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Optimal Location of a Single Facility with Circular Demand Areas

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Abstract—The single facility minimum location problem in Euclidean space has usually been studied with a certain number of discrete demand points. Some authors have also described the possibility of demand areas. In the present work, a new approach is offered to the optimal location of a single facility, which should serve a number of circular demand areas, each with uniform demand density, along with some discrete demand points. The effect of a circular demand area on the service facility at each stage of the Weiszfeld-like iterative procedure is evaluated for the three possible cases of the incumbent service point being outside, inside, or on the circumference of such a circle. Some limiting cases are considered, such as that of the demand area being very far from the service point to be optimally located. The amended Weiszfeld iterative procedure is described, and some numerical experience of solving such problems is reported.

Keywords—Single facility, Minimum location problem, Euclidean space, Circular demand area.

1. INTRODUCTION

The classical Weber [1] problem of optimally locating a service point in an area with m demand points is

$$\min_{x,y} f(x,y) = \sum_{i=1}^{m} w_i r_i(x,y), \tag{1}$$

where w_i is the weight of the i^{th} demand point located at (x_i, y_i), $r_i(x,y) = [(x-x_i)^2+(y-y_i)^2]^{1/2}$, the Euclidean distance between the i^{th} point and the point to be located (x, y). The Weber problem has been shown to be convex [2], and therefore, using an iterative procedure that brings us to a local minimum, yields in fact the global solution.

Let us consider the partial first derivatives of $f(x, y)$ in equation (1) with respect to x and y,

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{m} \frac{w_i(x-x_i)}{r_i}, \tag{2}$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{m} \frac{w_i(y-y_i)}{r_i}. \tag{3}$$

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It is obvious that a necessary condition for a point (x, y) to be a solution of the problem is that the sums in equations (2) and (3) are equal to zero. Due to the convexity, if such a point is found, it is a global solution. Thus, one looks for a point for which $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. Setting the expressions in (2) and (3) to zero yields

$$x = \frac{\sum_{i=1}^{m} [w_i x_i / r_i(x, y)]}{\sum_{i=1}^{m} [w_i / r_i(x, y)]}, \tag{4}$$

$$y = \frac{\sum_{i=1}^{m} [w_i y_i / r_i(x, y)]}{\sum_{i=1}^{m} [w_i / r_i(x, y)]}. \tag{5}$$

Of course, these equations cannot be solved in closed form. Weiszfeld [3] suggested an iterative procedure based on equations (4),(5). He wrote the N^{th} step in the iteration as

$$x^{N+1} = \frac{\sum_{i=1}^{m} \left[w_i x_i / r_i \left(x^N, y^N\right)\right]}{\sum_{i=1}^{m} [w_i / r_i(x^N, y^N)]}, \tag{6}$$

$$y^{N+1} = \frac{\sum_{i=1}^{m} \left[w_i y_i / r_i \left(x^N, y^N\right)\right]}{\sum_{i=1}^{m} [w_i / r_i(x^N, y^N)]}. \tag{7}$$

As a first guess for the solution, the "center of gravity" point is usually taken; i.e.,

$$x^0 = \frac{\sum_{i=1}^{m} w_i x_i}{\sum_{i=1}^{m} w_i}, \qquad y^0 = \frac{\sum_{i=1}^{m} w_i y_i}{\sum_{i=1}^{m} w_i}. \tag{8}$$

It has been pointed put [4] that Weiszfeld's method is, in fact, the "steepest descent" method with a stepsize determined by the denominator in equations (6) and (7). In order to see this, let us consider [5] the step determined by two consecutive points in the iteration,

$$x^{N+1} - x^N = \frac{\sum_{i=1}^{m} \left[w_i \left(x_i - x^N\right) / r_i \left(x^N, y^N\right)\right]}{\left[\sum_{i=1}^{m} w_i / r_i \left(x^N, y^N\right)\right]}, \tag{9}$$

$$y^{N+1} - y^N = \frac{\sum_{i=1}^{m} \left[w_i \left(y_i - y^N\right) / r_i \left(x^N, y^N\right)\right]}{\left[\sum_{i=1}^{m} w_i / r_i \left(x^N, y^N\right)\right]}. \tag{10}$$

Comparing this to equations (2) and (3), it is obvious that the process consists of going "downhill" along $-grad\ f$. It has been pointed out [4,5] that, in fact, the stepsize given in equations (9) and (10) is not necessarily the best, and that excluding cases in which the solution point coincides with a demand point, doubling the stepsize decreases the number of steps required to meet a given termination criterion, on the average by a factor of 2.

Equations (9) and (10) suggest the mechanical analogy in which $f(x, y)$ in equation (1) is looked upon as a scalar field [6] in analogy with the potential in a gravitational or an electric

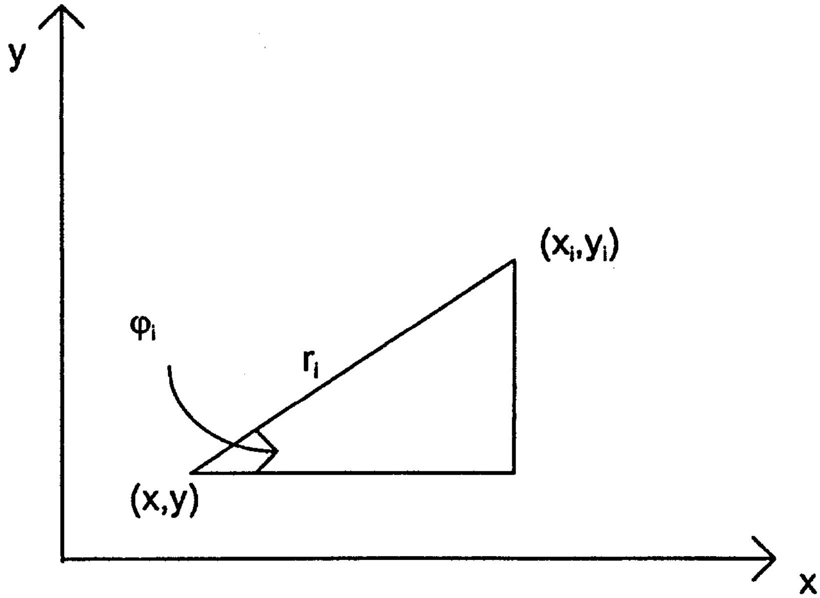

Figure 1. The i^{th} demand point as seen from a point (x, y). r_i is the Euclidean distance and φ_i is the angle between the horizontal and the line connecting the two points.

field. The partial first derivatives in equations (2) and (3) can be considered to be the x and y components of the resultant force acting on the service point which is to be located optimally.

Let us consider now the contributions of the i^{th} demand point to the expressions in equations (2) and (3). If we denote by φ_i the angle from the horizontal at which an observer at (x, y) views (x_i, y_i), the contributions of the i^{th} point are $w_i \cos \varphi_i$ and $w_i \sin \varphi_i$ to the x and y components of the gradient (see Figure 1). From the "physical" analogous point of view, the problem can be considered as follows. A "particle" is placed in a force field having m attracting points distributed in the two-dimensional space. The particle is attracted to each of the given points, the attraction to the i^{th} one being proportional to the weight of the point w_i. The direction of the force is along the line connecting the particle and the i^{th} point. The strength of this force element is *not* dependent on the distance between the particle and the demand point. The problem is that of finding a stationary point for the particle in this field. As opposed to the case of gravitational and electrostatic fields (in which cases the force is inversely proportional to the distance) where there is no such stationary point, a minimum point always exists [7] under the present circumstances. A mechanical analogous device was suggested by Pick as early as 1909 in an introduction to Weber's book [1] (see also [5]).

A number of authors considered different versions of the Weber problem with demand areas rather than discrete demand points. Love [8] discusses the problem of optimally locating a service facility among a number of population areas with Euclidean distances. The concrete example given is that of rectangular demand areas. Bennett and Mirakhor [9] point out some difficulties in Love's work. These include the complex expressions dealt with which result in a time consuming numerical procedure, the assumption of rectangular shapes, and the assertion of uniform densities. Instead they suggest looking at the centroid of the population region as a single point where the whole weight of the area in question is concentrated. Of course, finding the centroid may be rather difficult, depending on the shapes and weight distributions of the demand areas. Moreover, the centroid as calculated by Bennett and Mirakhor, namely determining $\bar{x}$, $\bar{y}$ as the first moment of the area about the appropriate axis divided by the area of the region, seems to be the appropriate choice for the squared-Euclidean distance problem rather than the Euclidean one.

Hillman and Rhoda [10] discuss the errors made in evaluating the effective distances while replacing a demand area by a point. Drezner [11] discussed the more general problem in which both the demand locations and the facilities to be located have circular shapes. He showed that the difference for the effective distance in the squared-Euclidean and Euclidean cases is not large, and gave approximate expressions for the latter. An extension to this work has been given by Carrizosa *et al.* [12].

In this work, we present a new approach to the problem of optimal location of a service point with respect to several circular demand areas. The concept of the pulling "force" of such an area is discussed, and the integration of the pulling force of such a circle is presented in a closed form, and with no approximations. In particular, the fact that in the Euclidean problem, the force of a demand point or area in a certain direction is independent of the distance is exploited. The pulling force of such a circle is presented as an expression depending only on a single parameter α, the ratio between the radius of the circle, and the distance from the point to be located to the center of the circle. Expressions are developed for the service point being outside, inside, and on the circumference of the circle. An explicit expression for the cost function under these circumstances is given. Also, a combination of demand circles and discrete demand points can be considered. Numerical examples of the solution of this problem with different demand circles are discussed.

2. THE "FORCE" DUE TO A CIRCULAR DEMAND AREA

2.1. The Attracting "Force" of a Uniform Weight Demand Circle on a Facility outside the Circle

Let us consider the pulling force of a full circle of radius R, with a constant density ρ on a service facility point located outside the circle, at a distance a from the center of the circle. Let us look at a direction determined by an angle φ from the line connecting the point P and the center O of the given circle as shown in Figure 2. Let us consider next an infinitesimal additional angle $d\varphi$, and let us calculate the attracting force of the part of the circle enclosed by the half-lines at the angles φ and $\varphi + d\varphi$. Since $d\varphi$ is infinitesimally small, we can consider all the enclosed area in the circle as being in the same direction φ. Since all the demand points in the same direction contribute in a similar way to the attracting force, as pointed out above, we can aggregate all the enclosed area as "pulling" in the same direction φ with a force proportional to the area in question. In order to do so, we have to calculate the lengths m and n, the distances between P and the close and far intersections, respectively, of the half-line with the circle. The line through P at an angle φ from the x-axis can be written as

$$y = -(a - x) \tan \varphi. \tag{11}$$

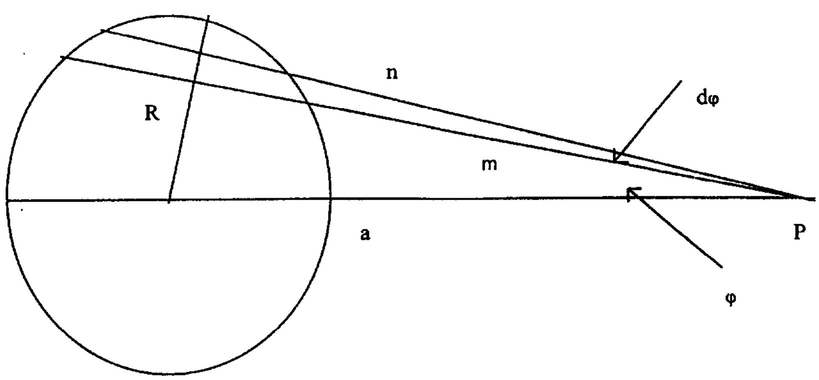

Figure 2. A circular demand area with a radius R as seen from a point P outside the circle at a distance a from the center of the circle. An infinitesimal area within the circle is shown, determined by the angle φ from the horizontal line and an additional infinitesimal angle $d\varphi$.

In order to find the two points of intersection, we can write the equation of the circle centered at the origin and insert y from equation (11),

$$x^2 + (a-x)^2 \tan^2\varphi = R^2. \tag{12}$$

Solving equation (12) for x yields

$$x_{1,2} = \frac{2a\tan^2\varphi \pm \sqrt{4a^2\tan^4\varphi - 4\left(1+\tan^2\varphi\right)\left(a^2\tan^2\varphi - R^2\right)}}{2\left(1+\tan^2\varphi\right)}. \tag{13}$$

Using some elementary trigonometric identities, we can get

$$x_1 = a\sin^2\varphi + \cos\varphi\sqrt{R^2 - a^2\sin^2\varphi}, \tag{14}$$

$$x_2 = a\sin^2\varphi - \cos\varphi\sqrt{R^2 - a^2\sin^2\varphi}. \tag{15}$$

Inserting these into equation (11) yields $y(x_1)$ and $y(x_2)$ of the two intersections, from which m and n can easily be calculated,

$$m = a\cos\varphi - \sqrt{R^2 - a^2\sin^2\varphi}, \tag{16}$$

$$n = \frac{a^2 - R^2}{m} = \frac{a^2 - R^2}{a\cos\varphi - \sqrt{R^2 - a^2\sin^2\varphi}}. \tag{17}$$

The enclosed area can be found as the difference between the long and short sectors, the areas of which are $(1/2)n^2\,d\varphi$ and $(1/2)m^2\,d\varphi$, respectively. The enclosed area is, therefore, given by

$$\frac{1}{2}\left(n^2 - m^2\right)\,d\varphi. \tag{18}$$

Let us consider now the component of the pulling force of the enclosed area in the x-direction. We have to multiply the area in (18) by the area-density ρ and by $\cos\varphi$. We can add here the effect of the symmetric area under the x-axis, and we get the contribution of the two infinitesimal areas as

$$\rho\left(n^2 - m^2\right)\cos\varphi\,d\varphi. \tag{19}$$

We can insert now n and m from equations (16) and (17) and integrate over φ for getting the total contribution of the circle to the pulling force along the x-axis. The integration over φ will be from 0 to $\sin^{-1}(R/a)$, and the pulling force along the x-axis, I, will be given as

$$I = \rho\int_0^{\sin^{-1}(R/a)} \left\{ \left[\frac{a^2 - R^2}{a\cos\varphi - \sqrt{R^2 - a^2\sin^2\varphi}}\right]^2 - \left[a\cos\varphi - \sqrt{R^2 - a^2\sin^2\varphi}\right]^2 \right\}\cos\varphi\,d\varphi. \tag{20}$$

Note that the components of the force in the vertical direction cancel out due to the symmetry with respect to the x-axis. Expression (20) can be written as

$$I = \rho a^2\int_0^{\sin^{-1}(R/a)} \left\{ \left[\frac{1-(R/a)^2}{\cos\varphi - \sqrt{(R/a)^2 - \sin^2\varphi}}\right]^2 - \left[\cos\varphi - \sqrt{\left(\frac{R}{a}\right)^2 - \sin^2\varphi}\right]^2 \right\}\cos\varphi\,d\varphi. \tag{21}$$

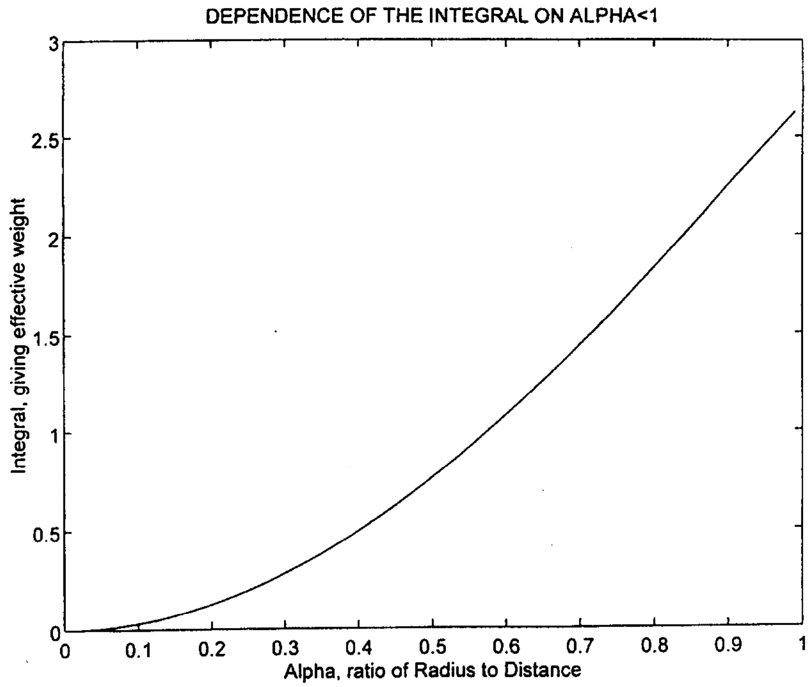

Figure 3. The calculated values of the integral in equation (21) for $\alpha < 1$, i.e., for a point located outside a given demand circle.

The integral cannot be performed analytically, but can easily be calculated numerically for any given value of the single parameter $\alpha = R/a$. Figure 3 shows the calculated values of the integral for values of α between 0 and 1. Note that the low value of the integral at low values of α does not mean that the effective weight is zero. In fact, small α is associated with large values of a, and in order to find the pulling force, one has to multiply the value of the integral by a^2.

Concerning the case of $R < a$, we can consider the limiting situation of $\alpha \to 0$ which is $R \ll a$. Here, we can consider the whole "mass" of the circle $\rho\pi R^2$ as being located at a single point, a distance a from the point P. We can, therefore, write

$$\lim_{\alpha \to 0} \rho a^2 \int_0^{\sin^{-1}\alpha} \{\ \}\, d\varphi = \rho\pi R^2, \tag{22}$$

where the curled brackets indicate the integrand in equation (21). We should, therefore, get

$$\lim_{\alpha \to 0} \frac{1}{\alpha^2} \int_0^{\sin^{-1}\alpha} \{\ \}\, d\varphi = \pi. \tag{23}$$

This has been checked in the numerical results and, indeed, the value of the integral divided by α^2 tended to π for $\alpha \to 0$. Another limit that will be of interest is that of expression (21) where $\alpha \to 1$. It is seen in Figure 3 that for $\alpha \to 1$, the value of the integral goes to ~ 2.66. This will be compared to the pulling force that will be calculated below (analytically) for a point on the circumference of the circle.

2.2. The Attracting "Force" When the Facility is inside the Circle

When the point P is inside a circle with a weight density ρ, the net pulling force in a given direction is proportional to the difference between the two sectors shown in Figure 4 in two opposite directions from the point P. The considerations for the present case of $\alpha > 1$ are very

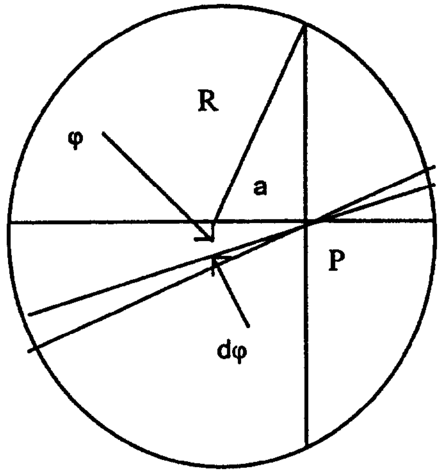

Figure 4. A circular demand area with a radius R as seen from a point P inside the circle at a distance a from the center of the circle. Two infinitesimal areas are shown, determined by the angle φ from the horizontal and an additional infinitesimal angle $d\varphi$. The difference between these areas is associated with the net contribution to the pulling force.

similar to the above-mentioned case of $\alpha < 1$, and we will discuss it only briefly. We get here

$$m = a\cos\varphi - \sqrt{R^2 - a^2\sin^2\varphi}, \tag{24}$$

$$n = \frac{R^2 - a^2}{a\cos\varphi - \sqrt{R^2 - a^2\sin^2\varphi}}. \tag{25}$$

Note that as opposed to the previous case, the integration here should go from 0 to $\pi/2$. We, therefore, get the net force

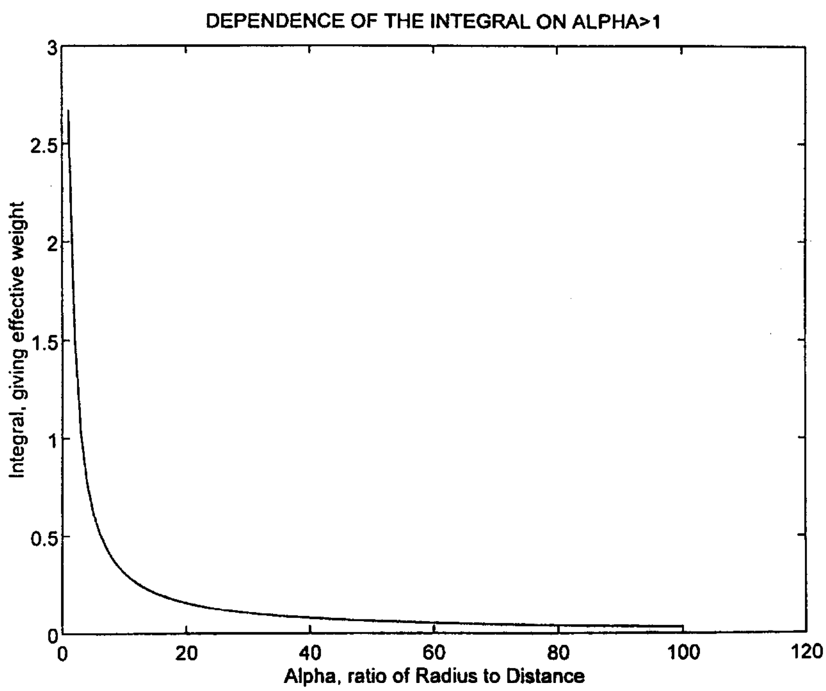

Figure 5. The calculated values of the integral in equation (26) for $\alpha > 1$, i.e., for a point located inside a given demand circle.

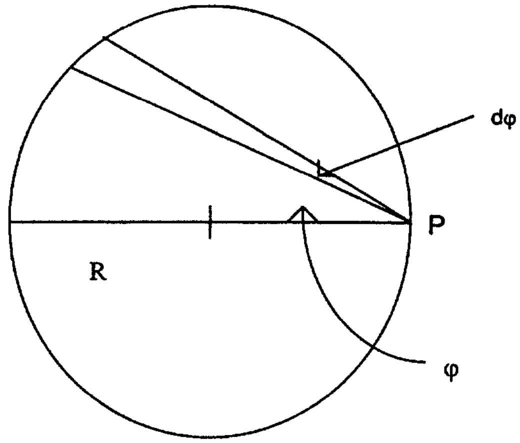

Figure 6. A circular demand area with a radius R as seen from a point P on the circumference. An infinitesimal area within the circle is shown, determined by the angle φ from the horizontal line and an additional infinitesimal angle $d\varphi$.

$$I = \rho a^2 \int_0^{\pi/2} \left[\left(\frac{\alpha^2 - 1}{\cos\varphi - \sqrt{\alpha^2 - \sin^2\varphi}} \right)^2 - \left(\cos\varphi - \sqrt{\alpha^2 - \sin^2\varphi} \right)^2 \right] \cos\varphi \, d\varphi. \tag{26}$$

The limiting case of $\alpha \to \infty$ is the situation where the point P is located very close to the center. Obviously, the limit of the integral for $\alpha \to \infty$ should be zero, which has indeed been seen in the calculations and depicted in Figure 5. Note that here too, when $\alpha \to 1$ (from above), the value of the integral tends to ~ 2.66.

2.3. The Attracting "Force" for a Point Located on the Circumference of the Circle

This situation is shown in Figure 6. Here, $\alpha = R/a = 1$, and the geometry is significantly simpler than in the two previous cases. The integral to be calculated here is

$$I = \int_0^{\pi/2} \rho (2R\cos\varphi)^2 \cos\varphi \, d\varphi, \tag{27}$$

which gives

$$I = 4R^2 \rho \int_0^{\pi/2} \cos^3\varphi \, d\varphi, \tag{28}$$

where the integral is 2/3 which immediately yields $I = (8/3)\rho R^2$.

It is quite obvious that both for equation (21), for a point outside the circle, and equation (26), for a point inside the circle, one can expect that

$$\lim_{\alpha \to 1} I = \frac{8}{3} \rho R^2. \tag{29}$$

This was indeed shown in Figures 3 and 5 where the value of the integral was calculated for α approaching unity from above and from below.

3. THE STEP-SIZE IN THE ITERATIVE PROCEDURE AND THE COST FUNCTION

The iterations to be performed while solving such a problem with uniform circular demand areas are, obviously, an extension to the iterations given above in equations (9) and (10) which was utilized when the demand was concentrated in discrete points with weight w_i. Note again that the numerators on the right-hand side of equations (9) and (10) can be written as $\sum w_i \cos \varphi_i$ and $\sum w_i \sin \varphi_i$, respectively (see Figure 1), as explained in the introduction above. The numerators, which were extended to the "pulling forces" I in equations (21),(26), and (29) in the present case of uniform circular demand area, determine, when all the circles are taken into account, the direction in which each step of the iterative process goes. As for the size of the step, it is determined by the expression appearing in the denominators of both equations (9) and (10). An important point to consider is that the scalar appearing in the denominator, which in the discrete case is $\sum w_i/r_i(x^N, y^N)$, should not be associated with each circle when we move to the case of demand circles, but rather, the sum over all the circles should be taken. The obvious reason is that in the discrete case, *all* the points are to be taken in the denominator expression, and therefore, in the case of several circles, all the points in all the circles are to be considered. In the following, we are going to calculate, however, the expressions for single circles with respect to an incumbent point (x^N, y^N) for the three cases of this point being outside, on the circumference, and inside the circle. Since the expressions in the denominators are scalars, once we have these expressions for all the relevant circles, we can add them up and insert them into the denominators of the x and y components of the "force" as shown below.

Another important point that deserves some attention is the extension of the cost function (1) to the present case of circular demand areas. This, again, will be the sum over the demand circles when the incumbent service point may be inside, on the circumference, or outside each of the given circles. The geometry of determining the elements of the cost function differs from that of the "force" components, but resembles the evaluation of the expressions extending equations (9) and (10) to the present case of circular demand areas. Therefore, we shall discuss the evaluation of these two scalars together for each of the three geometrical situations, namely, when the current service point is outside, inside, and on the circumference of the demand circle.

3.1. The Cost Function and Weight Factors for a Facility outside a Circle

The expressions to be developed here, namely the element of the cost function (1) and the weight factor appearing in the denominator of equations (9) and (10), differ from those given above of the force elements in one basic point. Whereas in the evaluation of forces we had to add together all the elements in a given direction from the incumbent point as seen in Figures 2, 4, and 5, here we have to add the contribution from points which are at a given distance r from the incumbent point. This is shown in Figure 7 for the case where the current service point is outside the demand circle. Here, again, we are dealing with a point $P(x^N, y^N)$ located at a distance a from the center of a circle with a radius R, $R < a$. Let us consider now the infinitesimal area enclosed by an arc at a distance r $(a - R < r < a + R)$ from the point P, an infinitesimally close arc at $r + dr$, where dr is an infinitesimal addition to r, and by the given circle, as shown in Figure 7. All the points within the infinitesimal area will contribute in a similar way to the expressions emerging from $\sum w_i r_i$ in the cost function (equation (1)) or from $\sum w_i/r_i$ in the denominator of equations (9) and (10). We should, therefore, find the infinitesimal area between the two arcs shown, multiply it by r or divide it by r, and integrate over r from $a - R$ to $a + R$.

Considering the angle α between the line connecting P to the center of the circle and the radius-vector r shown, the length of the arc is given by $2\alpha r$. The infinitesimal area in question will then be $2\alpha r\, dr$, and if we take a weight density of ρ for the circle, the infinitesimal weight to be considered is $2\alpha\rho r\, dr$. Using simple trigonometry, we get

$$\alpha = \cos^{-1}\left(\frac{a^2 + r^2 - R^2}{2ar}\right). \tag{30}$$

As stated above, we have to multiply the infinitesimal weight by r in order to get the contribution to the cost function. This yields

$$F = 2\rho \int_{a-R}^{a+R} r^2 \cos^{-1}\left(\frac{a^2 + r^2 - R^2}{2ar}\right) dr. \tag{31}$$

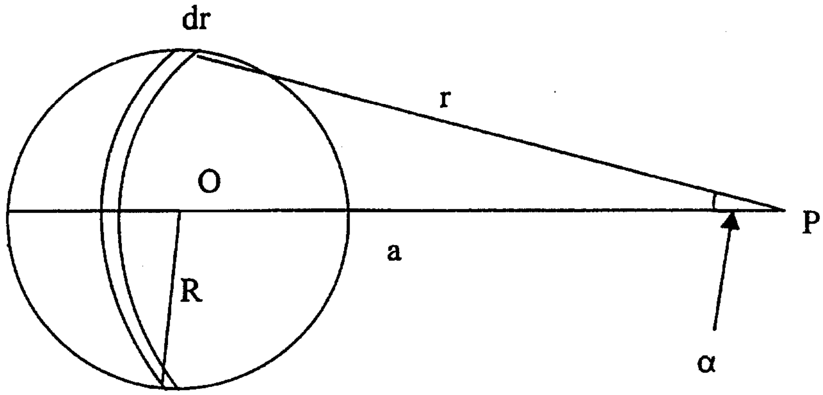

Figure 7. A circular demand area as seen from a point P outside the circle as in Figure 2. An infinitesimal area within the circle is shown, all points of which are at the same distance r from P.

As for the contribution to the denominator term, we have to divide the infinitesimal weight by r and integrate over r, which yields

$$W = 2\rho \int_{a-R}^{a+R} \cos^{-1}\left(\frac{a^2 + r^2 - R^2}{2ar}\right) dr. \tag{32}$$

3.2. The Cost Function and Weight for a Facility inside or on the Circumference of a Circle

The considerations here are analogous to the previous case, and therefore, we do not show the geometry or the details, but rather give the results. For $R > a$, we get

$$F = 2\rho \int_{R-a}^{R+a} r^2 \cos^{-1}\left(\frac{a^2 + r^2 - R^2}{2ar}\right) dr, \tag{33}$$

and the denominator term is

$$W = 2\rho \int_{R-a}^{R+a} \cos^{-1}\left(\frac{a^2 + r^2 - R^2}{2ar}\right) dr. \tag{34}$$

These two functions are not analytically integrable, but their numerical integration at each stage of the iteration is straightforward.

As in the case of the evaluation of the "force" elements, the situation with the current service point being on the circumference of the relevant circle is geometrically simpler. We get here, in a similar way,

$$F = 2\rho \int_{0}^{2R} r^2 \cos^{-1}\left(\frac{r}{2R}\right) dr, \tag{35}$$

for the contribution to the cost function, and

$$W = 2\rho \int_{0}^{2R} \cos^{-1}\left(\frac{r}{2R}\right) dr, \tag{36}$$

for the contribution to the denominator factor. The latter expression is easily integrable, which yields

$$W = 4\rho R. \tag{37}$$

3.3. The Iterations and the Total Cost Function

Suppose we have a problem with J circles, each of which has a density weight of ρ_j, $1 \le j \le J$. As a starting point for the iterations, we can take expressions like (8) for $j = 1, \dots, J$, where for each circle we concentrate the whole weight in the center of the circle, and thus the initial x^0, y^0 are found. The next step is evaluating the total denominator term W from its components W_j using equations (32),(34), and (37), depending on the location of the current point with respect to the different circles. This can be written as

$$W = \sum_{j=1}^{J} W_j, \tag{38}$$

where W_j are the individual elements related to the different circles.

Once x^0, y^0, and W are known, the iterations are now given by

$$x^{N+1} = x^N + \frac{\sum_{j=1}^{J} I_j \cos \Phi_j}{W}, \tag{39}$$

$$y^{N+1} = y^N + \frac{\sum_{j=1}^{J} I_j \sin \Phi_j}{W}, \tag{40}$$

where I_j are found by equations (21),(26), or (29), depending on the position of the current point (x^N, y^N) and each of the J circles, and where Φ_j are the angles between the lines connecting the incumbent point P and the center of the j circles, and an arbitrary line going through P.

As for the total cost function, its value at each stage is given by

$$F = \sum_{j=1}^{J} F_j, \tag{41}$$

where F_j are determined by equations (31),(33), and (35), again, depending on the relative position of the incumbent point and the different circles.

4. NUMERICAL EXPERIENCE

Several problems of ten demand circles with centers distributed over a 15×15 square have been solved. For simplicity and in order to compare the results of problems with a different nature, the centers were kept constant, whereas the other parameters of the problem (namely the different radii R_j and densities ρ_j) were varied. A schematic example (with only five demand circles) is shown in Figure 8. For each of the circles, the radius R_j and the distance to the incumbent point P, a_j are shown.

In the calculation, the effective weight of each of the circles has been evaluated at each stage, using the appropriate values of $\alpha_j (= R_j/a_j)$ and a_j, the distance from the current solution point. This was done by using equations (21) or (26), depending on whether the current solution point is inside or outside the j^{th} demand circle, respectively. It is obvious that when the incumbent solution point moves with the iterations, the effective weights change with the changing values of a_j. Also, it may happen that the solution point gets in and out some of the demand circles during the iterative procedure.

A number of empirical conclusions have been drawn from the different cases. Note that the same termination criterion has been used, so that some comparison can be made between the different examples.

EXAMPLE 1. Some problems were solved with circles having very small radii (R_j). The results were compared to the solution of the original Weber problem with point demands located at the

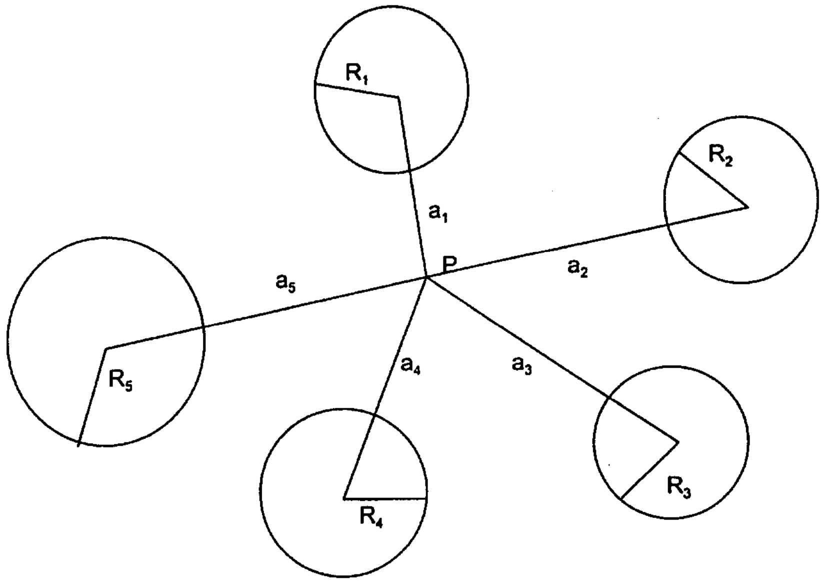

Figure 8. Five circular demand areas with radii $R_1, \ldots, R_5$, densities $\rho_1, \ldots, \rho_5$, and distances $a_1, \ldots, a_5$ from an incumbent service point P.

centers of the circles, and having appropriate weights. The final results were very close in the two problems, and so were the number of steps required to reach the same termination criterion. As pointed out above, previous works [4,5] indicated that doubling the stepsize reduces, on the average, the number of steps needed to reach a given termination criterion by a factor of 2, provided that the solution point does not coincide with one of the demand points. This was found to be the case here as well, where the demand circles were small relative to the distance between them. Typically, for the 10 points problem, 30–40 steps were required when the original Weiszfeld stepsize was used, and 15–20 steps when the double stepsize was utilized, to reach the required termination criterion. It should be reiterated that in cases of coincidence or near coincidence between a demand point and the solution, the original Weiszfeld method converged significantly faster.

EXAMPLE 2. If everything else was kept constant and the density of one of the circles ρ_K was increased, a value was reached where the solution point coincided, or nearly coincided, with the center of the K^{th} demand circle as could be expected. This was the case for both small and large radius circles.

EXAMPLE 3. In cases where the radii of the demand circles were relatively large, there was obviously an overlap between circles. The solution procedure went as straightforward as before. Note that in an overlap region, the density is assumed to be the sum of the densities of the two (or more) circles in question. An important point here is that the original Weiszfeld stepsize was found to be the best here rather than, say, the double stepsize or, for that matter, any other stepsize along the steepest descent direction. This is not too surprising since we often have here coincidence between the solution point and a demand area. Furthermore, for the same termination criterion, the procedure required significantly fewer steps here than in the discrete (or nearly discrete, i.e., small circles) case, and typically, the process terminated in 2–4 steps.

EXAMPLE 4. Hybrid problems of large demand circles and very small ones (practically discrete points) were also solved with no extra difficulty. It is to be noted that the effective weight of a circle is associated with $\pi \rho_j R_j^2$, and therefore, if small radii are taken, ρ_j are to be increased significantly if one wishes to solve problems with more or less comparable weights. No specific conclusion has been reached whether the original or double Weiszfeld stepsize is better. Generally speaking, however, it appears better to use the original stepsize. The reason for this is that in the cases where the double stepsize is optimal, using the original one requires typically twice as many steps to reach the termination criterion. However, in cases where the original stepsize is better (e.g., with large demand circles), using the double stepsize may increase the number of required steps by factors of 10–100 and perhaps more, to reach the same termination criterion.

5. CONCLUSION

In the present work, a new method is given for the solution of the Weber problem with circular demand areas. The method is an extension to the Weiszfeld method, and is based on evaluating the effective "force" exerted by each circle that depends on the radius of the circle and its distance from the current point in the iteration, as well as the density of the circle. The distance may be larger than, equal to, or smaller than the radius. The effective "forces" of the different circles vary from one step to another during the iterative procedure.

The method includes cases of different demand circles. Also included is the case of overlapping circles and, therefore, cases of geometric demand areas consisting of a combination of different circles are also taken into account. Also considered were hybrid cases of continuous circles and discrete points that may appear in certain problems. This includes cases in which the discrete demand points are embedded inside the continuous demand circles.

One can think of future extensions to the present method. This may include problems in more than two dimensions, e.g., demand spheres in three dimensions. One can also think of a nonuniform distribution of the density in a circle, as well as different demand areas, other than circles. Here too, the integration should be done in the same way, namely, taking into consideration the fact that all the weight in a certain direction contributes to the "force" in that direction. However, for the calculation of the cost function F and the denominator W, points of equal distance from the incumbent solution point are to be aggregated. Obviously, with these cases of different geometrical demand areas and nonuniform densities, the mathematical complication is significantly higher than in the present case of uniform circles.

Another rather straightforward extension is the multifacility location problem where the demand is distributed over circles in the same way as in the single facility problem described here.

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