Trans. Sci. 22, 157-161 (1988)

Thus a lower objective function value is achieved by assigning the destination i to hub k instead of hub t if

$$\left(\sum_{j\in K} W_{ij} + \sum_{p\in T} W_{ip} + \sum_{\substack{r\neq k\\ \neq t}} \sum_{l\in Z_r} W_{il}\right) \times (C_{ik} - C_{it}) + \left(\sum_{p\in T} W_{ip} - \sum_{j\in K} W_{ij}\right)\underline{C}_{kt} + \sum_{\substack{r\neq k\\ \neq t}} \sum_{l\in Z_r} W_{il}(\underline{C}_{kr} - \underline{C}_{tr}) \leqslant 0 \qquad (A3)$$

where K and T are used in place of K' and T' since $W_{ii} = 0$.

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Conditional Minisum and Minimax Location-Allocation Problems in Euclidean Space

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The problems of minimax and minisum location-allocation in two-dimensional Euclidean space, where some fixed service centers already exist in the area in question, are treated. The method utilized is an extension to a previously reported algorithm for the solution of the unconditional problem and yields good local minima. In the minisum problem this is at the moment the only feasible way to obtain any solutions. In the minimax case, a method for finding optimal solutions has been developed in parallel. However, the latter can yield results only to problems of limited size. A possible combination of the two methods is suggested.

INTRODUCTION

The problem of the simultaneous location of a number of identical service facilities, with a given set of demand points distributed over a given area, has been investigated by many researchers (e.g., see [2, 5, 7–9, 11, 12]). Two versions of the problem have been dealt with, the minisum and the minimax problems. Problems of this kind have been solved on networks[13] and in continuous two-dimensional space. In the minisum version, only one work, by KUENNE and SOLAND[10] proposes an optimal solution approach to the multicenter problem in Euclidean space, limited to relatively small problems. DREZNER[6] has recently given another method which is limited to two service centers but can handle a large number of demand points.

As for minimax problems, a number of methods have recently been suggested for their optimal solution.[3,6,14,15] Two of these are quite efficient for a large number of demand points[3] and of service centers.[14]

All the other methods suggested in the literature yield solutions which in most cases are local minima. The best one can do is start the iterative procedure from different initial points and choose the best final result attained. At the moment, this may be the only feasible route to take for large minimax and for all minisum problems. A work by the present author[2] gave one such method, usable for both minimax and minisum problems. The essence of the method is the utilization of a differentiable approximation to the problems which are originally not everywhere differentiable. The approximation is then solved by using a powerful nonlinear programming technique. Some additional details of this method are discussed below since the subject matter of the present work is an adaptation of the same method for conditional problems.

The term "conditional" location has first been used by MINIEKA.[13] This pertains to the very realistic situation in which a number of service centers are already located among given demand points and the decision maker is considering the construction of some additional centers without altering the positions of the fixed original ones. Minieka solved this problem on graphs and with one additional center only. The present work seems to be the first to deal with the problem in Euclidean space. In parallel, an approach has been proposed by CHEN and HANDLER[4] which optimally solves the minimax conditional problem; this can be utilized at present to solve moderate size minimax problems.

SOLUTION OF THE MINISUM CONDITIONAL PROBLEM

THE LOCATION-ALLOCATION unconditional minisum problem is formulated as[2]

$$\min_{x_J,y_J} \sum_{i=1}^{n} w_i \min_j [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}, \quad J = 1, \ldots, m \tag{1}$$

where (a_i, b_i), $i = 1 \ldots n$ are the given demand points, (x_j, y_j), $j = 1 \ldots m$ are the service centers to be optimally located, and w_i are the weights associated with the demand points. The meaning of (1) is that $\min_j$ selects for each demand point the closest service facility, $\sum_{i=1}^{n}$ sums all the weighted Euclidean distances and $\min_{x_J,y_J}$ means that the minimization is over the $2m$ variables $x_1, y_1, \ldots, x_m, y_m$. Thus, in the unconditional problem both j and J have the same span, from 1 to m. This is the main difference between the conditional problem and the unconditional one.[2] Out of the m service facilities we denote the number of variable centers by k, therefore, the number of fixed service centers is $m - k$. The formulation of the conditional minisum problem is then only a slight modification of that of the unconditional one,

$$\min_{x_J,y_J} \sum_{i=1}^{n} w_i \min_{j=1\ldots m} [(a_i - y_j)^2 + (b_i - y_j)^2]^{1/2}, \quad J = 1, \ldots, k \tag{2}$$

the difference being that now j goes from 1 to m whereas J from 1 to k. Let us denote by r_{ij} the Euclidean distance between the ith demand point and the jth service center, namely, $r_{ij} = [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}$. If we choose a large enough number N we get a good approximation for (2) by solving (see also [1])

$$\min_{x_J,y_J} \sum_{i=1}^{n} w_i [\sum_{j=1}^{m} r_{ij}^{-N}]^{-1/N}, \quad J = 1, \ldots, k. \tag{3}$$

This is a differentiable function in the $2k$ variables x_1, $y_1, \ldots, x_k, y_k$ and can be minimized using standard nonlinear programming methods. The "large" number N has been chosen to be 200 and for the minimization, a quasi-Newton method, the Broyden-Fletcher-Shanno (BFS) one has been utilized as previously with the unconditional problem.[2] Even for one variable service center addition to the given fixed centers, the problem is not convex.

All the problems solved involved equiweighted demand points distributed in random over a 100 × 100 square, with three fixed service centers located at (10, 10), (10, 90), (90, 10). All the runs reported have been performed on the Tel Aviv University Cyber 170-855 CDC computer. The results are shown in Table I. The table gives four solutions of the two-additional-centers problem. Four different results are found but they are all within 1.1% from each other. The next entries in the table are 3-, 4- and 5-centers problems (two of each). The difference between the values of the local minima obtained are 5.5%, 0.4% and 3.8% in the 3-, 4- and 5-centers problems, respectively. The last entry in the table refers to a large problem, with seven variable centers in addition to the three fixed ones.

SOLUTION OF THE MINIMAX CONDITIONAL PROBLEM

THE CONDITIONAL location-allocation minimax problem can be stated as

$$\min_{x_J,y_J} \max_i w_i \{\min_j [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}\} \quad J = 1, \ldots, k \tag{4}$$

where (a_i, b_i), (x_j, y_j) and w_i have the same meaning as in the previous section. As shown in [2], a differentiable approximation to (4) can be written, namely,

$$\min_{x_J,y_J} \sum_{i=1}^{n} w_i \{\sum_{j=1}^{m} [(a_i - x_j)^2 + (b_i - y_j)^2]^{-N/2}\}^{-1} \quad J = 1, \ldots, k. \tag{5}$$

In the problems solved, the same demand points as in Section 2 are distributed in random on a 100 × 100 square and the three fixed service centers are located again at (10, 10), (10, 90), (90, 10). Problems of 30, 40, 50, 100 and 200 demand points have been solved each with 1, 2, ..., 6 variable centers. Some of these problems have also been solved with a method capable of finding optimal solutions.[4] The comparison of the results with the optimal ones (attained for relatively small problems only) provides insight to the question of how close to the optimal solution one can get by using the method proposed here.

The results of some representative problems are summarized in Table II. The demand points have been chosen in such a way that the set of, say, 30 demand points is a subset of the set of 40 demand points, which in turn is a subset of the 50 demand points problem. The entries in the table indicate the final values of r_p, the solution of (5). An exclamation mark means that the solution is known (from [4]) to be a global optimum. In other cases, the number given with the percentage sign denotes the deviation from the

TABLE I

Results of Some Conditional Minisum Problems with 100 Demand Points and 3 Fixed Centers at (10, 10), (10, 90), (90, 10). F Is the Value of the Objective Function

No. of Additional Centers	Initial Guess	Solution	F	CPU Time (sec)
1	(60, 60)	(65, 67)	2349	1.288
1	(90, 90)	(65, 67)	2349	1.312
2	(40, 40), (80, 80)	(40.77, 49.59), (74.60, 70.22)	1970	0.845
2	(40, 80), (80, 40)	(69.67, 75.20), (53.95, 35.51)	1983	1.439
2	(90, 20), (20, 90)	(51.69, 29.31), (71.80, 70.28)	1984	1.051
2	(50, 50), (90, 90)	(34.64, 53.24), (74.30, 68.86)	1963	1.174
3	(30, 60), (60, 30) (80, 80)	(31.32, 56.01), (63.12, 28.08) (70.97, 76.14)	1656	1.633
3	(90, 60), (60, 90), (40, 40)	(77, 67) (49.91, 90.26) (51.92, 31.70)	1748	3.918
4	(50, 20), (20, 50) (90, 50), (70, 90)	(55.70, 20.59), (31.32, 56.01) (81.17, 56.66), (63.10, 84.43)	1451	2.879
4	(20, 20), (20, 80) (80, 20), (20, 80)	(55.70, 20.59), (31.32, 56.01) (79.52, 59.22), (59.57, 88.45)	1445	3.156
5	(40, 90), (90, 90), (50, 50) (40, 40), (90, 40)	(43.37, 94.01), (74.66, 73.10) (31.32, 56.01), (55.70, 20.59) (80.36, 75.55)	1310	4.386
5	(50, 25), (50, 75), (50, 50) (25, 50), (25, 75)	(55.65, 19.42), (78.19, 64.74) (41.18, 55.55), (10.92, 50.98) (53.03, 91.09)	1263	4.421
7	(50, 10), (10, 50), (50, 90) (90, 50), (85, 85), (40, 40) (60, 60)	(55.55, 19.42), (10.92, 50.98) (48.53, 94.04), (80.36, 47.44) (81, 83), (40.28, 56.10) (73.36, 67.67)	1090	19.368

TABLE II

Conditional Minimax Location-Allocation Problems: Computational Results

Centers	Points				
	30	40	50	100	200
1	36.12 (2.1%) (0.65 sec)	36.46! (0.82 sec)	36.46! (0.96 sec)	39.12! (2.32 sec)	41.23! (2.34 sec)
2	30.15 (1.0%) (0.68 sec)	30.15 (1.0%) (1.14 sec)	30.15 (1.0%) (1.33 sec)	32.56 (3.2%) (2.19 sec)	33.53 (0.18%) (4.19 sec)
3	30.15 (31.0%) (1.13 sec)	30.15 (31.0%) (1.74 sec)	30.15 (18.3%) (2.08 sec)	30.81 (2.2%) (4.28 sec)	31.62 (4.9%) (8.71 sec)
4	19.53 (1.5%) (1.43 sec)	22.36! (2.03 sec)	22.36!!! (2.15 sec)	27.86 (5.9%) (4.31 sec)	31.62 (20.2%) (10.91 sec)
5	19.235! (1.17 sec)	22.36 (13.5%) (1.72 sec)	22.36 (5.24 sec)	29.09 (4.90 sec)	31.62 (31.5%) (8.84 sec)
6	19.235 (11.1%) (1.64 sec)	22.36 (16.2%) (1.72 sec)	22.36 (5.25 sec)	31.62 (9.58 sec)	27.66 (8.44 sec)

global optimum when the latter is known. The results of the problem with 50 demand points and four additional centers is of special interest. The optimal method[4] behaved in a somewhat capricous manner and in this particular case, did not reach the global minimum and stopped at a feasible solution with $r_p = 22.69$. The present result of $r_p = 22.36$ is thus better and therefore the optimum should be $r_p \leq 22.36$. Furthermore, since the optimal solution of the 40-point problem is the same, it is obvious that in the 50-point problem $r_p \geq 22.36$ since the 40-point set constitutes a subset of the 50 points. The conclusion that $r_p = 22.36$ is the optimum value could have been reached only through a combination of the two methods.

DISCUSSION

A METHOD has been developed for the solution of minisum and minimax conditional location-allocation problems. The main importance of this nonoptimal method can be summed up as follows:

1. The proposed algorithm is the only one available for solving the minisum conditional problem. In fact, the situation with the unconditional problem is not much better: as indicated above, a number of nonoptimal methods exist for solving the unconditional problem, but the only optimal methods known[6,10] are limited to small problems.
2. In the minimax problem where an optimal alternative exists[4] the method proposed here makes possible the solution of larger problems, not manageable by the optimal method. The solution of weighted problems can also be carried out, both in the minimax and minisum cases.
3. For the minimax problem a combination of the heuristic method proposed here and of the optimal method is strongly recommended. One such combination has been described above for the optimal solution of the problem with 50 demand points and four service centers. Another combination of the two methods can be effected as follows. The optimal method is based on finding feasible solutions and trying to improve them. The method proposed here provides us with feasible solutions even for relatively large problems. Such feasible solutions can be used as a good starting point for a further improvement and possibly proof of optimality for large problems using the relaxation method.[4]

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