Production to Order and Off-Line Inspection when the Production Process is Partially Observable

Abraham Grosfeld-Nir
Academic College of Tel-Aviv-Yaffo, 29 Melchet, Tel-Aviv 61560, Israel, agn@mta.ac.il; and Faculty of Management, Tel-Aviv University, Tel-Aviv, 69978, Israel

Eyal Cohen
Department of Industrial Engineering, Tel-Aviv University, Tel-Aviv, 69978, Israel, coheneyl@hotmail.com

Yigal Gerchak
Department of Industrial Engineering, Tel-Aviv University, Tel-Aviv, 69978, Israel, ygerchak@eng.tau.ac.il

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ABSTRACT

This study combines inspection and lot-sizing decisions. The issue is whether to INSPECT another unit or PRODUCE a new lot. Demand need to be satisfied in full, by conforming units only. The production process may switch from a "good" state to a "bad" state, at constant rate. The proportion of conforming units in the good state is higher than in the bad state. The true state is unobservable and can only be inferred from the quality of units inspected. We thus update, after each inspection, the probability that the unit, next candidate for inspection, was produced in good state. That "good-state-probability" is the basis for our decision to INSPECT or PRODUCE.

We prove that the optimal policy has a simple form: INSPECT only if the good-state-probability exceeds a control limit. We provide a methodology to calculate the optimal lot size and the expected costs associated with INSPECT and PRODUCE. Surprisingly, we find that the control limit, as a function of the demand (and other problem parameters) is not necessarily monotone. Also, counter to intuition, it is possible that the optimal action is PRODUCE, after revealing a conforming unit.

Keywords: Production, Off-Line Inspection, MLPO, Binomial Yield, Rigid Demand, POMDP.
1. INTRODUCTION

A recurring choice one has to make in many situations is whether to continue trying to exploit existing material or abandon it and produce or procure a batch of a new material. The trade-off is between spending time and/or effort on the old material, which may turn out to be futile, and incurring the cost of processing new material. This work is concerned with a scenario of this type, involving the choice between continuing the inspection of an old lot and producing a new one, whose size has then to be selected.

We address a fairly general batch production system where an imperfect machine can deteriorate during the production of a batch. All units delivered to the customer need to be inspected, and their (perfect) inspection provides some indirect information on the state of the machine when it produced that particular unit. After inspecting a unit, the choice is between inspecting another (if available) and producing another batch. The order has to be delivered in full. The goal is to minimize the total production and inspection costs of fulfilling the order.

Production to order ("rigid demand") with random yields, where multiple lots may be needed, has been investigated since the 50's; for a recent review of these Multiple Lotsizing Production to Order problems, see Grosfeld-Nir and Gerchak (2004). Inspection issues were incorporated into such MLPO problem by Grosfeld-Nir, Gerchak and He (2000), but the assumption there was that the machine initially produces only good units and if it at some point gets "out of control" all subsequent units are unusable. That is a special case of the model analyzed here, which allows the machine to be imperfect before the change, and not completely imperfect after.

The production process can be in either a "good" (in-control) or a "bad" (out-of-control) state. The true state is not directly observable and can only be inferred from the quality of the output. States are probabilistically deteriorating with constant failure rate: if the process is in the good state while producing a unit (within a batch), there is a constant probability that it will deteriorate to the bad state while producing the next unit. Once the process enters the bad state it remains there until the whole batch is produced. Units produced in either state are classified as either conforming or defective. The difference between the two states is in that the probability of obtaining a conforming unit in the good state is higher than the probability to obtain a conforming unit in the bad state.
After the entire lot has been produced, inspection commences. We assume that units are inspected one at a time, in the order they were produced. Inspection is perfect, i.e., after a unit has been inspected it becomes clear with certainty whether the unit is conforming or defective. After inspecting a unit the manufacturer, if suspecting that the process has already moved to the bad state, may choose to scrap all remaining non-inspected units and produce a new lot. Defective units, as well as non-inspected units, are considered worthless. The objective is to minimize the expected total of production and inspection costs required to satisfy the demand.

The problem above can be separated into two inter-related problems: "the production problem" which deals with specifying the lot size to be produced; and "the inspection problem" which, given the qualities of all previously inspected units, must select between INSPECT (the next unit) and PRODUCE (a new lot). Similar to Anily and Grosfeld-Nir (2005) we formulate the inspection problem as a "Partially Observed Markov Decision Problem" (POMDP) with two "time parameters": the remaining demand and the number of non-inspected units. However, while Anily and Grosfeld-Nir considered a single production run with shortage cost, we assume rigid demand, a situation often necessitating multiple production runs.

As we will formulate the inspection problem it will become apparent that the choice between INSPECT and PRODUCE will depend upon the probability that the unit which is next candidate for inspection was produced in the good state. This "good-state probability" is periodically updated based on the quality of each inspected unit. We note that within the realm of structured POMDPs it is common that the optimal policy has simple control-limit structure, which in our setting would be: "inspect the next unit if and only if the good-state probability exceeds a control limit". However, since the POMDP discussed here is unusual in that it depends on two "time parameters" we cannot simply assume this property but must prove it directly.

1.2. RELATED LITERATURE

Literature related to our model can be found mostly in three areas: POMDP; Multiple Lotsizing Production to Order (MLPO); and Statistical Process Control (SPC).

The POMDP aspect is due to the underlying process, i.e., the unobservable states of the production process, which constitute a Markov chain (also referred to as the core process). Sondik (1971) and Bertsekas (1976) showed that a POMDP with a finite state core process is
equivalent to a Markov Decision Process (MDP) where the state space is a continuous probability distribution over the core process' states. One of the main structural results, when only two actions are available, is that the optimal policy has a single control limit (Albright 1979; Bertsekas 1976; Lovejoy 1987; and White 1979). As of now analytical methods to calculate the infinite horizon control limits are unknown. One exception is a method developed by Grosfeld-Nir (1996) for a two-state POMDP with uniformly distributed observations. For algorithms, solution techniques and bounds see Lovejoy (1991a and 1991b), White (1991), White and Scherer (1989) and Lauritzen and Nilsson (2001). For applications of POMDPs in a variety of areas, see Lane (1989), Monahan (1982) and references therein.

In addition to working with two time parameters, there is another distinction between our model and the POMDP literature. The traditional POMDPs focus on infinite horizon control limits while we are mostly interested in finite-horizon control limits: finite demand and finite number of units available for inspection.

MLPO models can be thought of as random yield models where one of the main issues is to determine production lot sizes (Yano and Lee 1995). The need for multiple production runs naturally arises when yield is random and demand is rigid. Grosfeld-Nir and Gerchak (2004) provide a review of such models. In considering rigid demand models, the two most relevant yield distributions are the binomial and the interrupted geometric (IG). For binomial yield, Beja (1977) proved that the optimal lot strictly increases in the demand, which also implies that the optimal lot is not smaller than the demand. General properties of the single stage problem with rigid demand can be found in Grosfeld-Nir and Gerchak (1996). IG yield was studied by Zhang and Guu (1998), and by Anily, Beja and Mendel (2002). For IG yield, it is known that the optimal lot never exceeds the demand and sometimes decreases in the demand; also, for large demand the optimal lot has an upper bound, independent of the demand. Only Grosfeld-Nir, Gerchak and He (2000) considered costly off-line inspection in conjunction with rigid demand.

We note that in our model the yield can be considered as binomial-IG. Anily, Grosfeld-Nir and Duanis (2005) label this yield as B-IG and explore several of its properties. In considering the production (lot-sizing) part of our problem, typically, for small demand levels, properties resemble what would be expected with the binomial distribution; while for large demand they resemble the IG distribution.

The fastest way to detect machine failures or out of control processes is on-line inspection. However, there are cases where on-line inspection is not feasible due to long inspection procedures and the need for special inspection equipment. In those cases off-line
inspection, which is the subject of this paper, is performed after completing the production of a lot, or at least the results do not start being known until the whole lot was completed.

Several articles were devoted to finding, using off-line inspection, the exact time when a process with constant failure rate became out-of-control. Hassin (1984) presented an optimal inspection procedure for a geometric process assuming that the last unit's quality is known. He, Gerchak and Grosfeld-Nir (1996) presented an optimal inspection procedure for the same process but without knowing the last unit's quality. Porteus (1986, 1990) introduced an interaction between process quality control and lot sizing. Raz, Herer and Grosfeld-Nir (2000) question popular offline inspection models where complete knowledge (the cost of rejecting a good unit or accepting a bad one is infinite) and zero defects (only the cost of accepting a bad unit is infinite) are assumed. They show that with realistic costs these models are not justified economically. Finkelstein et al. (2005) extended this work for the case where the production process can recover after a failure. They used dynamic programming in order to solve the problem.

In contrast to our model, mostly, these papers assume that all units produced while the process is in-control (out-of-control) are conforming (defective) and their main objective was to minimize the expected number of inspections required to detect the failure point. Several authors analyzed the same yield structure as in our model. Hald (1981) used Bayesian models to detect a process change point. Porteus (1997) used Bayes' rule to update the probability that the process entered the bad state. Calabrese (1995) used the POMDP methodology to solve, optimally, an SPC problem. Anily and Grosfeld-Nir (2005) used the same yield structure and analysis methodology for a problem with non-rigid demand.

1.3. SUMMARY OF RESULTS

We denote the demand by \( D \) and by \( K \) the number of not yet inspected units. Let \( x \) be the probability that the unit, which is the next candidate for inspection was produced in the good state; we refer to \( x \) as the good-state probability. We denote by \( R_{INS}^{INS}(x) \) the expected cost required to satisfy the demand \( D \), if there are \( K \) units available for inspection, the good-state probability is \( x \), the next action is INSPECT, and all future actions are optimal.

1) We prove that \( R_{INS}^{INS}(x) \) is piecewise linear decreasing concave (PLDC) in \( x \), and that there exists a unique control limit \( x^{*}_{D,K} \) such that the optimal action is INSPECT if and only if \( x \geq x^{*}_{D,K} \). We also show that while \( x^{*}_{D,K} \) decreases in \( K \) (not strictly) it is not necessarily monotone in \( D \).
2) Let \( \alpha \) denote the fixed setup cost and let \( \theta_0 \) (\( \theta_1 \)) denote the probability that a unit produced in the good (bad) state ends conforming. Counter to intuition we show that \( x^*_{D,K} \), as a function of \( \alpha \), \( \theta_0 \), or \( \theta_1 \), is not necessarily monotone.

3) Counter to intuition we show that the optimal action after detecting a conforming unit may be PRODUCE.

4) We provide a methodology to calculate the optimal lot and provide numerical results. Our numerical tests show that the optimal lot, as a function of \( D \), is not necessarily monotone, which is consistent with known results concerning the IG distribution. However, somewhat surprising, the optimal lot, as a function of \( r \) (1-\( r \) is the hazard rate), \( \theta_0 \) or \( \theta_1 \), is not necessarily monotone.

2. PRELIMINARIES

We denote by \( r \) the probability that the production process, which is in the good state when producing one unit, remains in this state when producing the next unit. That is, if \( Z \) is the state while producing one unit and \( \hat{Z} \) is the state while producing the next unit (within a lot), then

\[
P(\hat{Z} = \text{Good} \mid Z = \text{Good}) = r \quad \text{and} \quad P(\hat{Z} = \text{Good} \mid Z = \text{Bad}) = 0
\]  

We denote \( Y=0 \) (\( Y=1 \)) that a unit produced in state \( Z \) is conforming (defective). We assume that \( \hat{Z} \) is conditionally independent of \( Y \), given \( Z \). That is

\[
P(\hat{Z} = \text{Good} \mid Y = y, Z) = P(\hat{Z} = \text{Good} \mid Z)
\]

We denote \( \theta_0 \) (\( \theta_1 \)) as the probability that a unit produced in the good (bad) state ends conforming, i.e., \( \theta_0 \) and \( \theta_1 \) are defined by

\[
P(Y = 0 \mid Z = \text{Good}) = \theta_0 \quad \text{and} \quad P(Y = 0 \mid Z = \text{Bad}) = \theta_1
\]

Naturally we assume \( \theta_0 > \theta_1 \).

Suppose we inspected several units. We denote \( x \) as the probability that the unit which is the next candidate for inspection was produced in the good state; i.e., we define \( x \) as

\[
x = P(Z = \text{Good}),
\]

where \( Z \) is the state in which this unit was produced. We note that \( x \) is calculated given the quality of all previously inspected units. We refer to \( x \) as the "good-state probability". It will become apparent that the good-state probability is updated after each inspection and that it is the variable based on which decisions are made. (Some authors refer to \( x \) as "the information
state"). We assume that just before production begins the process is aligned to be in the good state. Thus, in considering the first unit, \(x=r\).

Let \(Y\) be the quality of the unit, which is next candidate for inspection (the unit produced in state \(Z\)). Note that
\[
P(Y=0) = \sum_{z=Good, Bad} P(Y=0|Z=z)P(Z=z)
\]
\[
= x \theta_0 + (1-x) \theta_1 \equiv p(x,0)
\]  
(3)
Thus, \(p(x,0)\) is the probability that a unit, with good-state probability \(x\), ends conforming. Similarly, \(p(x,1)=1-p(x,0)\) is the probability that this unit ends defective.

Finally, we define \(h(x,y)\) as
\[
h(x,y) = P(\hat{Z} = Good \mid Y = y)
\]
We interpret \(h(x,y)\) as the good-state probability associated with the unit produced in state \(\hat{Z}\); this probability is “updated” with the information that the observation \(Y=y\) has been obtained. The following properties of \(h(x,0)\) and \(h(x,1)\) can be found in Anily and Grosfeld-Nir (2005); also see Figure 1.

**LEMMA 1**
\[
h(x,0) = \frac{rx \theta_0}{x \theta_0 + (1-x) \theta_1} = \frac{rx \theta_0}{p(x,0)}, \quad \text{(4)}
\]
and
\[
h(x,1) = \frac{rx(1- \theta_0)}{x(1- \theta_0) + (1-x)(1- \theta_1)} = \frac{rx(1- \theta_0)}{p(x,1)}. \quad \text{(5)}
\]
**PROOF:** first note that
\[
h(x, y) = P(\hat{Z} = Good \mid Y = y)
\]
\[= P(\hat{Z} = Good \mid Y = y, Z = Good)P(Z = Good \mid Y = y) = rP(Z = Good \mid Y = y)
\]
Next, apply Bayes’ theorem, to obtain
\[
h(x, y) = \frac{rx P(Y = y \mid Z = Good)}{xP(Y = y \mid Z = Good) + (1-x)P(Y = y \mid Z = Bad)}
\]
**LEMMA 2**

Both \(h(x,0)\) and \(h(x,1)\) are strictly increasing with \(h(x,0)>h(x,1)\), \(0<x<1\), and \(h(x=1,0)=h(x=1,1)=r\). Also, \(h(x,0)\) is strictly concave while \(h(x,1)\) is strictly convex (Figure
1). In addition, suppose that \( r_0 > \theta_1 \), then \( x^0 = \frac{r_0 - \theta_1}{\theta_0 - \theta_1} \) is the nontrivial solution of \( h(x,0) = x \); the significance of \( x^0 \) will become apparent later.

We omit the proof.

![Diagram](image)

**FIGURE 1.** The functions \( h(x,0) \) and \( h(x,1) \).

2.1. OVERVIEW

Typically, the manufacturer needs to select between INSPECT (the next unit) and PRODUCE (scrap all remaining non-inspected units and produce a new lot). The available information is: the remaining demand = \( D \); the number of non-inspected units = \( K \), and the good-state probability \( x \). (The good-state probability summarizes the information concerning the quality of the already inspected units.)

It turns out that the problem can be decomposed into two inter-related problems, which we label "Production Problem", and "Inspection Problem". Briefly, the production problem is the problem of selecting the lot size to be manufactured whenever the action is PRODUCE; while the inspection problem concentrates on calculating the expected cost associated with the action INSPECT. The solution is recursive in both the demand and the number of units available for inspection, and depends upon the good-state probability.

More specifically, it is easy to solve the production problem for demand \( D=1 \). Because, then, all manufactured units will be inspected, one at a time, in the order they were produced, until a conforming unit is found. If all units are defective, another lot is manufactured. After solving the production problem for \( D=1 \), it is possible to solve the inspection problem (selection between INSPECT and PRODUCE) for \( D=1 \) and \( K=1 \) (only
one non-inspected unit). Then for D=1, and K=2, and so on. Thus the problem is solved for D=1 and any K.

Given the production and inspection problems have been solved for D=1 and any K it is possible to solve the production problem for D=2. Then solve the inspection problem for D=2 and any K (starting with K=1, then K=2 etc.). This leads to a solution of the production problem with D=3 and so on.

In Section 3 we formulate the inspection problem and show that a control limit policy is optimal. In Section 4 we solve the production problem and show an example where the action PRODUCE, after detecting a conforming unit, is optimal. In Section 5 we present numerical tests which reveal the dependency of the control limits and optimal lot on problem parameters. Section 6 provides concluding remarks and avenues for future research.

3. THE INSPECTION PROBLEM

We denote by \( V_D \) the minimal (optimal) expected cost that is required to satisfy a rigid demand D if the action is PRODUCE. Note that \( V_D \) is obtained if and only if an optimal lot, \( N_D \), is produced, and all subsequent actions are optimal. In Section 4 we explain how \( V_D \) and \( N_D \) are calculated.

We denote by \( R_{D,K}^{\text{INS}}(x) \) the expected cost required to satisfy the demand D, if there are K units available for inspection, the good-state probability is x, the next action is INSPECT, and all future actions are optimal. We denote by \( R_{D,K}^{\text{OPT}}(x) = \min\{R_{D,K}^{\text{INS}}(x), V_D\} \). The following is the mathematical statement of the inspection problem.

\[
R_{D,K}^{\text{INS}}(x) = \gamma + p(x,0)R_{D-1,K-1}^{\text{OPT}}[h(x,0)] + p(x,1)R_{D,K-1}^{\text{OPT}}[h(x,1)]
\]  

(6)

For \( D,K \geq 1 \), with the initial conditions \( R_{0,K}^{\text{OPT}}(x) = 0; \quad R_{D,0}^{\text{OPT}}(x) = V_D \).

We refer to \( R_{D,K}^{\text{INS}}(x) \) as the value function. Clearly, \( R_{D,K}^{\text{INS}}(x) \) is continuous in x. The following theorem specifies several important properties of the value function.

THEOREM 1

The value function, \( R_{D,K}^{\text{INS}}(x) \), \( D,K \geq 1, 0 \leq x \leq r \), is Piecewise Linear Decreasing Concave (PLDC).

We give here a short outline of the proof. For the complete proof, see the appendix.
(a) Referring to Equation 6, it is quite easy to verify that if $R^{OPT}_{D-1,K-1}[h(x,0)]$ is piecewise linear (PL) then also $p(x,0)R^{OPT}_{D-1,K-1}[h(x,0)]$ is PL. The same holds true for $p(x,1)R^{OPT}_{D,K-1}[h(x,1)]$. Thus, using induction, $R^{INS}_{D,K}(x)$ is PL.

(b) To show that the PL function, $R^{INS}_{D,K}(x)$, is concave in $x$ we prove that the intercepts of consecutive segments are increasing.

(c) To show that a concave function, $R^{INS}_{D,K}(x)$, $0 \leq x \leq 1$, strictly decreases in $x$ we prove that $R^{INS}_{D,K}(x)$ strictly decreases at $x=0$. (A concave function, $g(u)$, strictly decreases over $u \geq 0$, if and only if it strictly decreases at $u=0$.)

**THEOREM 2**

For each $D,K \geq 1$ there exists a unique control limit, $x^*_{D,K}$, such that the optimal action is INSPECT if and only if $x \geq x^*_{D,K}$. (We will sometimes write CLT(D,K) instead of $x^*_{D,K}$.)

**PROOF**

As $R^{INS}_{D,K}(x)$ strictly decreases in $x$, there exists at most one solution $x^*_{D,K} \in [0,r]$ to $R^{INS}_{D,K}(x) = V_D$, with $x^*_{D,K} = 0$ if there is no such solution. The situation is depicted in Figure 2.

![Figure 2](image-url)

**FIGURE 2.** A typical graph of the value function and the control limit.

It is quite easy to realize that the control limits decrease in $K$: simply observe that $R^{INS}_{D,K+1}(x) \leq R^{INS}_{D,K}(x)$ (the presence of an extra unit cannot hurt); recall that $x^*_{D,K}$ is the root.
of $R_{D,K}^{INS}(x) = V_D$ and that $x^{*}_{D,K+1}$ is the root of $R_{D,K+1}^{INS}(x) = V_D$; thus, $x^{*}_{D,K+1} \leq x^{*}_{D,K}$. On the other hand, numerical tests (Table 1) show that the control limits, as a function of $D$, are not necessarily monotone.

| $\alpha=40; \beta=1; \gamma=5; r=0.98; \theta_0=0.9; \theta_1=0.4; $ |
|---|---|---|---|---|---|---|---|---|---|
| $K$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $D$ | | | | | | | | | |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.4724 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0.5003 | 0.4177 | 0.0107 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0.4568 | 0.4037 | 0.3485 | 0.0575 | 0.0267 | 0 | 0 | 0 | 0 |
| 5 | 0.4445 | 0.3797 | 0.3367 | 0.2977 | 0.0775 | 0.0553 | 0.0402 | 0.0215 | 0 |
| 6 | 0.4501 | 0.3781 | 0.3269 | 0.2935 | 0.2648 | 0.0885 | 0.0699 | 0.0600 | 0.0507 |
| 7 | 0.4338 | 0.3710 | 0.3191 | 0.2819 | 0.2572 | 0.2363 | 0.0937 | 0.0749 | 0.0686 |

Table 1. Control Limits as a function of $D$ and $K$.

The fact that the control limits, as a function of $D$, are not necessarily monotone is counterintuitive: one would expect that if it is worthwhile to invest in producing a new lot when the demand is $D$, it would also be worthwhile to produce a new lot for higher demand. We note that this result is consistent with Grosfeld-Nir (1996) where the author constructs an example to show that for a "one time parameter" POMDP with uniform observations, finite horizon control limits, as a function of the time remaining, are not necessarily monotone. Also, in a recent study, Grosfeld-Nir (2005) shows that the same applies for a "one time parameter" POMDP with binomial observation.

4. THE PRODUCTION PROBLEM

We start by explaining how to calculate the optimal lot size and optimal expected cost when $D=1$, i.e., $N_1$ and $V_1$. After that, we will show how to calculate $N_D$ and $V_D$, for $D \geq 2$.

Some notation is required. We denote by $Y_1$, $Y_2$, ..., the quality of the first unit, second unit, etc. We write $Y_n=0$ ($Y_n=1$) to denote that the $n$-th unit produced is conforming (defective). We denote by $Q_n$ the probability that the first $n$ units are defective, i.e.,

$$Q_n = P(Y_1=Y_2=...=Y_n=1)$$

(7)

We write $h^2(r,1)=h[h(r,1),1]$ and in general $h^{k+1}(r,1)=h[h^k(r,1),1]$, $k \geq 1$ (with $h^1(r,1)=h(r,1)$). Note that $h(r,1)$ is the (conditional) probability that the second unit was produced in the good state given that $Y_1=1$; and, in general, $h^{n+1}(r,1)$ is the (conditional) probability that the $n$-th unit was produced in the good state given that $Y_1=...=Y_{n-1}=1$. (In
other words, \( h^{n-1}(r,1) \) is the good-state probability associated with the \( n \)-th unit, given that all previously inspected units are defective.)

**Lemma 3**

(a) \( p[h^{n-1}(r,1),1] = P(Y_n=1|Y_1=Y_2=\ldots=Y_{n-1}=1) = Q_n/Q_{n-1} \)

(b) \( Q_1 = p(r,1) \) and \( Q_n = p(r,1)p[h(r,1),1]p[h^2(r,1),1] \ldots p[h^{n-1}(r,1),1], n \geq 2. \)

(c) \( P(Y_1=Y_2=\ldots=Y_n=1, Y_{n+1}=0) = Q_n Q_{n+1}, n \geq 1. \)

**Proof**

(a) For the first equality note that, given that \( Y_1, \ldots, Y_{n-1} \) are defective, the good-state probability associated with the \( n \)-th unit is \( x=h^{n-1}(r,1) \). Recall that \( p(x,1) \) is the probability that a unit, with good-state probability =\( x \), ends defective. The second equality follows from (7).

(b) This part provides a simple (recursive) way to calculate \( Q_n \) and follows from (a); namely, that \( Q_n = Q_{n-1}p[h^{n-1}(r,1),1] \).

(c) Simply note that
\[
P(Y_1=\ldots=Y_n=1, Y_{n+1}=0) + P(Y_1=\ldots=Y_n=1, Y_{n+1}=1) = P(Y_1=\ldots=Y_n=1)
\]

4.1. THE PRODUCTION PROBLEM WITH \( D=1 \)

We denote by \( V_1(N) \) the expected cost if a lot of size \( N \) is produced whenever the demand is \( D=1 \) and the action is PRODUCE and, after that, units are being inspected, one at a time, in the order these units were produced, until a conforming unit is detected, or all units are exhausted. If all units are defective, again, a lot of size \( N \) is produced, and so on.

Note that, for \( D=1 \), inspecting until a conforming unit is found is optimal: it is futile to produce "extra units", i.e., units that would be scrapped if all previously inspected units turn defective, as such units will also be scrapped if a conforming unit is detected.

If the first unit produced is conforming the inspection cost is \( \gamma \); if the first unit is defective and the second is conforming the inspection cost is \( 2\gamma \); and so on. Thus, \( V_1(N) \) can be calculated via the following equation:

\[
V_1(N) = \alpha + \beta N + \gamma P(Y_1 = 0) + 2\gamma P(Y_1 = 1, Y_2 = 0) + \ldots +
+ N\gamma P(Y_1 = \ldots = Y_{N-1} = 1, Y_N = 0) + [N\gamma + V_1(N)]P(Y_1 = \ldots = Y_N = 1)
\]

Now use \( P(Y_1=0) = 1-Q_1 \) and Lemma 3(c) to obtain

\[
V_1(N) = \frac{\alpha + \beta N + \gamma[1 + \sum_{k=1}^{N} Q_k]}{1-Q_N}
\]
It is easy to find an upper bound, $N_1^{UB}$, for $N_1$. Thus, using (9) the optimal lot can be found via a search over $N$.

4.2. THE PRODUCTION PROBLEM WITH $D>1$

Assume that the problem has been solved for demand $d = 1, 2, \ldots, D-1$ and that $V_d$, as well as $R_{d,K}^{INS}(x)$, $d \leq D-1$, $K \geq 1$ are known. We will explain now how to calculate $N_D$ and $V_D$.

We define $V_D(N,S)$ as the expected cost if, after the action PRODUCE is selected, a lot $N$ is produced whenever the demand is $D$; and, after that, units are being inspected, one at a time, in the order they were produced. If inspection reveals that the first $S$ units are defective, the remaining units are scrapped and a new lot of size $N$ is produced, and so on. If, on the other hand, a conforming unit is found among the first $S$ units, an optimal policy for demand $D-1$ is followed throughout.

Similar to (8) we have

$$V_D(N,S) = \alpha + \beta N + [\gamma + R_{D-1,N-1}^{OPT}(h(r,0))]P(Y_1 = 0) +$$

$$+ [2\gamma + R_{D-1,N-2}^{OPT}(h[r(1),0])]P(Y_1 = 1, Y_2 = 0) + \ldots +$$

$$+ [S\gamma + R_{D-1,N-S}^{OPT}(h[r(S-1),0])]P(Y_1 = \ldots = Y_{S-1} = 1, Y_S = 0) +$$

$$+ [S\gamma + V_D(N,S)]P(Y_1 = \ldots = Y_S = 1)$$

Using (10) it is easy to calculate $V_D(N,S)$. Finally we define $V_D(N)$ and $V_D$ by

$$V_D(N) = \min_S \{V_D(N, S)\}$$

(11)

$$V_D = \min_N \{V_D(N)\}$$

(12)

We created a computer program which searches over $N=1, \ldots, N_D^{UB}$, where, $N_D^{UB}$ is an upper bound for $N_D$. For each $N$ we calculate $V_D(N,S)$ for $S=1, \ldots, N$. Thus, using (10)-(12) we find $V_D$ and $N_D$.

We denote by $S_D$ the optimal value of $S$ when the optimal lot, $N_D$, is used, i.e., $S_D$ is the value of $S$ minimizing $V_D(N_D,S)$. Note the interpretation of $S_D$: if, after producing an optimal lot, it is found that the first $S_D$ units are defective, the remaining units are scrapped and a new lot is produced.

Table 2 provides several values of $N_D$ and $S_D$ as a function of $D$. We note that $N_D$ is not necessarily monotone in $D$. This was expected as also for the simpler problem, without inspection (i.e., $\gamma=0$), the optimal lot is not necessarily monotone in the demand (see Grosfeld-Nir and Gerchak 1996 and Anily et.al 2002). Table 2 reveals that $S_D$ also is not necessarily monotone in $D$. 
\[ \beta=2, \; \gamma=3, \; r=0.98, \; \theta_0=0.9, \; \theta_1=0.4 \]

<table>
<thead>
<tr>
<th>( \alpha=1 )</th>
<th>( \alpha=8 )</th>
<th>( \alpha=10 )</th>
<th>( \alpha=12 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>( N_D )</td>
<td>( S_D )</td>
<td>( N_D )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
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<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>4</td>
<td>11</td>
</tr>
</tbody>
</table>

Table 2. \( N_D \) and \( S_D \) as a function of \( D \) for several values of \( \alpha \).

### 4.3. AN INTERESTING EXAMPLE

Intuition suggests that after detecting a conforming unit the manufacturer will prefer to inspect the next unit (if available) over producing a new lot. Surprisingly, we find that this is not always the case.

The first step toward understanding how this is possible is due to an observation made by Anily and Grosfeld-Nir (2005), who point out that after revealing a conforming unit, the good-state probability can decrease. Actually, it is easy to verify that if \( x>x^0 \) then, according to Lemma 2 and Figure 1, \( h(x,0)<x \). (Recall that \( x^0=\max\{0, (r_0-\theta_1)/(\theta_0-\theta_1)\} \).) Thus, after revealing a conforming unit, the good-state probability can decrease from above to below the control limit.

Consider a problem with the parameters of Table 3. Table 3 reports the optimal production lots, \( N_D, D\leq7; \) and the control limits \( x^{*}_{D,K}, K\leq3,D\leq7 \).

| \( \alpha=5; \; \beta=2; \; \gamma=20; \; r=0.8; \; \theta_0=0.98; \; \theta_1=0.784 \) |
|---|---|---|---|---|
| D | \( N_D \) | CLT(D,1) | CLT(D,2) | CLT(D,3) |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 0.1554 | 0 | 0 |
| 3 | 3 | 0.0494 | 0.0494 | 0 |
| 4 | 4 | 0 | 0 | 0 |
| 5 | 5 | 0 | 0 | 0 |
| 6 | 6 | 0 | 0 | 0 |
| 7 | 4 | 0.0012 | 0 | 0 |

Table 3. Optimal production lots and control limits.
Recall that the control limits decrease in $K$, and therefore $\text{CLT}(D,K)=0$ for $D \leq 7$ and $K > 3$.

Suppose now that $D=5$. Thus, according to Table 3, $N_{D=5}=5$. As before we denote by $Y_i$, $i=1,\ldots,5$ the quality of the $i$-th unit. In our example we consider a feasible scenario where $Y_1=0$ (conforming), $Y_2=1$ (defective), $Y_3=0$ and $Y_4=0$.

In Table 4 we calculate the good-state probability after each inspection and compare it to the control limits of Table 3. It turns out that after revealing that $Y_4=0$ the optimal action is PRODUCE.

<table>
<thead>
<tr>
<th>Before Inspection</th>
<th>$Y_1=0$</th>
<th>$Y_2=1$</th>
<th>$Y_3=0$</th>
<th>$Y_4=0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$K$</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\text{CLT}(D,K)$</td>
<td>0</td>
<td>0</td>
<td>0.0495</td>
<td>0.1554</td>
</tr>
<tr>
<td>Good-state probability</td>
<td>0.8000</td>
<td>0.6667~h(0.8000,0)</td>
<td>0.1250~h(0.6667,1)</td>
<td>0.1212~h(0.1250,0)</td>
</tr>
<tr>
<td>ACTION</td>
<td>INSPECT</td>
<td>INSPECT</td>
<td>INSPECT</td>
<td>INSPECT</td>
</tr>
</tbody>
</table>

Table 4. A scenario with optimality of PRODUCE after revealing a conforming unit.

5. NUMERICAL RESULTS

We performed many numerical tests to study the dependence of the optimal lots and control limits as a function of different problem parameters. Our numerical tests used a dedicated computer program. As the inspection and the production problems are inter-related, the way the program works is as follows:

1) Production: we calculate $N_1$ and $V_1$.

2) Inspection: we calculate $R^{\text{INS}}_{D=1,K}$, $K \geq 1$. (Naturally, we stopped at a value of $K$ thought to be large enough.)

3) Production: we calculate $N_2$ and $V_2$.

4) Inspection: we calculate $R^{\text{INS}}_{D=2,K}$, $K \geq 1$.

And so on. Mostly, we used variations of the following parameters, which we thought realistic: $r \approx 0.98$, $\theta_0 \approx 0.9$, $\theta_1 \approx 0.4$

The problem grows exponentially with the demand, thus we had to use some approximations. Our algorithm uses a grid of points (accuracy grows with the number of points used) in order to discretize the probability state - $x$. As the problem has linear characteristics, we used linear approximations to evaluate $R^{\text{INS}}$. 
Table 5. Processing Time for various settings (for a desktop PC with: 2.6 Ghz CPU, 512 MB Memory Capacity.)

Table 5 demonstrates some running times for typical parameters. The table shows how the problem grows exponentially with the demand and how the approximations reduce running times dramatically. Note that the accuracy compromised for the 10000 grid points for D=20 is around \(10^{-5}\).

5.1. DEPENDENCE OF THE OPTIMAL LOT ON PROBLEM PARAMETERS

Table 6 contains numerical results concerning the optimal lot \((N_D)\) for different values of the inspection cost \((\gamma)\), and the success probabilities \((\theta_0\) and \(\theta_1\)). These results indicate that \(N_D\) decreases (not strictly) in \(\gamma\). However, \(N_D\) is not necessarily monotone in \(\theta_0\) (\(\theta_1\)).
Table 7 reveals that the optimal lot as a function of the hazard rate \( r \) is also not necessarily monotone.

<table>
<thead>
<tr>
<th>Hazard rate ( r )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimal lot ( N_{D} )</td>
<td>60</td>
<td>60</td>
<td>59</td>
<td>58</td>
<td>57</td>
<td>56</td>
<td>53</td>
<td>13</td>
<td>22</td>
</tr>
</tbody>
</table>

Table 7. The optimal lot as a function of \( r \). (The other parameters used were: \( \alpha=40; \beta=1; \gamma=10; \theta_0=0.9; \theta_1=0.4; \) and \( D=20 \)).

5.2. DEPENDENCE OF THE CONTROL LIMITS ON PROBLEM PARAMETERS

Figure 3 demonstrates, again, that the control limits, as a function of \( D \), are not necessarily monotone. Figure 3 also affirms that the control limits decrease (not strictly) in \( K \).

In considering the feasible set of problem parameters one is tempted to use parameters so that \( x \approx 0 \) will imply that the next optimal action would be PRODUCE. However, we found that such an assumption would unnecessarily limit application and eliminate some interesting cases. Figure 3 demonstrates possible odd relationship between the control limits and the demand. In particular it is intriguing to observe that near \( D=30 \) the control limits become zero (for all \( K \geq 1 \)).

**FIGURE 3.** The control limits as a function of \( D \) for several values of \( K \). (The other parameters used were: \( \alpha=10; \beta=1; \gamma=1.5; r=0.98; \theta_0=0.9; \theta_1=0.4 \))

-18-
EXAMPLE 1

To understand, intuitively, why it is possible that \( \text{CLT}(D,K)=0 \) and \( \text{CLT}(D+1,K)>0 \) consider the following. Suppose that \( r=1; \theta_0=1; \theta_1=0.4; \alpha=40; \beta=1. \) Thus, for small \( D, \) \( N_D\approx D \) and \( V_D\approx \alpha+D+\gamma D. \) Hence, for \( D=1, V_{D=1}=41+\gamma, \) and for \( K=1 \)

\[
R_{D=1,K=1}^{\text{INS}}(x=0) = \gamma + (1-\theta_1)V_{D=1} = \gamma + (1-0.4)(41+\gamma)
\]

Therefore, if \( \gamma<82/3 \) the optimal action is \text{INSPECT}. Note that if \text{INSPECT} is optimal for \( x=0, \) it is optimal for \( 0\leq x\leq 1. \) We conclude that \( \text{CLT}(D=1,K=1)=0. \)

Consider the same problem with \( D=2. \) Then, \( V_{D=2}=42+2\gamma, \) and

\[
R_{D=2,K=1}^{\text{INS}}(x=0) = \gamma + \theta_1V_{D=1} + (1-\theta_1)V_{D=2} = 2.6\gamma + 41.6
\]

Therefore, if \( \gamma>2/3 \) the optimal action is \text{PRODUCE.}

Thus, overall, for \( K=1 \) and \( 2/3<\gamma<82/3, \) the \( \text{CLT}=0 \) if \( D=1 \) and \( \text{CLT}>0 \) if \( D=2. \) The reason for that is that for \( D=1 \) one hopes to save an additional setup while for \( D=2 \) an additional setup is unavoidable.

Naturally, we expected the control limits to be monotone in \( \theta_0 \) and \( \theta_1, \) and in the setup cost. However, as it turns out (Tables 8-10) this is not the case.

<table>
<thead>
<tr>
<th>( \theta_0 )</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control Limit</td>
<td>0.0000</td>
<td>0.5382</td>
<td>0.6064</td>
<td>0.6078</td>
<td>0.5699</td>
<td>0.5142</td>
<td>0.4800</td>
</tr>
</tbody>
</table>

Table 8. Control limits as a function of \( \theta_0. \) (The parameters used were: \( \alpha=40; \beta=1; r=0.98; \theta_1=0.4; \gamma=50; K=11; \) and \( D=10. \))

<table>
<thead>
<tr>
<th>( \theta_1 )</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control Limit</td>
<td>0.5181</td>
<td>0.5219</td>
<td>0.5142</td>
<td>0.4878</td>
</tr>
</tbody>
</table>

Table 9. Control limits as a function of \( \theta_1. \) (The parameters used were: \( \alpha=40; \beta=1; r=0.98; \theta_0=0.9; \gamma=50; K=11; \) and \( D=10. \))

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control Limit</td>
<td>0.2680</td>
<td>0.1809</td>
<td>0.1197</td>
<td>0.0983</td>
<td>0.0516</td>
<td>0.0520</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table 10. Control limits as a function of the setup cost (\( \alpha \)). (The parameters used were: \( \beta=1; r=0.98; \theta_0=0.9, \theta_1=0.4; \gamma=1.5; K=1; \) and \( D=8. \))
6. CONCLUDING REMARKS

We created a computer program to calculate the optimal lot, the control limits and various relevant costs. As, for now, there are no similar results to compare with, we invested a considerable effort to validate our numerical results. One way of testing our algorithm was to use \( r=1 \), which leads to a binomial problem with inspection; the same happens in using \( \theta_0=\theta_1 \). Thus, the results could be compared to Grosfeld-Nir, Gerchak and He (2000). Also, using \( \theta_0=1, \theta_1=0 \) and \( \gamma=0 \), the problem becomes the regular interrupted-geometric problem and the optimal lot could be compared to Anily et. al. (2002), and to Zhang and Guu (1998). Other methods of testing the algorithm focused on \( D=1 \) where more explicit methods can be developed.

Another way of checking our results was by examining the consistency with results obtained by Grosfeld-Nir (2005) who studied a "one time parameter" POMDP with binomial observations. This consistency also leads us to believe that properties of POMDP's with "two time parameters" are similar to those of traditional POMDP's.

As there is a growing interest in using POMDP techniques in different areas, including SPC and bioinformatics (Abbas and Holmes 2004, Beal et al. 2005), the investigation of elementary properties of such models becomes very important. Our paper reveals that researchers should be very careful about assuming "intuitive properties".

We feel that the most surprising result is that acting optimally it is possible that PRODUCE would be optimal after detecting a conforming unit. In fact, one of the co-authors lost a bet conjecturing this is impossible. The detection of various non-monotonic properties is also important for researchers applying algorithms to trust their results. (Many years ago, a co-author witnessed the abandonment of a project because of "unreasonable" numerical jumps in the control limits.)

Several directions for future research come to mind. We will temporarily label three of them as Imperfect Inspection; Unlimited Demand; and Three Actions.

Imperfect Inspection: Suppose there is a possibility of inspection error, where there is a chance that a unit classified as conforming is defective. The defective unit will be revealed later because of failing after a short time. The formulation could proceed by still using rigid demand for supplying a required amount of tested units and possibly incorporate shortage cost to prevent the shipment of an unreasonable number of defectives.

Unlimited Demand: Suppose the market demand for products exceeds the manufacturers' production capacity. Recall that, because of machine deterioration, optimally
only a limited lot size should be used. The problem becomes a non-trivial stationary production problem where one searches for the optimal lot and inspection procedure.

Three Actions: Suppose that at any instance the manufacturer can choose between: INSPECT; PRODUCE; and STOP: deliver the conforming units at hand and if their number is short of the demand pay a shortage cost. This model will combine our present model and the one developed by Anily and Grosfeld-Nir (2005).

We like to conclude saying that the computer program we developed is user friendly: one could use it simply by inserting the value of problem parameters. We will be happy to share it with interested researchers.
REFERENCES


APPENDIX

THEOREM 1

The value function, \( R^{\text{INS}}_{D,K}(x) \), is Piecewise Linear Decreasing Concave (PLDC).

The proof requires some preparations. Let

\[
\begin{align*}
    f_{D,K}(x) &\equiv p(x,0)R_{D^{-1},K^{-1}}^{\text{OPT}}(h(x,0)) \quad (A1) \\
    g_{D,K}(x) &\equiv p(x,1)R_{D,K}^{\text{OPT}}(h(x,1)) \quad (A2)
\end{align*}
\]

Therefore, (6) becomes

\[
R^{\text{INS}}_{D,K}(x) = \gamma + p(x,0)R_{D^{-1},K^{-1}}^{\text{OPT}}(h(x,0)) + p(x,1)R_{D,K}^{\text{OPT}}(h(x,1))
= \gamma + f_{D,K}(x) + g_{D,K}(x)
\quad (A3)
\]

LEMMA A1

(a1) Suppose that \( R_{D^{-1},K^{-1}}^{\text{OPT}}(x) \) is linear over an interval \([\delta_1,\delta_2]\), i.e.,\( R_{D^{-1},K^{-1}}^{\text{OPT}}(x) = A_0x + B_0, \quad \delta_1 \leq x \leq \delta_2 \); then \( f_{D,K}(x) \) is linear over \( x \) such that \( \delta_1 \leq h(x,0) \leq \delta_2 \).

(a2) Suppose that \( R_{D^{-1},K^{-1}}^{\text{OPT}}(x) \) is piecewise linear (PL) then \( f_{D,K}(x) \) is PL.

(b1) Suppose that \( R_{D,K}^{\text{OPT}}(x) \) is linear over an interval \([\omega_1,\omega_2]\), i.e., \( R_{D,K}^{\text{OPT}}(x) = A_1x + B_1, \quad \omega_1 \leq x \leq \omega_2 \); then \( g_{D,K}(x) \) is linear over \( x \) such that \( \omega_1 \leq h(x,0) \leq \omega_2 \).

(b2) Suppose that \( R_{D,K}^{\text{OPT}}(x) \) is piecewise linear (PL) then \( g_{D,K}(x) \) is PL.

PROOF

(a1) For \( h(x,0) \in [\delta_1,\delta_2] \), we have (note that \( p(x,0)h(x,0) = rx\theta_0 \)):

\[
    f_{D,K}(x) = p(x,0)R_{D^{-1},K^{-1}}^{\text{OPT}}(h(x,0)) = p(x,0)[A_0h(x,0) + B_0] = x[A_0r\theta_0 + B_0(\theta_0 - \theta_1)] + B_0\theta_1
\]

(a2) This follows from the continuity of \( f_{D,K}(x) \) and since by (a1) this function transform linear segments of \( R_{D^{-1},K^{-1}}^{\text{OPT}}(x) \) into linear segments of \( f_{D,K}(x) \).

(b1) For \( h(x,1) \in [\omega_1,\omega_2] \), we have (note that \( p(x,1)h(x,1) = rx(1-\theta_0) \)):

\[
    g_{D,K}(x) = p(x,1)R_{D,K}^{\text{OPT}}(h(x,1)) = p(x,1)[A_1h(x,1) + B_1] = x[A_1r(1-\theta_0) - B_1(\theta_0 - \theta_1)] + B_1(1-\theta_1)
\]

(b2) Similar to (a1).
**LEMMA A2**

\[ R_{D,K=1}^{\text{INS}}(x), \, 0 \leq x \leq 1, \, D \geq 1 \text{ is linear and strictly decreasing}. \]

**PROOF**

From (6) we have (Note that \( V_D > V_{D-1} \)):

\[
\begin{align*}
R_{D,K=1}^{\text{INS}}(x) &= \gamma + p(x,0)R_{D-1,K=0}^{\text{OPT}}(h(x,0)) + p(x,1)R_{D,K=0}^{\text{OPT}}(h(x,1)) \\
&= \gamma + p(x,0)V_{D-1} + p(x,1)V_D \\
&= x(\theta_0 - \theta_1)(V_{D-1} - V_D) + \gamma + \theta_1(V_{D-1} - V_D) + V_D
\end{align*}
\]

**COROLLARY A1**

The value function, \( R_{D,K}^{\text{INS}}(x), \, 0 \leq x \leq 1, \, D, K \geq 1 \) is PL.

**PROOF**

We will use induction over \( K \). By Lemma A2 \( R_{D,K=1}^{\text{INS}}(x) \) is linear. Note that hence \( R_{D,K=1}^{\text{OPT}}(x) = \min\{R_{D,K=1}^{\text{INS}}(x), V_D\} \) is PL.

Assume that \( R_{D,k}^{\text{INS}}(x), \, D \geq 1 \) is PL for \( k=1, \ldots, K-1 \); hence also \( R_{D,k}^{\text{OPT}}(x), \, k=1, \ldots, K-1 \) is PL. Note that by Lemma A1 \( f_{D,K}(x) \) is PL. Similarly, \( g_{D,K}(x) \) is PL; to conclude, by (a3), that also \( R_{D,K}^{\text{INS}}(x) \) is PL.

**PROPOSITION A1**

The value function, \( R_{D,K}^{\text{INS}}(x), \, 0 \leq x \leq 1, \, D, K \geq 1 \) is concave.

**PROOF**

We will use induction over \( K \). By Lemma A2 \( R_{D,K=1}^{\text{INS}}(x) \) is linear and strictly decreasing in \( x \); hence, \( R_{D,K=1}^{\text{OPT}}(x) = \min\{R_{D,K=1}^{\text{INS}}(x), V_D\} \) is concave.

Assume that \( R_{D,k}^{\text{INS}}(x), \, D \geq 1 \) is concave for \( k=1, \ldots, K-1 \); hence also \( R_{D,k}^{\text{OPT}}(x), \, k=1, \ldots, K-1 \) is concave. Note that to prove that a continuous PL function is concave it is sufficient to show that consecutive intercepts are increasing.

Let \( C_1, C_2, \ldots, \) be the intercepts of the first, second, \ldots segments of \( R_{D-1,K-1}^{\text{INS}}(x) \). Since by hypothesis \( R_{D-1,K-1}^{\text{INS}}(x) \) is concave, we have \( C_1 < C_2 < \ldots \)

Note that, from the proof of Lemma A1, the intercepts of \( f_{D,K}(x) \) are \( \theta_1 C_1, \theta_1 C_2, \ldots \). This proves that \( f_{D,K}(x) \) is concave.
Similarly, \( g_{D,K}(x) \) is concave, to conclude that \( R_{D,K}^{\text{INS}}(x) \) is concave.

To prove Theorem 1 it remains to show that \( R_{D,K}^{\text{INS}}(x), 0 \leq x \leq 1, D,K \geq 1 \) strictly decreases in \( x \). Note that the concave function \( R_{D,K}^{\text{INS}}(x) \) is strictly decreasing in \( x \) if it is decreasing at \( x=0 \).

**PROPOSITION A2**

The value function \( R_{D,K}^{\text{INS}}(x), 0 \leq x \leq 1, D,K \geq 1 \), strictly decreases at \( x=0 \).

**PROOF**

We will use induction over \( K \). By Lemma A2 \( R_{D,K=1}^{\text{INS}}(x), D \geq 1 \) is linear and strictly decreasing at \( x=0 \). Note that hence also \( R_{D,K=1}^{\text{OPT}}(x) = \min\{R_{D,K=1}^{\text{INS}}(x), V_D\} \) decreases (not strictly) at \( x=0 \).

Assume that \( R_{D,k}^{\text{INS}}(x), D \geq 1 \) strictly decreasing at \( x=0 \) for \( k=1,\ldots,K-1 \). Note that hence also \( R_{D,k}^{\text{OPT}}(x), k=1,\ldots,K-1 \) decreasing at \( x=0 \) (not strictly).

We will show that \( R_{D,K}^{\text{INS}}(x) < R_{D,K}^{\text{INS}}(0), 0 < x \leq 1 \). To conclude, using \( x \) belonging to the first segment, that \( R_{D,K}^{\text{INS}}(x) \) strictly decreases at \( x=0 \).

\[
R_{D,K}^{\text{INS}}(x) = \gamma + p(x,0)R_{D-1,K-1}^{\text{OPT}}(h(x,0)) + p(x,1)R_{D,K-1}^{\text{OPT}}(h(x,1)) \\
\leq \gamma + p(x,0)R_{D-1,K-1}^{\text{OPT}}(0) + p(x,1)R_{D,K-1}^{\text{OPT}}(0) \\
< \gamma + p(0,0)R_{D-1,K-1}^{\text{OPT}}(0) + p(0,1)R_{D,K-1}^{\text{OPT}}(0) = R_{D,K}^{\text{INS}}(0)
\]

The first inequality follows since, by hypothesis, \( R_{D,K-1}^{\text{OPT}}(t) \) (and \( R_{D-1,K-1}^{\text{OPT}}(t) \)) decreases in \( t \).

For the second inequality, note that

\[
p(x,0)R_{D-1,K-1}^{\text{OPT}}(0) + p(x,1)R_{D,K-1}^{\text{OPT}}(0) \\
= p(x,0)R_{D-1,K-1}^{\text{OPT}}(0) + p(x,1)R_{D,K-1}^{\text{OPT}}(0) + p(x,1)(R_{D,K-1}^{\text{OPT}}(0) - R_{D-1,K-1}^{\text{OPT}}(0)) \quad \text{(A4)}
\]

note that \( R_{D,K-1}^{\text{OPT}}(0) - R_{D-1,K-1}^{\text{OPT}}(0) \) is strictly positive and thus (A4) strictly decreases in \( x \).