there are other collections of algebraic integers which are not principal ideals but do satisfy the same two properties and that they will be of help for understanding factorization in its broadest setting. Thus, Dedekind defined an ideal M to be any collection of algebraic integers satisfying the following two properties:

(1) If $\alpha, \beta \in M$, then $\alpha + \beta \in M$ and $\alpha - \beta \in M$;
(2) If $\beta \in M$ and x is any integer in $\boldsymbol{D}$, then $\beta x \in M$.

Working out the details of a theory of ideals meant, above all, defining operations among the ideals, identifying properties of specific kinds of ideals (such as "prime ideals") and, of course, formulating generalized versions of the fundamental theorem of unique factorization as known to hold in the basic case of $\mathbb{Z}$. The various versions of the theory that Dedekind published along the years were meant to make the theorems less and less dependent on the need to choose *specific representatives* in the collection of numbers considered. The aim was to come up with proofs which would ideally be formulated solely in terms of the collections themselves. This was a quest for increasingly "structural", and less "computational" understanding of the mathematics involved here.

Let me give focus on a specific example in order to clarify this point. This involve some technicalities. As already mentioned, Dedekind's work was published in successive versions of a supplement to Dirichlet's *Vorlesungen*. This was very typical of Dedekind, who used to continuously reelaborate, polish and republish his results along the years. This is also very helpful for understanding the historical development of his ideas. One issue in which this development is particularly noticeable is the issue of prime ideals and of their multiplicities. In the first published version of his theory, dating from 1871, he formulated the main factorization theorem as follows:

> Every ideal is the *l.c.m.* of all the powers of prime ideals that divide it.[7, Vol. 3, p. 258]

The terms and concepts as used by Dedekind require some explanation, especially in what concerns the idea of the "power " of an ideal. Thus, an ideal A is for Dedekind a multiple of another ideal B (or "is divisible by B") whenever $A \subseteq B$. Remarkably, when speaking of the fields, Dedekind used the opposite notation, so that a field K' is a multiple of another field K whenever $K \subseteq K'$. This is just another hint (albeit subtle) to the different status that Dedekind accorded to these two different mathematical entities. An ideal P is prime if its only divisors are itself and the ideal U, U being $i(1)$ (or the collection of all algebraic integers in the field considered). Notice that in present-day terms, Dedekind's prime ideals are called "maximal ideals", while an ideal is called nowadays prime if for any product of ideals contained in it, at least one of the factors is also contained in it. Domains in which every ideal can be uniquely written as a product of prime ideals are called nowadays "Dedekind domains". It was Noether, of course, who indicated the importance of such domains.[4, pp. 225–237] It can be proved that a Dedekind domain is a Noetherian ring with unit, which is also integrally closed, and in which any non-zero prime ideal (in the second sense of the term) is also maximal.[17, pp. 600 ff.]

So, how did Dedekind define the "power" of an ideal at this stage? In order to do this he needed first to generalize for ideals the concept of "norm" that Kummer had introduced for ideal numbers. Thus, since an algebraic integer η in $\boldsymbol{D}$ is a root of a polynomial of degree n with integer coefficients, let $\eta_2, \eta_3, \ldots \eta_n$ be the other $n-1$ roots of the same polynomial. Then, the product $N(\eta) = \eta \cdot \eta_2 \cdot \ldots \cdot \eta_n$ is called the norm of η. In the second place he needed a more workable definition of prime ideals, and he thus introduced the following, equivalent alternative definition:

> An ideal P is prime if for every product $\alpha\beta \in P$, either $\alpha \in P$ or $\beta \in P$.

It is easy to see why this definition would characterize a prime ideal in Dedekind's setting, but it is also important to understand why Dedekind would not be satisfied with it and would try to eliminate it form future versions of the theory. The reason is that the definition involves the choice of particular numbers within the ideal. At this point, however, he had no better strategy, and with the help of this definition he could introduce the notion of "simple ideals" which he needed for handling multiplicities. For one thing, using this defintion he could prove the following result:

> Let μ be any non-zero integer such that $N(\mu) \neq 1$. Then there exists a number ν, such that if P is the collection of all roots π of the congruence $\nu\pi \equiv 0 \ (mod\ \mu)$, then P is a prime ideal.[7, Vol. 3, pp. 255-256]

And now, once again, Dedekind turned this *property* of P into a *definition*:

> A prime ideal P is said to be "simple" if there exists a number $\nu \in P$, such that for any number $\pi \in P$, $\nu\pi \equiv 0 \ (mod\ \mu)$, where μ is any non-zero integer such that $N(\mu) \neq 1$.

These simple ideals are convenient since they allow for a rather strighforward defintion of powers. Indeed, given a simple ideal P and an integer μ as above, then clearly also $N(\mu^r) \neq 1$. Thus:

> Given any rational integer r, the roots of the congruence $\nu\pi \equiv 0 \ (mod\ \mu^r)$ also constitute an ideal; this ideal is called the r-th power of the prime ideal P.

This is a useful definition, no doubt, but notice that in order for it to make sense it is necessary that the power r depend only on the ideal P, and not on the particular choice of the pair μ, ν. Dedekind proved that this is the case (with a minor mistake in the proof that he later corrected).[7, Vol. 3, p. 419] And yet, the definition still requires considering specific members in the ideal, as do the following fundamental properties that are needed to prove the main factorization theorem:

(1) If $s \geq r$, then P^r divides P^s.
(2) For every integer μ, there is always a highest power r of P such that μ is contained in P^r (or "a highest power of P in μ").
(3) If P^r and P^s are the highest powers of P in μ,η respectively, then P^{r+s} is the highest power of P in $\mu\eta$.
(4) P is the only prime ideal which is a divisor of all powers of a given simple ideal P.
(5) If all the powers of a (non-zero) prime simple ideal containing a given integer $\mu \neq 0$, also contain an integer η, then η is divisible by μ.

Since the collection of all multiples of an integer μ constitute the principal ideal $i(\mu)$, this last, fundamental property can be equivalently formulated as follows: every principal ideal $i(\mu)$ is the *l.c.m.* of all powers of simple ideals containing μ. To conclude this series of results, Dedekind also showed that every prime ideal is indeed a prime simple one, and thus one may finally speak only of prime ideals, while forgetting about the simple ones. Further, if all the powers of the prime ideals that divide a given ideal M, divide also the principal ideal $i(\delta)$, then M divides the principal ideal $i(\delta)$. With this results at hand he was able to prove the fundamental factorization theorem as formulated above.

Dedekind's definition and use of simple ideals in order to deal with multiplicity of factors in the first version of his theory epitomizes the inherent reasons for attempting improved versions. Simple ideals were not defined through an abstract property, as a special class of prime, or of other kind of ideals, but rather as collections of integers satisfying a specific kind of congruence which required selecting specific numbers. Dedekind was able to couch his new factorization theorem in ideal-theoretical language, but in an important sense he was continuing with the the traditional outlook which he wanted to overcome. Dedekind worked hard in the subsequent versions in order to omit this concept as well as some other similar ones. For lack of space, I will not give any details about the interesting ways in which Dedekind transformed his concepts and his results. The interested reader will find such a detailed account in my book. Still, it is interesting to illustrate with a brief example taken from a later version, published in 1879.

Important progress was made when Dedekind reformulated the main concepts and theorems in terms of the product of two ideals. Dedekind had introduced this concept in the first version, but for some reason he did it just as an afterthought which he did not incorporate into the main body of results. Thus, given two ideals A, B then their product AB is nothing but the ideal containing all numbers of the form ab where $a \in A$ and $b \in B$. In this terms he now dealt with primes ideals in terms of the basic property that if a product of ideals is divisible by a prime ideal P, then at least one of the factors is divisible by P. Curiously, Dedekind continued to refer to specific numbers even when he could do without them. Thus for instance in the following result:

> An ideal A (or a number α) is divisible by an ideal D (or a number δ) if and only if all the powers of prime ideals of D (or of δ) appear also in A (or in α).[7, Vol. 3, p. 312]

In the last published version of the theory the product of ideals was the starting point of the discussion and moreover, based on it, Dedekind actually defined a complete arithmetic of ideals, including operations such as division and negative powers. This allowed him to avoid defintions and proofs based on choices of specific numbers and to develop a theory with a distinct structurally-oriented flavor. An emblematic results found in this version is the following:

> Given a chain of modules $A_1, A_2, \ldots A_n, \ldots$ contained in a given finitely-generated module N, and such that A_i is contained in A_{i+1} for all indexes i, then there is an index k such that $A_i = A_k$, for all $k > i$.

This is already quite close to the kind of formulations that one may find later on in the work of Noether, particularly because of what appears here as an early version of the ascending chain condition (*a.c.c.*). Even more clear is the approach that arises in a theorem where this condition is applied. Dedekind defined the *Ordnung* of an ideal A as the quotient $A{:}A$, and he proved the important result that an algebriac number is an algebraic integer if and only if it is contained in the *Ordnung* of some ideal A. The proof relies on constructing a determinant whose rows contain the coordinates of a specific set of numbers, with respect to a given basis. Dedekind stated that "this proof is not satisfactory because it depends upon specific choices of numbers, and, moreover, because the theory of determinants is alien to the proper content of the theorem."[7, Vol. 3, p. 527]. Moreover, from there he reformulated a partial version of the theorem which he then proved using the chain condition that he had previously introduced.

Dedekind's methodological choices were well-conceived and he was rather consistent in applying them. But it is also important to understand that they were quite *sui-generis* for his time and that they were not shared by many. Viewing the theory of determinant as alien to the investigation of factorization laws was a very strong statement at the time. Emmy Noether would be among the few who initially adopted the abstract and structural guidelines of his work and understood how powerful they could be if systematically applied in algebraic research. Among his contemporaries, Dedekind's views created reticence and his ideas were not always fully understood. Even among those mathematicians who were closer to him, his approach was sometimes received with suspicion. Clear evidence for this appears in a letter of 1893 written by Ferdinand Georg Frobenius (1849–1917) to Heinrich Weber (1843–1912), two mathematicians who were in close interaction with Dedekind. Weber was at the time writing a textbook on Algebra that became the standard one at the turn of the century, and Frobenius wrote to him the following:

> Your announcement of a work on algebra makes me very happy... Hopefully you will follow Dedekind's way, yet avoid the highly abstract approach that he so eagerly pursues now. His newest edition (of the *Vorlesungen*) contains so many beauty ideas, . . . but his permutations are too flimsy, and it is indeed unnecessary to push the abstraction so far. I am therefore satisfied, that you write the *Algebra* and not our venerable friend and master, who had also once considered that plan.

3. Dedekind's Fields and Hilbert's Numbers

Dedekind was not the only mathematicians who undertook to develop Kummer's ideas further into the more general realms of fields of algebraic numbers. Also Leopold Kronecker (1823–1891) played a key role in this important mathematical quest. In their respective works, Dedekind and Kronecker mutually complemented the theorems, proofs and techniques elaborated by each other. At the same time, they represented two rather different, and to some extent even opposed, mathematical approaches. Kronecker represented what may be called a more "algorithmic" approach, whereas Dedekind was the quintessential representative of the so-called "conceptual" approach. This is not intended to mean that Kronecker introduced no new, abstract and general concepts or that he derived no results from an adequate use of them. Nor is it meant to imply that one finds no computations in Dedekind. Rather, I refer to a matter of preferences. Of particular importance is the fact that Dedekind's perspective allowed for the indiscriminate use of infinite collections of numbers defined by general abstract properties, whereas Kronecker insisted on the need to prescribe the specific procedures needed to generate the elements of such collections and to determine whether or not two given elements were one and the same. Dedekind did not seek or require such procedures whereas Kronecker did not consider it legitimate to ignore them.

A decisive factor in transforming Dedekind's approach into the dominant one in algebraic number theory and related fields at the turn of the twentieth century (especially within the German context) was the publication in 1879 of Hilbert's *Zahlbericht*.[14] This "Report on Numbers" was initially commissioned by the *DMV*, the Association of German Mathematicians, as an up-to-date overview on the state of the art in the discipline. Hilbert indeed summarized the work of his predecessors but he also added many new results and sophisticated techniques that opened new avenues for research. These avenues were indeed pursued by many leading researchers in the decades to come. The choices made by Hilbert were strongly influenced by both Dedekind and Kronecker. Still, Hilbert allowed for a clear emphasis on the "conceptual" perspective embodied in the work of the former, over the "algorithmic" one of the latter.

Hilbert was keen on making explicit the significance of this emphasis. Thus he famously wrote in the introduction:

> It is clear that the theory of these Kummer fields represents the highest peak reached on the mountain of today's knowledge of arithmetic; . . . I have tried to avoid Kummer's elaborate computational machinery, so that here too Riemann's principle may be realized and the proof completed not by calculations but purely by ideas.[16, p. ix]

The approach underlying the *Zahlbericht* follows very closely most of Dedekind's definitions and realizes in a similar way the interrelation among the basic concepts and tools found at the heart of the theory. In addition, Hilbert was the first to use the term "ring" in this context, but his definition was meant to refer to a specifically number-theoretic (rather than abstract and general) situation: a ring is a system of algebraic integers of the given field, closed under the three operations of addition, subtraction and product.[14, p. 121] Hilbert also defined an ideal of a ring as any system of algebraic integers within the ring, such that any linear combination of them (with coefficients in the ring) belongs itself to the ideal.

Hilbert quoted in this context several results from Dedekind's theory of ideals of algebraic numbers. But like Dedekind before him, Hilbert's account did not suggest a more general notion of an "algebraic structure" the various instances of which should be investigated from a common perspective. He did not describe a ring as a group endowed with a second operation, or as a field whose division fails to satisfy a certain property. Such concepts were introduced in the text so as to be used in the specific setting relevant to each of them. Hilbert's ideals were always specific collections of algebraic numbers within fields of complex numbers rather than a distinguished kind of sub-ring.

Likewise remarkable is the fact that, in spite of his direct involvement with the theory of polynomials in the earlier stages of his career and his acquaintance with the main problems of this discipline,[13] Hilbert never attempted to use ideals as an abstract tool allowing for a unified analysis of factorization equally applicable both in fields of numbers and in systems of polynomials. This step, crucial for the later unification of the two branches under the abstract theory of rings, was first taken more than twenty years later by Noether. Obviously, the absence of such a step in Hilbert's work or in Dedekind's theories was not so much a consequence of technical capabilities, as it is one of motivations: an indication of the nature of their overall disciplinary conception of algebra, to which the general idea of algebraic structures as an organizing principle was foreign.

4. Hilbert's Axiomatization

Defining the main objects of enquiry in algebraic theories by way of abstract postulates is one of the basic methodological traits that we typically associate with modern structural algebra. In the British context such a trend was part of a well-entrenched tradition dating back to the mid-nineteenth century. It comprised the works mathematicians such as George Peacock (1791–1858), Duncan Gregory (1813–1844), Augustus De Morgan (1806–1871) and William Rowan Hamilton (1805–1865). But this British tradition was not part of, and was not soon to connect with, the kinds of developments that we are considering here. Within the German context, of course, one cannot discuss the issue of axiomatization without referring to Hilbert. His name has become inseparably associated, among many other things, with the "modern axiomatic approach", which in turn has been frequently linked to the view of mathematics as a formal game with symbols devoid of intrinsic meaning. A careful historical analysis of Hilbert's views, however, presents us with a more complex picture, the understanding of which is necessary within the present account.[3]

Hilbert developed his axiomatic method as a powerful and necessary tool for elucidating the deductive structure of existing, well-elaborated scientific theories, and for enhancing the ability to further develop them in view of possible conceptual difficulties. He did not consider his method as a purely formal tool to be used as starting point for developing new theories. As intended objects of application, he had in mind, above all, the classical theories of nineteenth-century mathematics and physics: geometry, arithmetic, mechanics (both classical and statistical), electrodynamics, and so on. Crucial for understanding his outlook is the following quotation taken from a lecture of 1905 in Göttingen:

> The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development. (Cited in [4, p. 162].)

His famous treatise on the foundations of geometry [15] has to be understood from this perspective. This was not an attempt to turn this field of knowledge into a formal game with empty symbols. Indeed, Hilbert's conception of geometry was essentially empiricist. It is from this perspective that we can also make sense of his enquiries into the foundations of physical theories.[5] In his work on the foundations of arithmetic, which came somewhat later, Hilbert did suggest a formalist approach but only as a device specifically crafted for the intended aim of achieving a finitist proof of consistency. This should not be taken as expressing an overall view about the essence of mathematics.

In his work in number theory, as well as in the theory of invariants, Hilbert went as far as possible in carefully sorting out the foundational principles of the impressive edifices that had been built by his predecessors. In doing so he came forward with notions and methodological principles that turned out to be fundamental for establishing the structural approach to algebra. But his own conceptions never followed this direction, in the sense championed by Noether, as a preferred one for the discipline. His work retained many of the essential features that characterized the classical, nineteenth-century views of the discipline, albeit taken to their most elaborate manifestations.

For example, Hilbert showed no interest in works related to the trend of "postulational analysis", that used his techniques for analyzing axiomatic systems, but which focused on the systems of axioms as such rather than on the ways in which they provided a grounding of well-developed theories.citescanlan In 1914 Abraham Fraenkel (1891–1965) , for instance, came forward with a definition of rings in terms as abstract postulates.(see below section 6). It is remarkable that we have no record of Hilbert expressing any interest on Fraenkel's definition, or on other works that went the same way.

It is also illuminating to take a look at the kinds of doctoral dissertations that Hilbert supervised (no less than sixty-eight throughout his career). Remarkably, not one of the dissertations dealt with topics that later came to be connected with modern algebra—such as abstract fields, rings, or the theory of groups in any of its manifestations. No less remarkable is the fact that, although five among the twenty-three problems that Hilbert included in his famous list of 1900 can be considered in some sense as belonging to algebra in the nineteenth-century sense of the word, none of them deals with problems connected with more modern algebraic concerns, and in particular not with the theory of groups or for the theory of ideals.

Perhaps the most striking remark that can be mentioned in this regard is that we do not have any record of Hilbert reacting to either Noether's work on rings or to van der Waerden's textbook *Modene Algebra* written under her direct influence (see below section 7). In the case of van der Waerden one may perhaps take into consideration the advanced age and poor state of health of Hilbert by 1930, the time of publication of the book. In the case of Noether, however, absence of publicly recorded reaction is truly remarkable. Hilbert's appreciation for Noether's abilities were enormous and he spared no efforts to promote her career. Hilbert and Noether, moreover, closely collaborated before 1920 in matters related to the general theory of relativity. When it came to her work on factorization in abstract rings, however, what we find is silence. I do not mean to imply that Hilbert was hostile to the kind of work that Noether was doing. He may have been positive about it, given that her articles of 1920 and 1926 appeared in the *Mathematische Annalen*, a journal whose policy Hilbert continued to dominate until the early 1930s. Still, the fact is that, in all what concerns ideals in abstract rings as a tool to investigate questions of factorization, Hilbert never became actively involved in any kind of activity related to her ideas and those of her circle, nor publicly expressed his interest in it.

5. Steinitz's Abstract Fields

A main turning point in this story came in 1910 with the work of Ernst Steinitz (1871–1928) on abstract fields. No doubt, this was a work that provided a direct source of inspiration for Noether. In order to understand its impact, we must first discuss briefly the classical locus for the early axiomatic definition of fields. The first place where fields were defined by means of postulates as an abstract group endowed with a second operation was in an article of 1893 by Heinrich Weber.[29] Strongly influenced by Dedekind's approach to Galois theory for algebraic equations (see below in section 8), Weber's article was devoted to presenting this theory in the most general terms known to that date. It introduced all the elements needed for establishing in general terms the isomorphism between the group of permutations of the roots of the equation and the group of automorphisms of the splitting field that leave the elements of the base field invariant.

Weber's joint discussion of groups and fields within one and the same framework suggested in a natural way the convenience of adopting an abstract formulation for both of them. Moreover, it stressed the importance of the interplay between what we can retrospectively see as the structural properties of both entities. Nevertheless, it is remarkable that, for all what seems innovative in it, this article had minimal direct influence on the algebraic research of other, contemporary mathematicians. Even more strikingly, it did not even influence the perspective adopted in Weber's own *Lehrbuch der Algebra* [30], whose first volume appeared in 1895, and which soon became a standard text in the field (until the publication in 1930 of van der Waerden's book). While it included many of the most important recent advances in algebra and the awareness to the possibility of defining central concepts by means of abstract postulates, this textbook preserved the main traits of the classical nineteenth-century image of algebra as the science of polynomial equations and polynomial forms.

The subject matter of Steinitz's article, "Algebraische Theorie der Körper", were the abstract fields as defined by Weber in 1893. The aim of the investigation, however, diverged significantly. In Steinitz's own words:

> Whereas Weber's aim was a general treatment of Galois theory, independent of the numerical meaning of the elements, for us it is the concept of field that represents the focus of interest. ... The aim of the present work is to advance an overview of all the possible types of fields and to establish the basic elements of their interrelations.[25, p. 5]

Steinitz also explained in detail the steps to be followed in order to attain this aim. First, it is necessary to consider the simplest possible fields. Then, one must study the methods through which from a given field, new ones can be obtained by extension. One must then find out which properties are preserved when passing from the simpler fields to their extensions.

The main source of inspiration for Steinitz's work came from the original research that Kurt Hensel (1861–1941) had conducted on the thoery of p-adic numbers. The reason for this is that p-adic numbers embodied a truly new kind of entity, full of mathematical meaning and interest, "which counts neither as the field functions nor as the field of numbers in the usual sense of the word." Moreover, the p-adic numbers do not comprise a field of numbers located somewhere between $\mathbb{Q}$ and $\mathbb{C}$ as had been all the number fields theretofore investigated. These numbers represented something totally new, which nevertheless shared some important properties with the better known domains of algebraic numbers, and this was precisely the kind of properties that Steinitz wanted to investigate in his abstract theory.

A central concept, whose importance Steinitz claimed to have realized while studying Hensel's theory, was that of the characteristic of the field. Weber in 1893, for instance, may have envisaged the possibility of considering fields of characteristic other than zero, but he certainly did not see the importance of pursuing it, and he did not have a clue on its possible significance. Steinitz showed for the first time that any given field contains a "prime field" which is isomorphic, according to the characteristic of the original field, either to the field of rational numbers or to the quotient field of the integers modulo p (p prime). Then, after thoroughly studying the properties of these prime fields, he proceeded to classify all possible extensions of a given field and to analyze which properties are passed over from any field to its various possible extensions. Since every field contains a prime field, by studying prime fields, and the way properties are passed over to extensions, Steinitz would attain a full picture of the structure of all possible fields.

Steinitz's article embodies, in a nutshell and limited to the specific case of fields, the gist of the structural conception of algebra. As we will see now, in his articles on abstract rings, Fraenkel simply followed on Steinitz's footsteps example and asked very similar questions about the new abstract notion that he proposed to investigate. Emmy Noether then pursued the same lead in her work on ideals and factorization, taken it to its full-blown expression. Finally, Van der Waerden's book may be seen as the extended application of this paradigm to the entire discipline of algebra, considered now as the systematic investigation, form a unified point of view, of the whole variety of the different kinds of structures.

6. Noether's Abstract Rings

All of the important work done before Noether on questions related with factorization focused on fields of algebraic numbers which are sub-fields of $\mathbb{C}$. Accordingly, all the results were derived from the known properties of the real and complex numbers. Steinitz's contribution signified a change of seminal importance and yet he did not change the basic assumptions of the theory concerning the domains of algebraic numbers in which factorization was investigated. In the theory of polynomials, on the other hand, specific factorization theorems had also been proven and special techniques had been developed in the works of Hilbert, as well as in those of Emmanuel Lasker (1868–1941) and Francis Sowerby Macaulay (1862–1937) [13, 19, 20]. These results were proved while relying on specific properties of polynomials, which themselves derived from those of the real and complex numbers.

On the wake of Dedekind's work on algebraic fields, the role of ideals as a main tool for elucidating phenomena of factorization had become increasingly clear over the ensuing decades. Still, these ideals continued to be conceived as specific collections either of numbers within $\mathbb{C}$. The general concept of a ring, in turn,appeared as an offshoot of the idea of p-adic numbers. If instead of a prime number p, a composite number g is used as the basis for representing numbers the way Hensel did, a system is obtained where divisors of zero do appear. Hensel thought that such systems are devoid of interest, whereas his student Fraenkel undertook the exercise of formulating general axioms that define an abstract system with two operations, one of which does not necessarily comprise inverse elements.[11] Fraenkel's articles on this topic, however, remained at the most elementary level and did not attempt to use the general notion of ring as a tool for actual research.[2]

Against this background, Noether was the first to realize that rings, taken as a concept that may be defined in abstract terms via formal axioms, could provide a convenient conceptual framework for a general theory of ideals and factorization.[22, 23] Her work differed from Dedekind's in the greater generality of her results and in the much clearer axiomatic presentation of the ideas. Whereas Dedekind had opened the way to generalized factorization theorems by introducing new concepts, built by focusing on certain characteristic collections of *algebraic numbers*, Noether advanced one step forward and abandoned the restrictive framework of systems of numbers. She reformulated many of Dedekinds concepts in terms of collections of *abstract elements* of a formally defined ring. In these terms, she re-elaborated many of Dedekind's central results by focusing on properties of ideals in the ring such as the ascending chain condition (a.c.c.). Thus, her main factorization theorem was formulated as follows:

> In a ring with a.c.c. every ideal is representable as the reduced intersection of a finite number of indecomposable ideals (which are also primary); the number of such ideals and the collection of associated prime ideals is invariant for every given ideal, though probably the specific primary ideals used for the factorization are not.[22]

Both Dedekind and Hilbert had formerly identified the importance of such properties in the context of their earlier research on fields of numbers, but they had not pursued it systematically. At variance with them, Noether now conceived and applied chain conditions in the framework of her abstractly formulated theory. The very concept of ring was still so alien to contemporary mathematicians, that in her first article of 1921 Noether felt it necessary to prove its most elementary properties (e.g., that the identity element for multiplication in a ring is unique).

Over the later years of her life, Noether became interested in the non-commutative cases, thus pushing some of the central traits of her work into their most extreme expression. In the non-commutative case it is somewhat limitative to rely on the properties of the operations defined on the individual elements of the abstract ring. Decomposition theorems in this case are best proved purely in terms of inclusion properties of sub-domains. Proofs of this kind should reveal, in Noether's view, the real structure of the ideals of the ring.

Noether's abstractly conceived concepts provided a natural framework in which conceptual priority may be given to the axiomatic definitions over the numerical systems considered as concrete mathematical entities. With Noether, then, the balance between the genetic and the axiomatic point of view begins to shift more consciously in favor of the latter. This new balance was a necessary condition for the redefinition of the conceptual hierarchies, and for the establishment of a new image of knowledge. The notion of a structure would now dominate algebraic research and the various number systems would appear as particular instances of it. Nevertheless, Noether's axiomatic conception, perhaps because of her own deep acquaintance with the classical aims of concrete algebraic research, remained close to Hilbert's own. For Noether, the axiomatic analysis of concepts is only one of two complementary aspects, rather than the exclusive essence of mathematical research. Thus she was quoted as saying:

> In mathematics, as in knowledge of the world, both aspects are equally valuable: the accumulation of facts and concrete constructions and the establishment of general principles which overcome the isolation of each fact and bring the factual knowledge to a new stage of axiomatic understanding. (Quoted in [4, p. 249])

Noether pursued the study of abstract rings as an object of interest in itself and used it as the main conceptual framework of algebra. Her work was designed along the lines of Steinitz's treatment of abstract fields ten years earlier. But she had a greater overall impact on algebra than Steinitz, if only because it showed that

Steinitz's program applied not only for the particular case worked out by him, but for many other significant cases as well. Whereas group theory was the first algebraic discipline to be abstractly investigated, field theory became the first discipline that arose from the research of numerical domains into an abstract, structural subject. Subsequently, research on ideals in an abstract ring, as pursued by Noether, consolidated the idea that a more general conception lay behind all of this: the conception that algebra should be concerned, as a discipline, with the study of algebraic structures in general.

The intrinsic mathematical virtues of Noether's work appear as obvious in retrospect, but it also seems clear that the great influence that she was able to exert can be explained by the quantity and the quality of her Göttingen students.[18] Neither Dedekind, nor Steinitz, Fraenkel, Lasker or Macaulay—regardless of their personal abilities, or lack thereof, to create a stable group of students around them and to communicate to them their own ideas—could ever have profited from the opportunity to work out their research in conditions similar to those enjoyed by Noether in Göttingen with some many brilliant students and collaborators around her.

7. Van der Waerden's *Moderne Algebra*

Fundamental for understanding the impact of Noether's work in algebra is to turn attention to the seminal textbook published in 1930 by the Dutch mathematician Bartel Leendert van der Waerden (1903–1996) under the title of *Moderne Algebra*. This textbook signified a true paradigm-shift in the way that the discipline of algebra, its aims and methods, was conceived. Like many other good textbooks, this one presented a synthesis of a large number of recent works that called for a unified and systematic presentation of the topics it considered. But as van der Waerden himself indicated, algebraic knowledge had not only grown dramatically over the preceding decades. A fundamental change had also affected the very understanding of the discipline as a whole. He thus wrote:

> The recent expansion of algebra far beyond its former bounds is mainly due to the "abstract", "formal", or "axiomatic" school. This school has created a number of novel concepts, revealed hitherto unknown inter-relations and led to far-reaching results, especially in the theories of *fields* and *ideals*, of *groups* and of *hypercomplex numbers*. The chief purpose of this book is to introduce the reader into this whole world of concepts.[27, p. 9] (Italics in the original).

But in his book, van der Waerden did much more than just introducing the reader into a new world of concepts and innovative techniques. His presentation involved an original insight of far-reaching consequences, namely, the realization that a certain family of abstract mathematical notions (groups, rings, fields, etc.), defined via sets of formal axioms, should be best seen as comprising various instances of one and the same underlying idea, namely, the general idea of an algebraic structure. Under the new approach pursued in the book, the aim of algebraic research would become now the in-depth elucidation of the individual kinds of structures, based on the recurrent use of several common fundamental concepts, questions and techniques (e.g., isomorphisms, homomorphisms, quotients, residue classes, composition series and direct products, etc.), and the search after similar kind of mathematical results concerning each of them. Strange as it may sound nowadays, this fundamental insight had not been definitely achieved, let alone in a textbook, before van der Waerden's.

It is important to emphasize that nowhere in the book did van der Waerden state what is an algebraic structure, either at the general, non-formal level or by means of the introduction of some rigorously defined mathematical concept. Rather, he just worked out in detail, chapter after chapter, the basic concepts and properties relevant to each of the domains he included under the general notion of structure. Neither did he specify a list of main tools to be repeatedly used in the investigation of the individual structures. Rather, he just put to work these tools under a single methodological perspective, thus yielding a unique and innovative view of what algebra is all about.

One fundamental innovation implied by van der Waerden's approach was a redefinition of the conceptual hierarchy underlying the discipline of algebra. The various system of numbers were not considered here, as was the case with previous textbooks, as the underlying foundation over which the entire edifice of algebra, including the properties of polynomials, was to be erected. Rather, it was the other way round. The real or the rational numbers were now conceived as particular cases of abstract algebraic constructs. The concept of a field of fractions could be defined, for instance, for integral domains in general, and the rational numbers were thus realized as a particular case of this general kind of construction. Real numbers were defined in purely algebraic terms as a "real field". Additional, non-algebraic properties such as continuity or density could simply be ignored as part of the characterization of the real numbers in the newly defined algebraic context.

The task of finding the real and complex roots of an algebraic equation and of understanding their mutual interrelations, which had been the hard core of algebra over the previous century, was relegated in van der Waerden's book for the first time to a subsidiary role. Three short sections in his chapter on Galois theory dealt with this specific application of the theory, and they did so without assuming any previous knowledge of the properties of real numbers. The new conceptual hierarchical underlying this structural view of algebra was illuminatingly visualized in a diagram (*Leitfaden*) appended to the Table of Contents, that indicated the logical interdependence of the chapters (Figure 1).

Van der Waerden had become acquainted with this world as a student of Noether in Göttingen and of Emil Artin (1898–1962) in Hamburg. A considerable part of the contents of the book was directly taken from their lectures. The main influence of Artin came through his innovative and strikingly structural approach to Galois theory and his work on "real fields". His influence on van der Waerden cannot be underplayed. But clearly the decisive one came from Noether.[28] Indeed, van der Waerden was but one of the many brilliant students that studied with Noether at Göttingen and that helped put forward her ideas and understanding their actual impact and scope.

8. Dedekind's Ideals and Dedekind's Cuts

In this final section, I would like to return to Dedekind and to asses the significance of his work on ideals from a broader perspective concerning his work in general. The perspective afforded by the image of algebra as a discipline of structures, as embodied in van der Waerden's book, may mislead us when trying to assess the significance of some important developments in nineteenth century

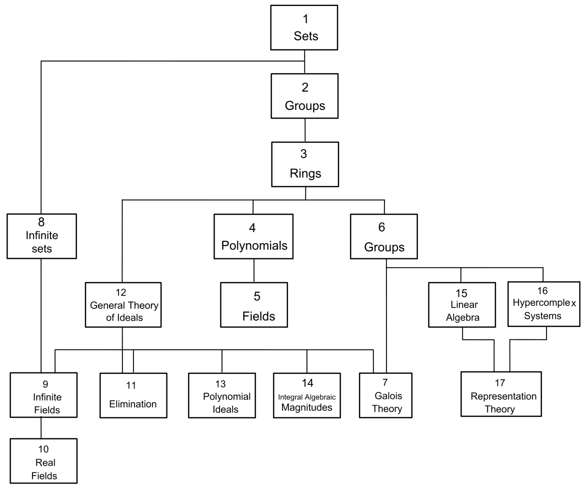

FIGURE 1. van der Waerden's *Leitfaden*

mathematics. Ideals and matrices, for instances, fall after 1930 under the common notion of an algebraic structure, but their historical developments followed separate paths whose eventual convergence could not have been envisaged at the time when they were evolving (see, e.g., [1]). Likewise, the history of number theory after Gauss can be traced along diverging paths of development. The path outlined in the previous sections, and which involved Dedekind, Kronecker and Hilbert, was accompanied by others, no less influential.[12] Likewise, Dedekind's work on ideals, even as it played a central role in the way that the discipline would eventually be conceived after Noether, is more adequately understood within the context of some other works of Dedekind himself.

The perspective that I want to suggest, as the proper one to understand of Dedekind's work on ideals in its proper historical context is the one that takes into account similar efforts that he devoted in his work on other systems of numbers. Dedekind came up with three different mathematical theories intended as laying the foundations of specific domains by means of a general conceptual tool specifically conceived for each of them. These were (1) "ideals" as the tool to analyze the question of factorization in domains of algebraic numbers, (2) "cuts" as the tool to analyze the question of continuity in the domain of real numbers, and (3) "chains" as the tool to analyze the question of induction in the domain of natural numbers. To a large extent we can also see Dedekind's approach to Galois theory in the same vein, with "groups" appearing as the conceptual tool with the help of which one could analyze the question of solvability by radicals in the theory of polynomials.

Dedekind's theory of "chains" as a way to discuss the foundations of the arithmetic of natural numbers appeared in a booklet first published in 1893 under the title of *Was sind und was sollen die Zahlen*, or, roughly, *What are numbers and what should they be?*.[9] For lack of space I will not go into any detail about this work (see [6, pp. 257–259]). I want to devote here some more detailed comments to Dedekind's theory of cuts as a way to understand the broader context of his theory of ideals, in a unified underlying methodology. Dedekind published his theory of cuts in 1872, in a famous booklet entitled *Continuity and irrational Numbers*.[8]

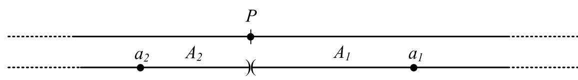

FIGURE 2. The straight line as a continuum

Dedekind sought to elucidate the riddle of continuity, and in particular the possibility to understanding in what sense, whereas both $\mathbb{Q}$ and $\mathbb{R}$ are dense, only the latter is taken to be continuous.

Dedekind started by considering the mathematical situation where the idea of continuity is straightforward. He focused on some simple, acknowledged properties which are evident in this situation, and then turned these *properties* into a *definition*. The natural candidate for doing this was the straight line, and the property on which he focused as the one that makes the line a "continuous" mathematical entity is both surprising and seemingly self-evident. Indeed, if we take any point P on the line, we see that P divides the line into two parts, A_1 to the right of P and A_2 to its left (see Figure 2). These two parts satisfy three simple geometric properties:

(1) A_1 and A_2 are disjoint;
(2) The union of A_1 and A_2 (adding the point P to either A_1 or A_2) yield the entire straight line;
(3) Any point a_2 belonging to A_2 is always to the left of any point a_1 belonging to A_1.

Dedekind simply transferred this situation into the arithmetic domain by deifning a "cut" in similar terms. Given any set of numbers $\boldsymbol{S}$ endowed with some relation of order "$<$", he defined a "cut" as a pair (A_1, A_2), of subsets of $\boldsymbol{S}$, such that the same three conditions hold:

(1) $A_1 \bigcap \mathrm{A}_2 = \phi$;
(2) $A_1 \bigcup \mathrm{A}_2 = \boldsymbol{S}$;
(3) If $a_2 \in A_2$ and if $a_1 \in A_1$ then $a_2 < a_1$.

Now, much as the principal ideals were a straightforward case, so also here, if we take $\mathbb{Q}$ as an already defined system of numbers, we can think of the straightforward example of a cut (A_1, A_2) on $\mathbb{Q}$ generated by a rational number (e.g., 2):

$$A_1 = \{x\epsilon\mathbb{Q} \ / \ x > 2\}; \ A_2 = \{x\epsilon\mathbb{Q} \ / \ x \leq 2\}.$$

This in itself does not take us too far away. But much the same as the strength of the theory of ideals lay in the fact that besides the principal ideals there are also ideals that are not principal, so also here we have cuts of $\mathbb{Q}$, which are not generated in the same straightforward manner as (A_1, A_2) above. Thus, for instance the cut (B_1, B_2) defined as follows:

$$B_1 = \{x\epsilon\mathbb{Q} \ / \ x > 0 \ \& \ x^2 > 2\}; \ B_2 = \{x\epsilon\mathbb{Q} \ / \ (x > 0 \ \& \ x^2 \leq 2) \text{ OR } x \leq 0\}.$$

This is the cut that allows constructing, out of the existing system $\mathbb{Q}$, a new entity not previously found in $\mathbb{Q}$, and which we can call $\sqrt{2}$.

Dedekind went on to define $\mathbb{R}$ as the collection of all cuts that can be defined on $\mathbb{Q}$, and he also introduced in a natural way a full arithmetic of the cuts. In particular, he was able to point out the sense in which $\mathbb{R}$ is continuous and $\mathbb{Q}$ is not, namely, that the cuts defined in $\mathbb{R}$ do not add any new number not already found in the system.

A final important issue to consider, and that sheds additional light on the main stresses of the present account, is the approach followed by Dednekind in his treatment of Galois theory. In a series of lectures taught at Göttingen in 1856–57, he was among the first to attempt a systematic clarification of the theory. Following closely on Galois's original approach, Dedekind stressed above all the parallel relationship between the Galois group and its subgroups, on the one hand, and the field of rationals and its successive extensions by addition of roots, on the other hand. But while Dedekind saw in the interrelations between subfields of the system of complex numbers (and certainly not of abstract fields) as the main *subject matter* of this theory, groups, on the contrary, afforded for him no more than a *tool*—a very effective and innovative tool, to be sure, but still a tool.[4, pp. 76–80] This remained his approach when he discussed the theory again in the introductory section to his 1894 version of the theory of ideals. In this sense, fields and groups were, in Dedekind's treatment of Galois theory, different kinds of mathematical entities, much in line with the classical conceptions that characterized nineteenth-century German algebra.

I would like to conclude by summarizing as follows: Emmy Noether's work on ideals and factorization, as part of an abstract theory of rings, represented the peek of a line of development that started with Dedekind. As fully deployed in van der Waerden's book of 1930, Noether's view of algebra implied a new overall conception of the discipline, where concepts such as groups, fields, modules, and rings are seen as individual instances of a more general notion, that of algebraic structures. This new conception implied also a fundamental change in the basic hierarchy of ideas, whereby the various domains of numbers and the realm of polynomials loose their conceptual priority as basic entities on which all of algebraic knowledge is based. Instead, they become objects of study as particular cases of the more general structures whose elucidation becomes now the main task of algebra. One cannot exaggerate the importance of Dedekind as the initiator of the genealogy of ideas that led to Noether's innovations. And yet ... not all of what pertains to her work is already found in the work of Dedekind ... Dedekind had indeed introduced many of the basic ideas that will lead to the new conception of algebra, with important consequences for mathematics at large. But Dedekind himself, as well as Hilbert after him, remained within the core conceptions that characterized classical, late nineteenth-century mathematics, and did not modify the basic conceptual hierarchy underlying it. Fields and algebraic integers were seen as the basic entities and the study of their properties was the subject-matter of "higher arithmetic". Likewise the study of polynomial forms, their invariants, and the question of theirr solvability was the subject-matter of "algebra". Ideals and groups were in this conception a different kind of beasts: innovative, powerful tools, that allowed for a deeper understanding of what for mathematicians like Dedekind or Hilbert remained the fundamental entities of mathematics.

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