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AN IMPROVED METHOD FOR THE SOLUTION OF THE PROBLEM OF LOCATION ON AN INCLINED PLANE (*)

by Reuven CHEN [1]

Abstract. — *A method for solving the problem of minimizing the work expended in sliding blocks along an inclined plane to a central location is considered. Significant improvement in the iterative procedure is offered in cases where the centroid does not coincide with a demand point. Also, an alternative to the iterative procedure is suggested for cases of such coincidence. This approach adds an insight to the model and also helps solving the one dimensional centroid problem.*

Keywords : Location; inclined plane; centroid; log harvesting.

Résumé. — *Nous considérons une méthode visant à supprimer le travail nécessaire pour faire glisser des blocs le long d'un plan incliné jusqu'à un lieu central. Nous offrons une amélioration significative de la procédure itérative dans les cas où le centroïde ne coïncide pas avec un point de demande. Nous suggérons en outre une seconde procédure itérative pour les cas où une telle coïncidence se présente. Cette attaque permet de mieux approfondir le modèle, et apporte une aide à la résolution du problème du centroïde à une dimension.*

1. INTRODUCTION

An interesting problem of log harvesting which has to do with the skidding of logs on an inclined plane, has been formulated by Hodgson *et al.* [1], as follows. Given an inclined plane with an angle θ. The x axis is taken in the cross-slope direction and the y axis in the upslope direction. N blocks of masses m_i are located at demand points (a_i, b_i) for $i=1, \ldots, N$ and are to be slid to an unknown centroid (x, y). Given a friction coefficient μ, the total work expended to collect the logs at (x, y) is [1],

$$W(x, y)=\mu g \sum_{i=1}^{N} m_i \{ [(x-a_i)^2 \cos^2\theta + (y-b_i)^2]^{1/2} + (y-b_i) \tan\theta/\mu \}. \quad (1)$$

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(1) School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israël.

In order to minimize the total work, one solves

$$\min_{x,\, y} W(x, y). \tag{2}$$

For a horizontal plane ($\theta=0$), this reduces to the well known Weber-Fermat problem

$$\min_{x,\, y} \sum_{i=1}^{N} w_i R_i(x, y), \quad \text{where } R_i(x, y)=[(x-a_i)^2+(y-b_i)^2]^{1/2} \tag{3}$$

and where w_i are weights associated with the i-th demand point which are replaced in the present case by the masses m_i. This has been solved by Weiszfeld [2] and further investigated by Kuhn and Kuenne [3], Kuhn [4], and Ostresh [5, 6]. The iterative procedure for the solution consisted of steps such as

$$\begin{aligned} x_{r+1} &= \left[\sum_{i=1}^{N} w_i a_i / R_i(x_r, y_r)\right] \Big/ \left[\sum_{i=1}^{N} w_i / R_i(x_r, y_r)\right] \\ y_{r+1} &= \left[\sum_{i=1}^{N} w_i b_i / R_i(x_r, y_r)\right] \Big/ \left[\sum_{i=1}^{N} w_i / R_i(x_r, y_r)\right]. \end{aligned} \tag{4}$$

This is well defined, as long as (x_r, y_r) is not a demand point. The process can be initialized at different points, a popular one being the "center of gravity" (*see* e. g. equation 5 in reference [7]). When $(x_r, y_r)=(a_k, b_k)$ for some k, one defines the two dimensional vector

$$F_k = \begin{bmatrix} \sum_{i\neq k} w_i (x_r - a_i) / R_i(x_r, y_r) \\ \sum_{i\neq k} w_i (y_r - b_i) / R_i(x_r, y_r) \end{bmatrix}. \tag{5}$$

If $|F_k| < w_k$, the point (a_k, b_k), is the solution.

If $|F_k| \geqq w_k$, the incumbent solution should be pulled out of the "trap" at (a_k, b_k), and the iterative procedure continued as before, until a desired termination criterion is attained [3, 4].

For the present location problem on an inclined plane, Hodgson *et al.* [1] suggested the following variation of the Weiszfeld iterative process

$$x_{r+1}=\left[\sum_{i=1}^{N} m_i a_i / R_i(x_r, y_r, \theta)\right] / f(r, \theta)$$
$$y_{r+1}=\left\{\left[\sum_{i=1}^{N} m_i b_i / R_i(x_r, y_r, \theta)\right]-(\tan\theta/\mu)\sum_{i=1}^{N} m_i\right\} / f(r, \theta) \tag{6}$$

where

$$f(r, \theta)=\left[\sum_{i=1}^{N} m_i / R_i(x_r, y_r, \theta)\right] \quad \text{and} \quad R_i(x_r, y_r, \theta)$$

stands for

$$[(x_r-a_i)^2\cos^2\theta+(y_r-b_i)^2]^{1/2}.$$

For the case where $(x_r, y_r)=(a_k, b_k)$, they define

$$G_k=\left[\frac{\sum_{i\neq k} m_i(x_r-a_i)\cos^2\theta/R_i(x_r, y_r, \theta)}{\sum_{i\neq k} m_i(y_r-b_i)/R_i(x_r, y_r, \theta)+(\tan\theta/\mu)\sum_{i=1}^{N} m_i}\right] \tag{7}$$

and claim to have proven that the solution is at $(x_r, y_r)=(a_k, b_k)$ if $m_k>|G_k|$.

In the present work a variation of this method is proposed. In cases where the solution does not coincide with a demand point, the number of steps needed to reach a given termination criterion is reduced substantially with no additional complication in the performance of each step. A correction seems to be needed to the expression (7) for identification of the cases where the solution coincides with one of the demand points. A method is offered for identifying such a coincidence prior to the beginning of the iterative process. In addition, the insight achieved by the present approach can be used for a better understanding of the one dimensional problem. Also, the solution of the location-allocation problem can be accelerated since it consists of a repeated application of the single facility problem.

2. IMPROVING THE ITERATIVE PROCEDURE

It has been pointed out [3 – 6] that the original Weiszfeld method (4) is, in fact, a steepest descent method with a stepsize determined by the denominator of (4), namely $\sum_{i=1}^{N} w_i/R_i(x_r, y_r)$. In fact, equations (4) can be written as

$$
\begin{aligned}
x_{r+1} &= x_r - \sum_{i=1}^{N} [w_i(x_r - a_i)/R_i(x_r, y_r)]\Big/\left[\sum_{i=1}^{N} w_i/R_i(x_r, y_r)\right] \\
y_{r+1} &= y_r - \sum_{i=1}^{N} [w_i(y_r - b_i)/R_i(x_r, y_r)]\Big/\left[\sum_{i=1}^{N} w_i/R_i(x_r, y_r)\right]
\end{aligned}
\tag{4}
$$

which indeed gives (x_{r+1}, y_{r+1}) by going from (x_r, y_r) along the steepest descent direction with a stepsize defined by the mentioned denominator. Ostresh [5, 6] showed that a substantially faster convergence is reached by doubling the stepsize given in (4'). Chen [7, 8] gave a proof based on a work by Cohen [9] who had shown that the best step to be taken in a steepest descent method without search is

$$x_{p+1} = x_p - \theta_p \nabla W_p \tag{8}$$

where x is an n dimensional co-ordinate vector, ∇W_p is the gradient in the p-th iterative step and

$$\theta_p = 2/(\lambda_{\max} + \lambda_{\min}) \tag{9}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of the Hessian, respectively. Chen [7, 8] has pointed out that in the two dimensional problems, one always has

$$\lambda_{\min} + \lambda_{\max} = \partial^2 W/\partial x^2 + \partial^2 W/\partial y^2 \tag{10}$$

which yields in the Weber problem the iterative step

$$
\begin{aligned}
x_{r+1} &= x_r - 2\sum_{i=1}^{N} [w_i(x_r - a_i)/R_i(x_r, y_r)]\Big/\left[\sum_{i=1}^{N} w_i/R_i(x_r, y_r)\right] \\
y_{r+1} &= y_r - 2\sum_{i=1}^{N} [w_i(y_r - b_i)/R_i(x_r, y_r)]\Big/\left[\sum_{i=1}^{N} w_i/R_i(x_r, y_r)\right].
\end{aligned}
\tag{4''}
$$

In the Weber problem, this has been found to be, on the average, twice as fast as the original Weiszfeld method for solution points which do not coincide with a demand point.

This improvement of the Weiszfeld method may be extended to the case of the single facility location on an inclined plane in the following way. We shall consider the objective function $W_1(x, y) = W(x, y)/\mu g$. Repeating the procedure of finding the gradient and the sum of the diagonal terms in the Hessian, one gets the iterative step as

$$x_{r+1} = x_r - 2\left\{\sum_{i=1}^{N} m_i [\quad]^{-1/2}(x_r - a_i)\right\} \Big/ \left\{\sum_{i=1}^{N} m_i [\quad]^{-3/2} R_i^2\right\}$$

$$y_{r+1} = y_r - 2\left\{\sum_{i=1}^{N} m_i [\quad]^{-1/2}(y_r - b_i) + \left[\tan\theta \sum_{i=1}^{N} m_i/\mu\right]\right\} \Big/ q(r, \theta) \tag{11}$$

where

$$q(r, \theta) = \sum_{i=1}^{N} m_i \cos^2\theta [\quad]^{-3/2} R_i^2 (x_r, y_r),$$

and where [ ] stands for

$$[(x_r - a_i)^2 \cos^2\theta + (y_r - b_i)^2].$$

As shown in the table given below, the solutions of problems on the inclined plane were reached on the average in less than half the number of steps as by the method given by Hodgson *et al.* [1], for solutions not coinciding with demand points. It has also been found that the Hodgson method was superior for the convergence of the process when such coincidence occurs. However, as shown below, such solutions can be separately identified rather easily.

3. IDENTIFICATION OF OPTIMA COINCIDING WITH VERTICES

It is quite obvious that all the versions of the iterative equations have a basic difficulty when the minimum point (x, y) coincides with one of the demand points (a_k, b_k). In fact, it is quite surprising that the Weiszfeld method and its extension by Hodgson *et al.*, perform so well in the vicinity of the solution at (a_k, b_k). It seems, however, by far superior, to identifiy at the outset this situation rather than to resort to the iterative procedure. To do so, instead of providing a formal rigorous proof, we follow an approach mentioned by Kuhn [3]. Let us consider "pseudo-forces" which are in fact the components of the gradient related to the different given points (a_i, b_i). Intuitively, in the Weber problem on the horizontal plane these represent the "pulling effect" that each demand point exerts on the location (x, y). A demand point (a_k, b_k) would be the optimum location if in a circular δ vicinity of the point, its pulling force is larger than the resultant of the pulling forces of the other points. As mentioned above, in the case of the radial functions in the original Weber problem, the condition is $m_k > |F_k|$ where F_k is the gradient vector (5). In the present case of the inclined plane, the deviation from this situation of radial cost functions is seen both in the $\cos^2\theta$ term multiplying $(x-a_i)^2$ and the $\tan\theta$ term in equation (1). Due to this deviation from radial behaviour, it seems incorrect to compare m_k to $|G_k|$ as defined by Hodgson *et al.*, and in fact, it is rather easy to show an example where $m_k < |G_k|$ whereas (a_k, b_k) is still the solution. In order to replace this condition by a correct one, we resort to a transformation used by Hodgson *et al.* in their proof of convergence, *i.e.*

$$\hat{x} = x\cos\theta, \qquad \hat{a}_i = a_i\cos\theta \tag{12}$$

which changes the objective function to

$$\min \hat{W}(\hat{x}, y) = \sum_i m_i [(\hat{x}-\hat{a}_i)^2 + (y-b_i)^2]^{1/2} + (\tan\theta/\mu)\sum_i m_i (y-b_i). \tag{13}$$

This transformed expression is radial only as far as the first term is concerned. At the k-th point, the gradient of the other $N-1$ points is

$$\hat{G}_k = \begin{bmatrix} \sum_{i\neq k} m_i (\hat{a}_k - \hat{a}_i)/R_{ik} \\ \sum_{i\neq k} m_i (b_k - b_i)/R_{ik} + (\tan\theta/\mu)\sum_i m_i \end{bmatrix} \tag{14}$$

where

$$R_{ik} = [(\hat{a}_k - \hat{a}_i)^2 + (b_k - b_i)^2]^{1/2}.$$

Due to the radial nature of the transformed problem, Kuhn's condition can be applied, namely, m_k should be compared to the absolute value of the resultant of the pulling pseudo-forces of the other demand points.

It is to be noted, however, that these pseudo-forces consist of the radial components of the $N-1$ other demand points (denoted in the sum of $i\neq k$), as well as $(\tan\theta/\mu)\sum_{i=1}^{N} m_i$ which includes the k-th point. The external pseudo-force includes in this case the gravitation related components of *all* the demand points. The condition for the solution to coincide with the k-th demand point is therefore

$$m_k > |\hat{G}_k|. \qquad (15)$$

$\hat{G}_k$ can be written more explicitly as

$$\hat{G}_k = \begin{bmatrix} \sum_{i \neq k} m_i (a_k - a_i) \cos\theta / R_{ik} \\ \sum_{i \neq k} m_i (b_k - b_i)/R_{ik} + (\tan\theta/\mu) \sum_i m_i \end{bmatrix} \qquad (14')$$

where $R_{ik} = [(a_k - a_i)^2 \cos^2\theta + (b_k - b_i)^2]^{1/2}$.

Apart from the trivial division by the constant $g\mu$, the expression (14′), the absolute value of which has to be compared with m_k, differs from the expression by Hodgson *et al.* only by $\cos\theta$ appearing in the x component rather than $\cos^2\theta$ as given by them. Numerical results confirm that the condition (15) is indeed the correct one.

The result given here is obviously in variance with that given by HWH. It appears that the incorrect element is in the proof of their theorem 2 where $g\mu m_k$ is to be replaced by $g\mu m_k (z_1^2 \cos^2\theta + z_2^2)^{1/2}$. This term is, obviously, direction dependent, and therefore, it seems very difficult to derive from it a simple condition for the coincidence of the solution and a demand point.

4. NUMERICAL RESULTS

Similarly to Hodgson *et al.*, a problem with 20 demand points randomly distributed on a 100×100 square has been solved with the same friction coefficient and the same angles as reported by them. The demand points were not the same as in their example. Similarly to their case, in the instances with relatively high θ, the solution occurred outside the 100×100 square, at points with negative y values. The problems were solved both by the method given by Hodgson *et al.*, (HWH column in the table) and by the present method. It can readily be seen that the number of steps is better than halved, on the average, by using the present method whereas the computational effort per step is unchanged. Concerning the time factor, it is reduced on the average by about 25% only since each iteration is preceded by a check

TABLE

μ	θ°	Location		HWH		Present	
		x	y	Time (ms)	Steps	Time (ms)	Steps
0.1	0.	41.00	50.00	17	19	1	0
	1.085	40.66	38.93	16	18	13	7
	2.170	39.65	28.55	17	21	15	11
	3.255	43.15	17.34	19	28	15	15
	4.340	46.27	7.29	18	26	17	20
	5.425	48.78	−50.39	38	89	27	46
0.3	3.173	40.70	39.22	16	18	13	7
	6.346	39.65	28.98	17	21	14	11
	9.519	42.75	18.32	19	27	15	15
	12.691	46.61	8.34	26	50	20	28
	15.864	48.74	−45.15	38	86	24	41
0.5	5.047	40.76	39.73	16	18	13	7
	10.095	39.66	29.75	17	21	13	10
	15.142	42.09	19.95	18	24	15	14
	20.189	47.00	9.00	17	20	2	0
	25.237	48.67	−36.63	36	80	22	34
0.7	6.648	40.83	40.37	16	18	13	7
	13.297	39.66	29.75	16	20	14	10
	19.945	41.33	21.79	16	18	15	14
	26.594	47.00	9.00	17	20	2	0
	33.242	48.57	−27.24	33	73	19	26
0.9	7.978	40.90	41.06	16	17	12	7
	15.955	39.83	31.79	17	20	13	10
	23.933	40.64	23.52	17	21	15	14
	31.910	47.00	9.00	34	72	2	0
	39.880	48.46	−18.54	32	67	17	19

whether each of the 20 demand points happens to be the optimum. In cases of such coincidence, however, denoted by 0 steps in the table, the gain in time is very significant. Thus, in this example, all cases included, the gain in time is about 30% as compared to HWH. This gain in the number of steps, as well as computation time should be significant when the location-allocation problem is to be solved. The obvious reason is that in most cases, this problem is solved by a repeated solution of the single facility one.

5. THE ONE DIMENSIONAL PROBLEM

Hodgson *et al.* have also treated the location problem where all the logs are located along the y-axis. They gave an example of nine logs thus located and the angles (with $\mu=0.5$) for having a multiple solution, namely, the angles at which any intermediate location between two given adjacent logs is the solution. Using the present ideas, this can be generalized as follows. It is obvious that in the gradient [equation (7) with correction], the x component is nil whereas the y component reduces to

$$G_y = \sum_i m_i \operatorname{sign}(y_r - b_i) + (\tan\theta/\mu) \sum_i m_i. \tag{16}$$

This is so since $R_i(x_r, y_r, \theta)$ reduces here to $|y_r - b_i|$. $|G_y|$ has to be compared with the value of m_k and the location of m_k is the solution if

$$m_k > \Big| \sum_{i \neq k} m_i \operatorname{sign}(b_k - b_i) + (\tan\theta/\mu) \sum_i m_i \Big|. \tag{17}$$

In cases where m_k equals the right hand side, then if the resultant pseudo-force involved is positive, the whole range between the point k and its nearest neighbour upwards is a solution. If equality holds and the resultant is negative, the range between the k-th point and its down-slope neighbour is the solution. The reason is that we have a whole range in which the total gradient is zero. The cases given by HWH are seen to be included in the special case of $\mu = 0.5$, and m_i being all equal. Note that in this one dimensional case, changing the position of any log without varying $\operatorname{sign}(y_r - b_i)$ [or $\operatorname{sign}(b_k - b_i)$], does not change the result since it does not alter expressions (16) and (17).

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