Pergamon

Computers Math. Applic. Vol. 35, No. 9, pp. 25–31, 1998

0898-1221/98 $19.00 + 0.00

PII: S0898-1221(98)00054-6

A Note on Convergence in the Single Facility Minisum Location Problem

J. BRIMBERG
Department of Business Administration
Royal Military College of Canada
Kingston Ontario, Canada K7K 5LO

R. CHEN*
School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University
Tel-Aviv 69978, Israel

(Received September 1997; accepted October 1997)

Abstract—The single facility minisum location problem requires finding a point in $\mathcal{R}^N$ that minimizes a sum of weighted distances to m given points. The distance measure is typically assumed in the literature to be either Euclidean or rectangular, or the more general l_p norm. Global convergence of a well-known iterative solution method named the Weiszfeld procedure has been proven under the proviso that none of the iterates coincide with a singular point of the iteration functions. The purpose of this paper is to examine the corresponding set of "bad" starting points which result in failure of the algorithm for a general l_p norm. An important outcome of this analysis is that the set of bad starting points will always have a measure zero in the solution space ($\mathcal{R}^N$), thereby validating the global convergence properties of the Weiszfeld procedure for any l_p norm, $p \in [1, 2]$.

Keywords—Single facility minisum location problem, l_p norm, Convergence, Singular points.

1. INTRODUCTION

The continuous single facility minisum location problem is one of the classical models in location theory. It comes under various names such as the Fermat-Weber problem. (For an interesting review of the long history of this problem, the reader is referred to [1].) The objective is to find a point in $\mathcal{R}^N$ which minimizes the sum of weighted distances to m given points. The given points represent customers or demands in a practical setting, the new point denotes a new facility such as a warehouse which needs to be located, and the weighted distances are cost components associated with the interactions such as flow of goods between the new facility and its customers.

We consider the case where distances are measured by an l_p norm. The resulting minisum model is formulated as follows:

*Author to whom all correspondence should be addressed.

$$\min\ W(x) = \sum_{i=1}^{m} w_i l_p(x - a_i), \tag{1}$$

where $a_i = (a_{i1}, \ldots, a_{iN})^\top$ is the known position of the i$^{\text{th}}$ demand point, $i = 1, \ldots, m$; $x = (x_1, \ldots, x_N)^\top$ is the unknown position of the new facility; $w_i > 0$ is a weighting constant, $i = 1, \ldots m$; and the distance between the points x and a_i is given by an l_p norm; i.e.,

$$l_p(x - a_i) = \left[\sum_{t=1}^{N} |x_t - a_{it}|^p\right]^{1/p}, \qquad \forall x \in \mathcal{R}^N,\ p \geq 1. \tag{2}$$

The restriction on the parameter p to a value greater than or equal to unity ensures that $l_p(\cdot)$ has the properties of a norm. We also note that the l_p norm generalizes the popular Euclidean and rectangular norms for which $p = 2$ and 1, respectively.

The minisum model with Euclidean distances has been studied extensively in the literature. An iterative solution technique to solve the Euclidean case was first proposed by Weiszfeld [2], and rediscovered several years later by Miehle [3], Kuhn and Kuenne [4] and Cooper [5]. The algorithm, which is based on the first-order necessary conditions for a stationary point of the objective function, is readily extended to l_p distances (e.g., see [6]).

Global convergence of Weiszfeld's procedure was proven in a seminal paper by Kuhn [7] for Euclidean distances, and the results were extended by Brimberg and Love [6] to the l_p norm where p can assume any value in the closed interval [1,2]. Brimberg and Love [6,8] also show that convergence may not occur for $p > 2$. In all cases, global convergence of the sequence of iterates assumes that none of them coincides with a singular point of the iteration functions.

Extension of the minisum problem from Euclidean to l_p distances has, aside from theoretical interest, a practical significance. When the parameter p takes on any value in the closed interval [1,2], a continuum of distance measures is covered spanning the popular rectangular and Euclidean norms. The statistical superiority of the l_p norm as a distance measure is well known (e.g., see [9,10]; also see Table I in [6] for empirically obtained p values of several road networks). We conclude that a generalization of the minisum problem to l_p distances allows a more accurate representation of real systems.

To avoid the singularities of the iteration functions, a hyperbolic approximation of l_p norm may be used. This smoothing function allows global convergence of the iterative sequence for all starting points, as shown by Morris [11] for any $p \in [1, 2]$. However, as noted by Brimberg and Love [6], the resulting solution may be considerably removed from the optimal solution of the original unapproximated model. Thus, it is useful to consider the problem in the original form given in (1).

An open question concerning the main result of [7] for Euclidean distances was posed by Chandrasekaran and Tamir [12]. These authors provide counter-examples demonstrating that a continuum set of starting points may exist which will terminate the Weiszfeld algorithm at a singular point a_i. Brimberg [13] resolves this question by proving that a necessary and sufficient condition for this set of "bad" starting points to be denumerable is that the convex hull of the demand points $a_1, \ldots, a_m$, be of full dimension (N).

The nature of the set of starting points which result in an iterate coinciding with a singular point remains an unresolved issue when the Weiszfeld procedure is extended to a general l_p norm. Our objective then will be to investigate this question. In the next section we review the generalized iterative procedure for solving the model given in (1). Subsequently, we derive some analytical results to show that the set of starting points in question may not be denumerable, but will always have a measure zero in the solution space, $\mathcal{R}^N$. This outcome extends the results by Brimberg [13] for the Euclidean case. This also validates the generalized iterative procedure for l_p distances, since termination of the sequence at a singularity for randomly-chosen starting points is seen to be an unlikely event.

2. REVIEW OF THE GENERALIZED WEISZFELD PROCEDURE

If the optimal solution x^* occurs at a differentiable point of the objective function, it follows that the first-order partial derivatives $\frac{\partial W(x^*)}{\partial x_t}$, $t = 1, \ldots, N$, exist and must be zero. Furthermore, since W is a convex function, any stationary point thus identified must be a global optimum. The system of equations to solve can be written in the following form:

$$\sum_{i=1}^{m} w_i \operatorname{sign}(x_t^* - a_{it}) \frac{|x_t^* - a_{it}|^{p-1}}{[l_p(x^* - a_i)]^{p-1}} = 0, \qquad t = 1, \ldots, N, \tag{3}$$

where $x^* = (x_1^*, \ldots, x_N^*)^\top$. Unfortunately, this system of equations does not afford a closed-form solution. However, substituting $(x_t - a_{it}) = \operatorname{sign}(x_t - a_{it})\,|x_t - a_{it}|$ and rearranging the terms suggests the one-point iterative scheme below for finding x^*:

$$x_t^{q+1} = \frac{\sum_{i=1}^{m} w_i \,|x_t^q - a_{it}|^{p-2}\, a_{it}/[l_p(x^q - a_i)]^{p-1}}{\sum_{i=1}^{m} w_i \,|x_t^q - a_{it}|^{p-2}/[l_p(x^q - a_i)]^{p-1}}, \qquad t = 1, \ldots, N, \tag{4}$$

where the superscript $q = 0, 1, 2, \ldots$, denotes the iteration number, and x^0 is any arbitrary starting point in $\mathcal{R}^N$.

To simplify the notation, we rewrite the iteration function for the x_t coordinate as follows:

$$f_t(x) = \frac{1}{s_t(x)} \sum_{i=1}^{m} y_{it}(x) a_{it}, \qquad t = 1, \ldots, N, \tag{5}$$

where

$$y_{it}(x) = \frac{w_i \,|x_t - a_{it}|^{p-2}}{[l_p(x - a_i)]^{p-1}}, \qquad i = 1, \ldots, m, \quad t = 1, \ldots, N, \tag{6}$$

and

$$s_t(x) = \sum_{i=1}^{m} y_{it}(x), \qquad t = 1, \ldots, N. \tag{7}$$

Letting

$$S_t = \bigcup_{i=1}^{m} \{x \mid x_t - a_{it} = 0\}, \qquad t = 1, \ldots, N, \tag{8}$$

it is immediately obvious from the functional form of the iteration functions that $f_t(x)$ is defined and infinitely differentiable $\forall\, x \in \mathcal{R}^N \backslash S_t$. The set of singular points of the iteration functions is given by

$$S = \begin{cases} \bigcup_{t=1}^{N} S_t, & 1 \le p < 2, \\ \{a_1, \ldots, a_m\}, & p \ge 2. \end{cases} \tag{9}$$

As noted in Brimberg and Love [6], we see that a basic difference between the original Weiszfeld procedure for Euclidean distances and its extension to the l_p norm is that in the former case, the singularities of the iteration functions occur only at the fixed points, whereas when $p < 2$, they occur at mN hyperplanes of the form $x_t = a_{it}$.

In order to define the iterative procedure at all points $x \in \mathcal{R}^N$, we specify the following mapping, denoted symbolically as $T : x \to T(x)$:

(a) for $1 \le p < 2$,

$$x_t^{q+1} = \begin{cases} f_t(x^q), & \text{if } x^q \notin S_t, \\ a_{it}, & \text{if } x_t^q = a_{it}, \text{ for some } i \in \{1, \ldots, m\}, \end{cases} \qquad t = 1, \ldots, N; \tag{10}$$

(b) for $p \ge 2$,

$$x^{q+1} = \begin{cases} (f_1(x^q), \ldots, f_N(x^q))^\top, & \text{if } x^q \ne a_i, i = 1, \ldots, m, \\ a_i, & \text{if } x^q = a_i, \text{ for some } i \in \{1, \ldots, m\}. \end{cases} \tag{11}$$

As shown in [6], the mapping in (10) is continuous, while the mapping in (11) is continuous for $p = 2$, and discontinuous in general at the fixed points a_i for $p > 2$. Referring to (5)–(7), (10), and (11), it also follows that the iterates x^{q+1}, $q = 0, 1, 2, \ldots$, will all fall in a bounded hypercube containing the fixed points, given by

$$\min_i \{a_{it}\} \leq x_t \leq \max_i \{a_{it}\}, \qquad t = 1, \ldots, N. \tag{12}$$

Furthermore, for $p = 2$ (Euclidean distances), the iterates will fall in a more compact region given by the convex hull of the fixed points.

Suppose that for some dimension t, $a_{it} = b_t$, a constant, $\forall i = 1, \ldots, m$. Referring to the map T, we see that $x_t^{q+1} = b_t$, $\forall q = 0, 1, 2, \ldots$. This trivial case will be disregarded by assuming henceforth that the problem has been reduced to the minimum number of dimensions required.

3. ANALYSIS OF SINGULARITIES

The iterative procedure discussed above will fail in general to achieve an optimal solution if one of the iterates x^q coincides with a singular point of the iteration functions $f_t(x)$, $t = 1, \ldots, N$. For this reason we classify the sequence x^q, $q = 0, 1, 2, \ldots$, as *regular* if $x^q \notin S$, $\forall q$, and *nonregular* otherwise, where S is the set of singular points defined in (9). Brimberg and Love [6] prove that convergence to an optimal solution will occur for any *regular* sequence generated by the map T, when p has a value in the closed interval [1,2]. (The iterative procedure would not normally be used for $p = 1$, since this special case can be solved easily as N one-dimensional median problems (e.g., [14]).) We wish to show now that the set of points comprising all the *nonregular* sequences has a measure zero in the solution space. Such a result will validate the solution algorithm, since it implies that the probability the algorithm will fail for a randomly-chosen starting point tends to zero as the precision of the computations is increased.

We begin the analysis by calculating the first-order partial derivatives of f_t:

$$\frac{\partial f_t(x)}{\partial x_j} = \frac{1}{s_t(x)} \left\{ \sum_i \frac{\partial}{\partial x_j}[y_{it}(x)]a_{it} - \frac{\sum_i y_{it}(x)a_{it}}{\sum_i y_{it}(x)} \sum_i \frac{\partial}{\partial x_j}[y_{it}(x)] \right\}, \qquad \forall j, t = 1, \ldots, N, \tag{13}$$

where the summations are understood to be over the index set $\{1, \ldots, m\}$. Referring to (5), the preceding relation simplifies to

$$\frac{\partial f_t(x)}{\partial x_j} = \frac{1}{s_t(x)} \sum_{i=1}^{m} \left(\frac{\partial [y_{it}(x)]}{\partial x_j} \right) \cdot (a_{it} - f_t(x)), \qquad \forall j, t. \tag{14}$$

Letting

$$v_{itj}(x) := \frac{\partial}{\partial x_j}[y_{it}(x)], \qquad \forall j, t, \tag{15}$$

we obtain

$$v_{itj}(x) = \frac{(1-p)w_i \operatorname{sign}(x_j - a_{ij}) \, |x_j - a_{ij}|^{p-1} \, |x_t - a_{it}|^{p-2}}{[l_p(x - a_i)]^{2p-1}}, \qquad j \neq t, \tag{16}$$

and

$$v_{itt}(x) = \frac{-w_i \operatorname{sign}(x_t - a_{it}) \, |x_t - a_{it}|^{p-3}}{[l_p(x - a_i)]^{p-1}} \left[(2-p) + \frac{(p-1) \, |x_t - a_{it}|^p}{[l_p(x - a_i)]^p} \right]. \tag{17}$$

Thus, the $N \times N$ Jacobian matrix,

$$f'(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} = D(x)V(x), \tag{18}$$

where

$$D(x) = \begin{bmatrix} (s_1(x))^{-1} & 0 & \dots & 0 \\ 0 & (s_2(x))^{-1} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & (s_N(x))^{-1} \end{bmatrix} \tag{19}$$

is a diagonal matrix with positive-valued diagonal elements, and

$$V(x) = \begin{bmatrix} \sum_{i=1}^m v_{i11}(x)(a_{i1} - f_1(x)) & \dots & \sum_{i=1}^m v_{i1N}(x)(a_{i1} - f_1(x)) \\ \vdots & & \vdots \\ \sum_{i=1}^m v_{iN1}(x)(a_{iN} - f_N(x)) & \dots & \sum_{i=1}^m v_{iNN}(x)(a_{iN} - f_N(x)) \end{bmatrix}, \tag{20}$$

where the v_{itj} are defined in (16) and (17).

Note that for $p = 2$ (Euclidean distances), we have

$$s_t(x) = s(x) = \sum_{i=1}^m \frac{w_i}{l_2(x - a_i)}, \qquad t = 1, \dots, N, \tag{21}$$

and

$$v_{itj}(x) = u_{ij}(x) = \frac{-w_i(x_j - a_{ij})}{(l_2(x - a_i))^3}, \qquad \forall t, j. \tag{22}$$

Thus, for $p = 2$,

$$f'(x) = \frac{1}{s(x)} \left[\sum_{i=1}^m u_{i1}(x) \cdot (a_i - f(x)) \cdots \sum_{i=1}^m u_{iN}(x) \cdot (a_i - f(x)) \right], \tag{23}$$

where $f(x) = (f_1(x), \dots, f_N(x))^\top$ denotes the vector of iteration functions. We see that each column j of $f'(x)$ reduces in the Euclidean case to a weighted sum of the vectors $(a_i - f(x))$, $i = 1, \dots, m$, where the weights $u_{ij}(x)$ are independent functions of x. This leads to the observation in Brimberg [13] that the matrix $f'(x)$ is a singular matrix if the set of given points $\{a_1, \dots, a_m\}$ is contained in an affine subspace of $\mathcal{R}^N$; or otherwise said, the invertibility of $f'(x)$ requires that the convex hull of $\{a_1, \dots, a_m\}$ has full dimension N.

For the non-Euclidean case, the more complicated form of $f'(x)$ (see (18)–(20)) gives the following stronger result.

LEMMA 1. *If $p \neq 2$, the Jacobian matrix $f'(x)$ will always be invertible everywhere except at a subset of points of measure zero in R^N.*

PROOF. For $p = 1$, it follows from (16) and (17) that $f'(x)$ is a diagonal matrix with nonzero diagonal elements $\forall x \notin S$. For $p > 1 (p \neq 2)$, we observe that any element (t, j) of $f'(x)$ is a weighted sum of the components $(a_{it} - f_t(x))$, $i = 1, \dots, m$, where the individual weights $v_{itj}(x)/s_t(x)$ are unique functional forms of x. It follows that the determinant of $f'(x)$ (denoted by $\det(f'(x))$) is a continuously-varying infinitely differentiable function of x in any open subset V of $\mathcal{R}^N$ which does not intersect $\bigcup_{t=1}^N S_t$. Hence, $\{x \mid \det(f'(x)) = 0\}$ must have a measure zero in V. We conclude that $f'(x)$ will be invertible everywhere except at a subset of points of measure zero in $\mathcal{R}^N$. ∎

We are now able to present the main result of the paper.

THEOREM 2. *For all $p \geq 1$, the sequence generated by the map T will be regular ($x^q \notin S$, $q = 0, 1, 2, \dots$) except for a set of starting points $\{x^0\}$ which has measure zero in $\mathcal{R}^N$.*

PROOF. Three cases are considered.

(a) $1 \leq p < 2$. The hyperplanes defined by $\{x \mid x_t - a_{it} = 0\}$, $i = 1, \dots, m$, $t = 1, \dots, N$, divide $\mathcal{R}^N$ into at most $(m + 1)^N$ cells or hypercubes. Let V_k denote an open set in $\mathcal{R}^N$

formed by such a hypercube (less its boundaries), $k = 1, \ldots, M$, where $M \leq (m+1)^N$. We have that f is continuous and infinitely differentiable in each V_k.

Let $x_0 \in V_k$ such that $f(x_0) = b$, where $b \in S$. By Lemma 1, and the fundamental inverse function theorem of calculus (e.g., see [15, p. 354]), it follows that a neighborhood $N(x_0)$ around x_0 can be constructed such that $f(x)$ provides a one-to-one mapping $\forall x \in N(x_0)$. We conclude that $\{x \mid f(x) \in s, x \in V_k\}$ comprises a countable number of hypersurfaces with measure zero in V_k. Hence, $\{x \mid f(x) \in S, x \in \mathcal{R}^N \backslash S\}$ has measure zero in $\mathcal{R}^N$.

Letting $T^q(x)$ denote the q^{th} iterate of T, it now follows by the same argument that $\{x \mid T^q(x) \in S, q \leq K\}$ has measure zero in $\mathcal{R}^N$ for any finite number of iterations K. We conclude that the set of starting points $\{x^0\}$ which will produce a nonregular sequence, must have measure zero in $\mathcal{R}^N$.

(b) $p = 2$. For the Euclidean case, it can be shown that the set of starting points $\{x^0\}$ which will terminate the sequence at some demand point a_i after a finite number of iterations is denumerable, if and only if, the convex hull of the set of demand points has full dimension N [13]. On the other hand, if this convex hull is contained in an affine subspace of $\mathcal{R}^N$, it follows by extension of the inverse function theorem that $\{x \mid T(x) = a_r\}$ will no longer be denumerable but will have measure zero. We conclude as in Case (a) that the set $\{x^0\}$ which will produce a nonregular sequence must always have measure zero in $\mathcal{R}^N$.

(c) $p > 2$. Using a similar reasoning as in Case (a) and nothing that S now contains a finite number of points, we conclude that the set $\{x^0\}$ which will produce a nonregular sequence must always be denumerable.

Combining the results in (a)–(c), it follows that for any $p \geq 1$, the map T generates a regular sequence for all but a set of starting points of measure zero in $\mathcal{R}^N$. ∎

4. CONCLUSIONS

This paper reviews a generalization of the Weiszfeld iterative procedure for solving the single facility minisum location problem with l_p distances. Global convergence of this algorithm was proven by Brimberg and Love [6] for any value of p in the closed interval [1,2] under the proviso that none of the iterates coincides with a singular point of the iteration functions. Otherwise, the sequence is termed nonregular and the algorithm fails in general.

Based on the invertibility of $f'(x)$, the Jacobian matrix of the iteration functions, we have shown that the set of starting points $\{x^0\}$ which will result in a nonregular sequence, always has a measure zero in the solution space ($\mathcal{R}^N$). This property validates the generalized Weiszfeld procedure, since the probability that the algorithm will fail for randomly-chosen starting points tends to zero as the precision of the computations is increased.

Computational experience on a series of random problems in $\mathcal{R}^2$ confirms this result. The problem sizes varied from small to large, with $m = 10$, 30, 100, and 1000 demand points. For each example, the p values ranged from 1.1 to 2.0 in increments of 0.1; as well several values greater than 2 were used up to $p = 25$. Homogeneous weights ($w_i = 1, \forall i$) and random weights were applied in each case. In one group of experiments, the demand points were generated uniformly in a 100×100 square. In a second group, the runs were repeated with the demand points arranged in clusters. A third group had the demand points generated uniformly in a 100×500 rectangle. Double precision arithmetic was used in the computations. In all cases, a regular sequence was generated for randomly-chosen starting points.

REFERENCES

1. G.O. Wesolowsky, The Weber problem: History and perspectives, *Location Science* **1**, 5–23, (1993).
2. E. Weiszfeld, Sur le point pour lequel la somme des distances de n points donnés est minimum, *Tôhoku Mathematical Journal* **43**, 355–386, (1937).
3. W. Miehle, Link-length minimization in networks, *Operations Research* **6**, 232–243, (1958).

4. H.W. Kuhn and R.E. Kuenne, An efficient algorithm for the numerical solution of the generalized Weber problem in spatial economics, *Journal of Regional Science* **4**, 21–34, (1962).
5. L. Cooper, Location-allocation problems, *Operations Research* **11**, 37–52, (1963).
6. J. Brimberg and R.F. Love, Global convergence of a generalized iterative procedure for the minisum location problem with l_p distances, *Operations Research* **41**, 1153–1163, (1993).
7. H.W. Kuhn, A note on Fermat's problem, *Mathematical Programming* **4**, 98–107, (1973).
8. J. Brimberg and R.F. Love, Local convergence in a generalized Fermat-Weber problem, *Annals of Operations Research* **40**, 33–66, (1992).
9. R.F. Love and J.G. Morris, Modelling inter-city road distances by mathematical functions, *Operational Research Quarterly* **23**, 61–71, (1972).
10. R.F. Love and J.G. Morris, Mathematical models of road travel distances, *Management Science* **25**, 130–139, (1979).
11. J.G. Morris, Convergence of the Weiszfeld algorithm for Weber problems using a generalized distance function, *Operations Research* **29**, 37–48, (1981).
12. R. Chandrasekaran and A. Tamir, Open questions concerning Weiszfeld's algorithm for the Fermat-Weber location problem, *Mathematical Programming* **44**, 293–295, (1989).
13. J. Brimberg, The Fermat-Weber location problem revisited, *Mathematical Programming* **71**, 71–76, (1995).
14. R.F. Love, J.G. Morris and G.O. Wesolowsky, *Facilities Location: Models and Methods*, North-Holland, New York, (1988).
15. A.E. Taylor and W.R. Mann, *Advanced Calculus*, 2nd edition, Xerox College Publishing, Lexington, MA, (1972).