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SOLVING INFINITE p-CENTER PROBLEMS IN EUCLIDEAN SPACE USING AN INTERACTIVE GRAPHICAL METHOD

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Abstract—An algorithm is developed for the location of p service centers with one or more continuous demand areas. The method is based on the repeated solution of finite relaxation problems. Due to the infinite number of demand points distributed on arbitrarily shaped, bounded regions, an interactive computer graphical method is utilized. Here, the user has to determine the initial points to be included in the relaxation set, to inspect on the screen at each stage whether a displayed solution as demonstrated by circles covering the given demand region, is feasible, and if not, to choose a "good" demand point to be added to the relaxation set. Using a desktop PC, various problems were solved including locating optimally up to four centers over the area of Israel as a demand region.

Keywords: Center problems, interactive method, planar location model.

INTRODUCTION

One of the popular criteria used to optimally locate new centers which are to serve given demand points is the minimax criterion. Different versions of this problem appear in the literature with different metrics (Chen and Handler, 1987; Drezner, 1984a, b; Elzinga and Hearn, 1972; Elzinga *et al.*, 1976; Vijay, 1985; Watson-Gandy, 1984) and on networks (see, for example, Handler and Mirchandani, 1979) for single and multiple service centers to be optimally located. In the present work, we are dealing with the solution of the p-center location problem in Euclidean space. The term p-center is used for the problem of locating p identical centers such that the demand points are served each by its closest service center. We are concerned here with the uncapacitated version of the problem which means that any call for service at a demand point can immediately be served by its closest center. Chen and Handler (1987); and Vijay (1985) show that the problem is identical in the equi-weighted situation to that of covering all the demand points by p circles, the radius of the largest of which is to be minimized.

The minimax criterion is utilized in different situations such as when emergency facilities are located, in which case one may not be interested in minimizing the total cost but, rather, the best (fastest) service to the farthest customer is of prime importance. The practical application of this problem is that of positioning emergency centers when the effective distances are Euclidean, which is the case in transportation problems where the vehicle employed is, say, a helicopter or a boat. In fact, it has been shown that in problems related to the travel of motor vehicles on dense road networks the distances can be approximated as being proportional to the Euclidean distance (Love, 1988). Also included are the interesting problems of covering a certain area by p broadcasting stations, amplification stations used for the coverage of given areas by cellular phones and radar installations. Coverage of given regions by p alarm sirens or lamps are also problems of the same sort. Another important application is that of optimally locating and finding the minimal number of defensive missiles with a given range, in order to cover a given region. In this situation, the p-center problem is solved, as here, for $p = 1, 2, 3, \ldots$ until a value of p is found for which the "p-radius" r_p is not larger than the range of the defensive missile. The p-radius is defined for a problem with a finite number of demand points as

$$r_p = \min_{X_p \subset E^2} \max_i w_i \min_j r_{ij} \tag{1}$$

where $X_p = \{(x_j, y_j)\}_{j=1}^{p}$, $r_{ij} = [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2} \{(a_i, b_i)\}_{i=1}^{m}$ are the locations of the n demand points, with associated weights w_i, usually assumed to be equal (unity), and $\{(x_j, y_j)\}_{j=1}^{p}$, are the p service centers to be located.

A special class of minimax location problems in Euclidean space with general constraints and with a finite number of demand points has been suggested and solved by Brady and Rosenthal (1980) for a single facility, and by Brady, Rosenthal and Young (1983) for the multi-facility p-center version. These problems were solved to optimality by an interactive user-computer graphical method.

In the present work we extend the Chen and Handler (1987) method of optimally solving the p-center problem in Euclidean space to cases in which there is an infinite number of demand points, distributed over one or more continuous regions. In this case, the finite set $\{(a_i, b_i)\}_{i=1}^{n}$ is extended to include any point (a, b) in the demand area, and w_i should be replaced by a continuous weight function. Since the weights in the minimax problem are usually considered to be equal in the first place, we will omit the weights, replacing them by unity. The present problem is now formulated as

$$r_p = \min_{X_p \subset E^2} \max_{(a,b) \in C} \min_j r_j \tag{2}$$

where $r_j = [(a - x_j)^2 + (b - y_j)^2]^{1/2}$, and C is the demand region which is a subset of E^2.

THE RELAXATION METHOD

Chen and Handler (1987) developed a relaxation method for solving (1). They used an idea, carried over from the solution of the p-center problem on a network (Handler and Mirchandani, 1979), of solving iteratively the p-center problem for a sub-set of the demand points. Once a feasible solution to the relaxed problem is found, its feasibility to the full-size one is checked. If the full-size problem is not covered, a point farthest from its closest center is added to the relaxed problem and the procedure is repeated. If the full-size problem is covered, one has a feasible solution in hand, and the process continues to try and find a better one. The algorithm terminates when it is shown that no better solution of the relaxation problem can be reached. Some more details on the relaxation method for solving problems with a finite number of demand points are given in a concise way in a recent work by Chen and Handler (1993).

The present work describes the extension of the relaxation method to solving (2). It has been shown in an example by Handler and Rozman (1985) that an attempt to use the relaxation method developed for a finite number of demand points to the continuous case may result in convergence to a non-optimal solution. Their counter-example which consists of the location of two service centers on a single link network is, in fact, a counter-example in the present Euclidean case as well. These authors suggest a remedy which is a crucial modification of the original relaxation algorithm. As described below, the same idea (with some important modifications) is adapted in the present Euclidean distance problem. A special feature of the present procedure is its interactive nature, capitalizing on the efficiency of the human pattern recognition. At different stages, a human operator sees the problem (the region to be covered) on the screen as well as the circles in the incumbent solution. The operator has to make decisions which, at different stages, can be one of the following:

1. At the beginning, choose the points to be included in the initial relaxation problem.
2. Check and see whether the whole demand area is covered by the current solution circles.
3. If not, choose a "good" uncovered point to be added to the relaxation set.

Human pattern recognition seems to be of great help in these stages, in particular when an area which is not nicely defined geometrically is to be checked for coverage. It is to be emphasized, however, that the method is optimal, and the difference between "good" and "bad" user choices results only in the speed of convergence to optimality.

THE ALGORITHM FOR PROBLEMS WITH AN INFINITE NUMBER OF DEMAND POINTS

In this section we give the details of the new algorithm for finding the optimal coverage of the infinite problem using relaxations. The first two steps are identical to those in the finite problem algorithm (see Chen and Handler, 1987) and the third and fourth are different. It is to be mentioned that the present version, which is crucial in the solution of the infinite problem, can, in fact, be used for the finite problem as well, but was found to be relatively inefficient for that case.

Step 1

Find an upper bound $\bar{r}_p$. This can be achieved by geometrical considerations or, alternatively, the optimal solution for $p-1$ centers can be used as $\bar{r}_p$. Also, choose a set R of m points (a_i, b_i), $i = 1, \dots, m$ to be in the initial relaxation problem.

Step 2

As explained by Chen and Handler (1987), one should consider all $\binom{m}{3}$ circles with any 3 points (out of the finite relaxation set) on the circumference, $\binom{m}{2}$ circles with any 2 points at the ends of a diameter, and m degenerate circles of zero radius located at the m demand points of the relaxation problem. Exclude circles built on triplets of points forming an obtuse triangle [e.g. see p. 110 in Rademacher and Toeplitz (1957) or p. 118 in Love *et al.* (1988)] as well as circles with a radius in excess of the current $\bar{r}_p$.

Step 3

For any of the circles remaining in Step 2 with a radius $c_j < \bar{r}_p$ we form a (0, 1) vector $\mathbf{q}$ with m components (m is the number of points of the current R). If we denote by q_i the ith component of the vector $\mathbf{q}$, we define $q_i = 0$ if $r_{ij} > c_j$ and $q_i = 1$ if $r_{ij} \leqslant c_j$, where r_{ij} is the distance from the ith demand point to the center of the jth circle. The set of all the vectors $\mathbf{q}$ makes up the matrix $\mathbf{B}$. We now define a decision vector $\mathbf{z}$ composed of binary decision variables z_j, where $z_j = 0$ implies that the center of the jth circle is a candidate center in the current solution. We now have to solve the Covering Problem (CP)

$$\begin{aligned} h &= \min e^t\mathbf{z} \\ \text{s.t.}\ldots\ldots\ \mathbf{Bz} &\geqslant e \\ z_j &\in \{0, 1\} \\ e^t &= (1, \ldots, 1). \end{aligned} \tag{3}$$

The optimal value h is then the minimal number of centers required to cover all demand points in R within a maximal distance less than $\bar{r}_p$. If $h > p$, there is no solution to the present relaxation problem which indicates that the last existing solution to the full-size problem is optimal. Otherwise, we have a better coverage to the relaxation problem. At this point we have to distinguish between the finite and infinite problems. In many of the methods used for solving the set covering problems, including the one utilized in the present work (see below), the set covering solution is reached by going through some feasible solutions. As long as the finite problem was concerned, it was found possible, and in fact, very useful, to stop the set covering procedure once a feasible solution was reached and check whether this was a feasible solution to the full-size problem and proceed from there. As mentioned above, this may not result in a convergence to the optimum in the infinite problem. A proof has been given for the parallel problem on networks in Handler and Rozman (1985) where if each relaxation problem is solved to optimality, the process does converge to an optimum of the full-size problem. This proof is translated to the present case of Euclidean space in the Appendix. In the computation, this can be performed in one of the following ways. We repeat Step 3 as given for the finite problem several times, where at each stage, once we get a coverage accompanied by $\bar{r}_p = \max_{j \in J} c_j$ where $J = \{j \mid z_j = 1\}$), we delete from the matrix $\mathbf{B}$ all the columns related to circles with radii greater than or equal to $\bar{r}_p$ and continue until no cover for R can be found. A very attractive alternative, which was employed here, is to deal with the relaxation problem $R \subset E$ in hand as any finite p-center problem, and solve it to optimality using a second relaxation $R_l \subset R$. The double relaxation problem with the demand set R_l is solved at each stage only to the point of finding a feasible solution. This double relaxation approach was found to perform very successfully.

Step 4

In the solution of the problem with a finite number of demand points, we check in this step to see whether the h centers found in Step 3 cover the full-size problem with radius $\bar{r}_p$. This means that for all i, we check if $\max_i \min_j r_{ij} \leqslant \bar{r}_p$ and return to Step 3. If not, we choose an uncovered point from the original demand points to add to R, update $m = m + 1$ and return to Step 2.

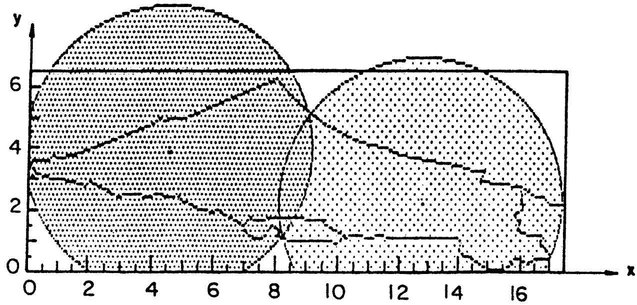

Fig. 1. The area of Israel, optimally covered by two centers.

Step 4 is changed in the present infinite algorithm in such a way that we, by inspecting the screen, check whether or not the *optimal* solution of Step 3 of the relaxed problem R is feasible to the full-size problem. If so, this is the optimal solution sought. If, however, the optimal solution to the first relaxation problem is not feasible to the full-size infinite problem, the user adds an uncovered demand point judged to be far from its closest center, and the algorithm returns to Step 2.

IMPLEMENTATION

The set-covering algorithm utilized here was that devised by Bellmore and Ratliff, as reported by Garfinkel and Nemhauser (1972). Although the results reported here are rather encouraging, it is possible that the use of more modern set covering methods (Balas and Ng, 1989; Sassano, 1989) would expedite the solution process, and enable the handling of larger problems. An IBM PC computer with 640K memory has been used. The program has been written in Turbo-Pascal Version 5, and included graphical software.

As pointed out before by Chen and Handler (1987, 1993), and as is rather typical of integer programming methods, the performance of the algorithm has been rather unpredictable. Contrary to what might be expected, no clear difference has been found between the behavior in the case of simple geometrical shapes such as a square and triangle on the one hand, and a "difficult" geometrical shape such as the map of Israel shown in Figs 1–3, and for the same number of service centers p. Upon further reflection, this may not be considered so strange since the map of Israel can be described as long and narrow. Such a shape may be a relatively easy case for the algorithm because the choice of centers to which a demand point can be assigned is limited to two neighbouring centers.[1] In further numerical experiments, a problem with a circular demand area turned out to be difficult to solve for $p > 2$. In any case, we depict here, for demonstration, the calculated results for the case of the map of Israel.

In the following examples, four points were chosen in the initial relaxation. The optimal solution found for the 2-center problem, shown in Fig. 1, was found with 16 points in the

[1]We would like to thank the anonymous referee who made this point.

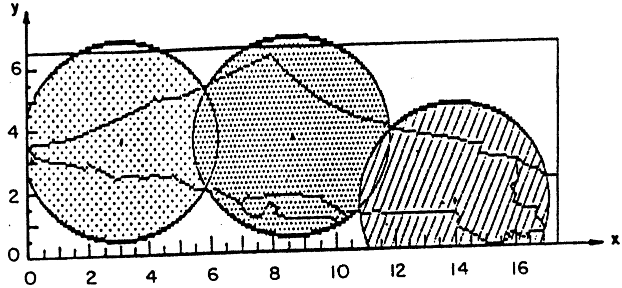

Fig. 2. The area of Israel, optimally covered by three centers.

final relaxation problem *R*. The centers are located at the points (12.77, 2.29) and (4.53, 3.96), and the value of the solution is $r_p = 4.63$. The total number of CP problems solved to reach the optimal solution was 161; the largest matrix had the size of 16 × 90. The total computation time was 8 min on the aforementioned PC computer.

Figure 2 shows the solution of the 3-center problem. The final relaxation problem *R* here included 26 points. The optimal locations of the centers were at (3.05, 3.64), (8.63, 3.62) and (13.98, 1.42), and the value of the solution is $r_p = 3.19$. The total number of CP problems solved here was 829; the largest matrix was of 43 × 245. The total computation time was 36 min.

The problem with four centers was optimally solved using 28 points in the final relaxation problem *R*. The optimal solution is depicted in Fig. 3. The optimal locations are now at (14.95, 2.01), (2.48, 3.74), (10.5, 2.37) and (7.19, 3.70), and the value of the solution is $r_p = 2.63$. The total number of CP problems solved here was 1108; the largest matrix was of 200 × 139. The total computation time was about 60 min.

It is to be mentioned that the set-covering program utilized in Garfinkel and Nemhauser (1971) produced all-integer cuts while looking for the solution. The number of cuts was

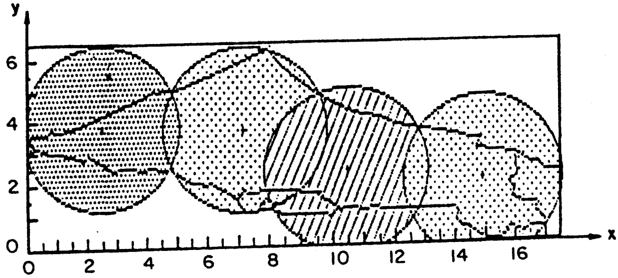

Fig. 3. The area of Israel, optimally covered by four centers.

usually very small (typically 0–2); however, at some unpredictable instances, very many cuts were produced before a solution could be found, which consumed too much of the computer resources. In the present program we define 100 cuts as "too many". In earlier versions, we stopped the run when this occurred. A very simple way was found which could overcome this difficulty in many cases. If the "too many cuts" situation occurs, a message to this effect is given to the user. The user retains the current upper bound $\bar{r}_p$, and chooses another relaxation which is subsequently solved. This happened while solving the 4-center problem, and has been overcome by choosing a different relaxation problem R. It appears that the good chances for by-passing the "too many cuts" situation by choosing a different relaxation are related to the rare occurrence of this undesirable situation.

In "difficult" problems of this kind, for example when a large number p of service centers are to be located and the optimal solution may be hard to find, one may be willing to accept a non-optimal solution. The acceptability of such a solution can be characterized by finding lower and upper bounds on the optimal solution. The optimal solution of a relaxation problem is, obviously, a lower bound $\tilde{r}_p$ on the optimal solution of the full-size (continuous in the present case) problem with the same number p of covering circles. Any coverage of the full-size problem is an upper bound on the optimal solution. At each stage, while checking for coverage, the program identifies the maximum distance r_{max} from a chosen set of demand points each to its closest center. With the incumbent centers, a set of p circles with a maximum radius r_{max} is viewed on the screen, and the user judges whether the whole area is covered. If so, r_{max} is the current upper bound. If not, r_{max} is increased until the p centers cover the whole demand area, and the new r_{max} is an upper bound. Once the upper and lower bounds are thus defined, the decision-maker may specify an acceptable error (say, 5%), and the program would calculate at each stage $100 \times (r_{max} - \tilde{r}_p)/\tilde{r}_p$, and stop if this is smaller than the allowed percentage error. In the examples above, however, the problems were solved to optimality.

DISCUSSION

It has been demonstrated that the infinite demand points problem can be solved for relatively "difficult" problems. Using the double relaxation technique and, where necessary, the change of relaxation when "too many cuts" occur, problems of covering the area of Israel by two, three and four centers were optimally solved. It was found that, usually, the difficulty in solving the problems did not depend on the complexity of the geometrical shape to be covered. Both in the cases of complex geometry and in "simpler" ones, it was found that the critical circles in the solution had approximately the same radii. This means that the p circles obtained had very similar radii r_p. This is in contrast to the case with a finite number of demand points where, typically, in an optimal solution, r_p, only the largest circle is "critical" whereas the other $p - 1$ circles could be shrunk without changing the optimality or the value of the solution. However, if the area to be covered consists of two or more disjoint continuous regions, the critical circles in the solution may not have the same radii.

A question may rise concerning the capability of the user to determine precisely whether a set of circles fully covers the given area. This problem has only a theoretical interest since, with the quite fine graphics used, the accuracy obtained has been adequate for all practical purposes.

It is quite obvious that the results discussed above are merely a demonstration of the applicability of the method. Much larger problems can be solved by using more modern CP covering methods, see for example, Balas and Ng (1989) and Rademacher and Toeplitz (1957), and, of course, more powerful computers. When a very large problem is to be solved, however, such that the optimal solution cannot be reached, the upper and lower bounds mentioned can yield a reasonably good solution.

An obvious further work that one can think of along the same lines is that of Chen and Handler (1993): the "conditional" location of p centers in which one optimally locates p centers to cover a given continuous region, when a number of existing centers are already distributed over the same area.

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APPENDIX

THEOREM. The modified relaxation algorithm, in which each relaxation problem is solved to optimality, converges to an optimal solution for the infinite *p*-center problem in Euclidean space.

Let us denote by r_p the optimal solution of the full-size problem, and by r_p^n the optimal solution of the relaxation problem where the relaxation set includes n demand points.

In order to prove the theorem, we will start with a set of lemmas.

LEMMA 1 $r_p^n \leq r_p^{n+1} \,\forall n$. While solving the CP (covering problem), we find an optimal solution for covering R. Obviously, adding a point to R may not decrease the solution. This lemma may not be true while using the previous algorithm, used for problems with finite number of demand points, in which at each stage, only a feasible solution of the relaxation problem is sought.

LEMMA 2 $\{r_p^n\}_{n=1}^{\infty}$ is a bounded sequence. In Step 1 of the algorithm we start with an upper bound $\bar{r}_p$. While looking for a solution, we do not take into consideration circles with radii $> \bar{r}_p$. Thus for all n, $r_p^n \leq \bar{r}_p$.

LEMMA 3. $\{r_p^n\}$ is a convergent sequence. By lemmas 1 and 2, $\{r_p^n\}_{n=1}^{\infty}$ is a nondecreasing and bounded from above sequence. Therefore, the sequence is convergent. Let us denote by $\mathbf{X}_p$ the vector of p centers. Let us define $l(\mathbf{X}_p) = \max_i \min_j r_{ij}$ where $(a_i, b_j) \in E^2$ $j = 1 \ldots p$. Let us define $l^n(\mathbf{X}_p) = l(\mathbf{X}_p)$ where R includes n points.

LEMMA 4. $r_p^n \leqslant r_p \leqslant l^n(\mathbf{X}_p)$. (a) $r_p^n \leqslant r_p$. It is obvious that the optimal solution for a relaxation problem (n points) is less than or equal to the optimal solution of the full-size problem.
(b) $r_p \leqslant l^n(\mathbf{X}_p)$. $l^n(\mathbf{X}_p)$ is a feasible solution to the full-size problem. Therefore, it is larger than or equal to the optimal solution of the full-size problem.

LEMMA 5. $\lim_{n\to\infty}(l^n(\mathbf{X}_p) - r_p^n) = 0$. Suppose, on the contrary, that $\lim_{n\to\infty}(l^n(\mathbf{X}_p) - r_p^n) = \delta > 0$, that is, we have a circle with a radius δ which is not covered by the circles found in the solution of the relaxation problem. Let us show that the number of times such an area can be left uncovered is finite. Let us denote the center of one of the uncovered circles with radius δ by pte. During the iterations, we add pte to R, and it will be covered in all the future iterations. In the next iteration, since pte is covered, part of the circle, the center of which is pte, is also covered, and therefore, according to the assumption above, there is another circle with radius δ which is not covered. By the assumption, at each iteration, at least one circle with radius δ remains uncovered. We thus have an infinite number of δ circles, as opposed to the original assumption that the area is bounded. The contradiction implies that the number of times $\delta > 0$ occurs is only finite, so that the original supposition cannot hold, and the lemma is established. We are now ready to complete the proof:

THEOREM. The modified relaxation algorithm converges to the optimal solution.

By lemma 5, $\lim_{n\to\infty}(l^n(\mathbf{X}_p)) = \lim_{n\to\infty} r_p^n$. By lemma 4, $r_p^n \leqslant r_p \leqslant l^n(\mathbf{X}_p)$, and therefore, $\lim_{n\to\infty} r_p^n = r_p$. This completes the proof of the theorem.