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phenomena. Introducing an arbitrary convention is surely accompanied by the loss of simplicity and sense.

[1]B. Townsend, Am. J. Phys. 51, 1092 (1983).

[2]H. Erlichson, private communication and Am. J. Phys. 53, 53 (1985).

[3]A. Einstein, *The Principle of Relativity* (Dover, New York, 1952), p. 39.

[4]H. Reichenbach, *The Philosophy of Space and Time* (Dover, New York, 1957), p. 132.

Optimal location of a service facility as a problem in basic mechanics

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The economic problem of locating optimally a service facility in a two-dimensional Euclidean space is equivalent to a problem of equilibrium between forces in statics. A mechanical analog, suggested at the beginning of the century, as well as more modern numerical methods for the solution of the problem, are discussed. The multifacility problem, which is an extension to the original one, is also mentioned. Cases in which the cost involved is nonlinearly dependent on the Euclidean distance are briefly referred to.

I. INTRODUCTION

The modern economic problem of the optimal location of one or several service facilities with respect to a given set of "demand" points (i.e., points that are to be served by one or more of the service facilities) has emerged from an old geometrical problem. The problem, first posed by Fermat in the 17th century and solved by Steiner in the 19th century, is the following: Given three points, find a point such that the sum of distances to the given points is minimal. The solution is (e.g., see Eilon *et al.*,[1] p. 39) that if all the angles in the triangle formed by the three given points are smaller than 120°, the optimal location is the point from which lines, drawn radially to each demand point, form three angles of 120°. If the triangle contains an angle greater than or equal to 120°, then the optimal location is at that vertex. Two extensions to this problem are to be considered. One is that several points are to be served by a single service facility and the other is that each pont is assigned a certain "weight." The problem will now be to minimize the cost function

$$f(x,y) = \sum_{i=1}^{m} w_i r_i(x,y), \tag{1}$$

where m is the number of given points, w_i is the weight of the ith point and $r_i(x,y) = [(x-x_i)^2 + (y-y_i)^2]^{1/2}$, the Euclidean distance between the ith point, having coordinates (x_i,y_i), and the point to be located (x,y). This problem, usually known as the Weber[2] problem, is considered to be a reasonably good model for the practical problem of situating a service facility so that the total travel cost is minimum. The weight involved reflects the economic importance of the demand point; in many cases it is proportional to the population in the ith site. The model is good in particular when the means of transportation can move in a straight line, e.g., boats, helicopters, or airplanes. The problem is entirely different when the transportation is along given roads; however, Love and Morris[3] showed that the Euclidean distance is quite a good measure to the road distance in a dense network of roads.

The possible nonlinearity of the cost with the radial distance has been considered by Cooper[4] who discussed the solution of the minimization of

$$f(x,y) = \sum_{i=1}^{m} w_i r_i^n, \tag{2}$$

which includes either superlinearity of the cost dependence on the distance for $n > 1$, sublinearity for $n < 1$, or the previous case of linearity for $n = 1$. Even more general cost functions have been considered by Katz[5] who took into account different cost functions for different "customers," $\phi_i(r_i)$, where ϕ_i are nondecreasing differentiable and convex (having a positive second derivative) functions of the radial distances r_i, so that the objective function to be minimized is

$$f(x,y) = \sum_{i=1}^{m} \phi_i(r_i). \tag{3}$$

Different generalizations to the original Weber problem have also been discussed in the literature. These include the case where the transportation is only on a given network (of roads) and that where the L_P norm is considered, namely, when minimization is of

$$f(x,y) = \sum_{i=1}^{m} w_i(|x-x_i|^P + |y-y_i|^P)^{1/P}, \tag{4}$$

the rectilinear norm being a special case in which $P = 1$.

In the present paper we deal with the solution of the radial cases, i.e., Eqs. (1)–(3) and the mechanical analog which can add much insight to the properties of the solution.

II. SOLUTION OF THE WEBER PROBLEM

In order to study the properties of the solution, the first derivatives of Eq. (1) are to be written, namely,

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{m} w_i(x - x_i)/r_i, \quad (5a)$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{m} w_i(y - y_i)/r_i. \quad (5b)$$

It is obvious that a necessary condition for a point (x,y) to be a solution of the problem is that the sums in Eqs. (5) are equal to zero. It should be noted that the two-dimensional function in Eq. (1) has the property of convexity (e.g., see Ref. 1, p. 45) which means that the Hessian (the matrix of second derivatives) is positive definite. The important property of a convex function is that if it has a local minimum, it is necessarily a global one. Thus our main aim is to find a point in which $\partial f/\partial x = \partial f/\partial y = 0$ which will automatically be the only solution of our problem.

Setting Eqs. (5a) and (5b) to zero yields

$$x = \sum_{i=1}^{m} [w_i x_i / r_i(x,y)] \Big/ \sum_{i=1}^{m} [w_i / r_i(x,y)], \quad (6a)$$

$$y = \sum_{i=1}^{m} [w_i y_i / r_i(x,y)] \Big/ \sum_{i=1}^{m} [w_i / r_i(x,y)]. \quad (6b)$$

Of course, these equations cannot be solved in closed form. Weiszfeld[6] suggested an iterative process based on Eqs. (6). He wrote the Nth step in the iteration

$$x^{N+1} = \sum_{i=1}^{m} [w_i x_i / r_i(x^N,y^N)] \Big/ \sum_{i=1}^{m} [w_i / r_i(x^N, y^N)], \quad (7a)$$

$$y^{N+1} = \sum_{i=1}^{m} [w_i y_i / r_i(x^N,y^N)] \Big/ \sum_{i=1}^{m} [w_i / r_i(x^N, y^N)]. \quad (7b)$$

As a first guess for the solution, the "center of gravity" point is usually taken, i.e.,

$$x^0 = \sum_{i=1}^{m} w_i x_i \Big/ \sum_{i=1}^{m} w_i; \quad y^0 = \sum_{i=1}^{m} w_i y_i \Big/ \sum_{i=1}^{m} w_i. \quad (8)$$

It has been pointed out in the literature[7] that Weiszfeld's method is merely the "steepest descent" method with a step size determined by the denominator in Eqs. (7). In other words, the process consists of going "downhill" along $-\operatorname{grad} f$.

This brings to mind the consideration of the mechanical analog in which $f(x,y)$ in (1) is looked upon as a scalar field analogous to the potential in a gravitational or an electric field. The first derivatives in Eqs. (5) can be looked upon as the x and y components of the resultant force acting on the service point which is to be located optimally. Let us consider the contributions of the ith demand point to the expressions in Eqs. (5). If we denote by θ_i the angle from the horizontal at which an observer at (x,y) views (x_i,y_i), the contributions of the ith point are $w_i \cos\theta_i$, $w_i \sin\theta_i$ to the x and y components of the gradient (see Fig. 1). From the "physical" point of view, the problem can be considered as follows. A "particle" is placed in a force field having m attracting points distributed in the two-dimensional space. The particle is attracted to each of the given points, the attraction toward the ith point being proportional to the weight of the point w_i. The direction of the force is along the line connecting the particle and the ith point. The strength of this force element is *not* dependent on the distance between the particle and the demand point. The problem is that of finding a stationary point for the particle in this field. As opposed to the cases of gravitational and electrostatic fields where no such stationary point exists,[8] a minimum point always exists in the present circumstances.[9,10]

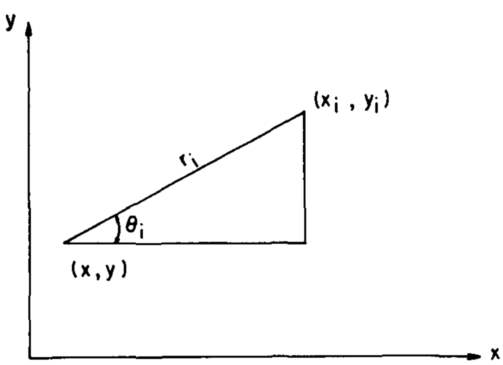

Fig. 1. The ith demand point (x_i, y_i) as seen from a point (x, y). r_i is the Euclidean distance and θ_i is the angle between the horizontal and the line connecting the two points.

It is to be noted that one of the possibilities for the solution is that it coincides with one of the demand points. This may cause some difficulty. The cost function, Eq. (1), is not differentiable at the demand points, (x_i,y_i) for $i = l,...,m$. The ways to bypass the difficulties involved in the Weiszfeld iterative procedure when the solution should coincide with a demand point or when one of the intermediate points in the iterative process happens to be the same as one of (x_i,y_i), have been discussed in the literature.[5,9] In fact, Katz[5] suggested to check first each of the demand points for optimality before proceeding to the Weiszfeld iterations, and that only if none of the demand points is found to be optimal should the iterative process be initiated. We shall reconsider this point in the framework of the physical picture below.

III. THE MECHANICAL ANALOG

As early as 1909, Pick suggested the following analog device in an introduction to Weber's book.[2] Although his approach was mainly intuitive, the rationale behind it was practically the same as that mentioned above of the potential field and force concepts. The mechanical analog is shown in Fig. 2. A map of the area in question is placed on a board and holes are drilled in the points denoting the demand locations. Strings are passed through the holes and weights proportional to the economic "weights" are hung on them. The other edges of the strings are tied together. In view of the above explanation, it is quite obvious that the stationary situation reached after possibly a few oscillations is the equilibrium point, namely, the solution of the minimization problem. Of course, the accuracy of the method is limited by the friction of the strings in the holes and anyway, it looks quite primitive when the alternative of an efficient numerical procedure is available. The mechanical analog gives, however, an insight to the properties of the solution.

As explained above the "force" elements due to each of the demand points are radial in direction, but independent of the distance between the demand point and the location of the service facility. An interesting implication of this point is that once the location of the solution is known, it is, in fact, also the solution of other problems in which each demand point can be anywhere along the line connecting the service facility point and the demand point. In the me-

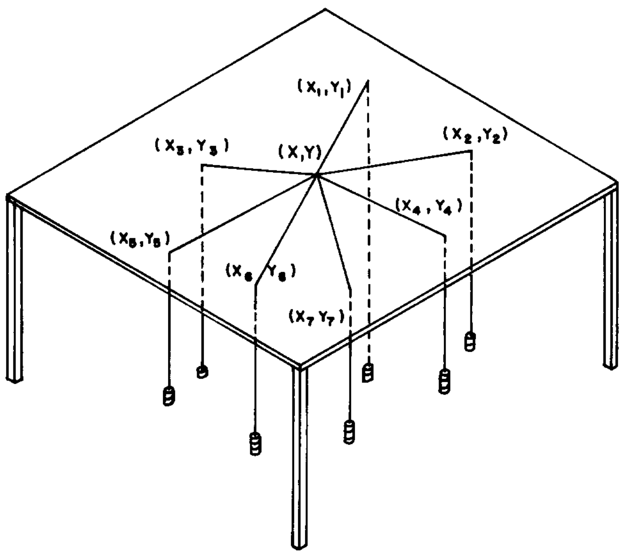

Fig. 2. Mechanical analog model of the Euclidean problem [after Eilon *et al.* (Ref. 1) and Francis and White (Ref. 10)].

chanical analog, this means that if (x,y) represents in Fig. 2 the optimal solution, i.e., the final location of the knot connecting the strings, the solution will remain the same if we move a hole along the line connecting it with the middle point. Of course, this does not help in the solution of the problem in the numerical iterative way, but it does give some information on the sensitivity of the solution to changes in the location of the demand points.

As pointed out in the literature, the mechanical analog can be used to demonstrate the majority theorem[10,11] which states that if at least one-half of the cumulative weight is associated with an existing facility, the optimum location for the service facility coincides with this facility. It is quite obvious that under these circumstances, the knot in the mechanical analog will end up at this point. In fact, this is a rather weak sufficient condition, and a much stronger necessary and sufficient theorem can be stated with the aid of the physical picture.

Suppose that we check whether the demand point k is the solution. We would like to show that a necessary and sufficient condition for k to be the optimum is that w_k, the weight associated with the k th demand point, is larger than the absolute value of the resultant of the "forces" of the other $m-1$ demand points at (x_k,y_k). A mathematical proof has been given by Katz[12]; however, one can use physical arguments as follows. If the sum of the forces exerted by the other $m-1$ points is $\mathbf{R}_k$, it will be nearly the same $\mathbf{R}_k$ in the very close vicinity of (x_k,y_k) due to continuity. The contribution of the k th point in the close vicinity, say points on a circle with a radius ϵ centered at (x_k,y_k) is a vector of magnitude w_k pointing at (x_k,y_k). The resultant $\mathbf{R}_k$ is least effective for "pulling" the facility (the knot in the mechanical analog) if they point at opposite directions. Thus, $|\mathbf{R}_k| > w_k$ is the condition for the service facility not to be at (x_k,y_k). If $|\mathbf{R}_k| < w_k$, the force of the demand point k is strong enough to pull the knot to this sink.

It has been pointed out above that the cost function is not differentiable at the demand points. As suggested by Katz[12] it is recommended to check each of the demand points for optimality by comparing $|\mathbf{R}_k|$ and w_k and proceed with the iterative process only if none of them turns out to be the optimum. It should be mentioned, however, that even if no given point is the optimum, it is possible that the iterative procedure will go through one of the demand points, which results in a failure of the process to converge at the optimum. As shown by Kuhn,[9] however, this is a very unlikely event since for all but a denumerable number of initial points in the iterations, the Weiszfeld process converges at the optimum. If the unlikely event does occur, namely, that the iterations lead us to a demand point which is known beforehand (by the abovementioned initial test) not to be the optimum, a small perturbation can be introduced in order to avoid the point of nondifferentiability and the iterative process is proceeded.

IV. GENERALIZATION OF THE WEBER PROBLEM

Two generalizations of the linear cost location problem will be briefly discussed here. One is the problem (3) mentioned above which takes into account general cost functions of the radial distance, $\phi_i(r_i)$. Of course, the mechanical analog of Fig. 2 cannot be utilized here, but extensions of the iterative process can be employed.[13,14] In this case, the force concept can be maintained; the force element of the ith demand point should be $d\phi_i/dr_i$, and pointing toward (x_i,y_i). The condition for the k th demand point to be the optimum is, in analogy with the previous case, $d\phi_i/dr_i > |\mathbf{R}_k|$ where $\mathbf{R}_k$ is the resultant of the forces of all the other $m-1$ points at (x_k,y_k). As pointed out already, the solution is unique only if $\phi_i(r_i)$ are strictly nondecreasing, differentiable, convex functions. The iterative process suggested[14] also specifies the step size to be used in the steepest descent method, and, for example, it is shown that a step size twice as large as that given by Weiszfeld yields much better convergence in the original Weber problem.

Another extension is that of locating a number of facilities, the multifacility problem. In this case, each demand point may have a different weight with respect to each of the facilities. Furthermore, one can assume that the different service facilities are expected to serve one another, and therefore, an attractive force should be assumed to exist between each pair of service facilities (in the absence of interaction between service facilities the problem simply reduces to p single location problems, p being the number of service facilities). A mechanical analog has been suggested to this case based on representing fixed points by fixed pegs and movable points by movable pegs, interconnected by strings. The details are given in a paper by Miehle,[15] however, as pointed out by Miehle, friction prevents the use of the method for this multifacility problem for a large number of service facilities. It has been shown, however, that the problem is convex,[10] therefore any numerical iterative process of the minimization problem in $2p$ variables should yield the optimal solution.

Another multifacility problem, usually referred to as the location–allocation problem[16] will only briefly be mentioned. A number of identical service facilities are to be located among m given demand points in an optimal manner in such a way that each demand point will be served by the closest service facility. A number of methods have been suggested for the solution, the main difficulty being that the problem is not convex which means that iterative methods may end up at local minima which are not necessarily global. It is to be noted, however, that for each of these local solutions, every one of the service facilities is in equilibrium under the influence of the "forces" exerted by the demand points allocated to it. Some heuristic methods have been given in the literature which yield "good" but not necessarily optimal solutions.

V. SUMMARY

The Weber problem and some of its extensions of optimal location of service facilities in two-dimensional Euclidean space have been introduced. The physical analog in which one considers the total cost function as a potential field has been pursued. Thus an optimal solution is shown to be a point where the forces derived from the potential (cost) field are in equilibrium. The possibility that one of the demand points is the solution as well as the cases in which another point is optimal are discussed in view of this physical concept. In the simple case of cost elements which are proportional to the radial distance, it is shown that the forces are independent of the distance to the demand point, they are in the direction of the demand point, and they are proportional to the weights of these points. The implications of this kind of force behavior are considered. The numerical methods used as well as the mechanical model employed in the past are presented. The extension to three-dimensional problems seems quite straightforward.[14]

[1]S. Eilon, C. D. T. Watson-Gandy, and N. Christofides, *Distribution Management. Mathematical Modelling and Practical Analysis* (Griffin, London, 1971).
[2]A. Weber, *Uber den Standort der Industrien* (Tübingen, 1909) [*Alfred Weber's theory of the location of industries*, translated by C. J. Friendrich (University of Chicago, Chicago, 1929)].
[3]R. F. Love and J. G. Morris, Manage. Sci. **25**, 130 (1979).
[4]L. Cooper, J. Reg. Sci. **8**, 181 (1968).
[5]I. N. Katz, SIAM J. Appl. Math. **17**, 1224 (1969).
[6]E. Weiszfeld, Tohoku Math. J. **43**, 355 (1937).
[7]L. M. Ostresh, Oper. Res. **26**, 597 (1978).
[8]E. M. Purcell, *Electricity and Magnetism*, Berkeley Physics Course (McGraw–Hill, New York, 1963), Vol. 2, p. 62.
[9]H. W. Kuhn, *Nonlinear Programming*, edited by J. Abadie (North-Holland, Amsterdam, 1967), p. 37; Math Prog. **4**, 98 (1973).
[10]R. L. Francis and J. A. White, *Facility Layout and Location: An Analytical Approach* (Prentice-Hall, Englewood Cliffs, NJ, 1974).
[11]C. Witzgall, Natl. Bur. Stand. (U.S.), Rep. No. 8388 (1965).
[12]I. N. Katz, Math. Prog. **6**, 89 (1974).
[13]R. Chen, Comput. Oper. Res. (in press).
[14]R. Chen, Comput. Math. Appl. **10**, 87 (1984).
[15]W. Miehle, Oper. Res. **6**, 232 (1958).
[16]R. Chen, Naval Res. Log. Q. **30**, 449 (1983).

Wigner's rotation revisited

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The separation of Wigner's rotation from the three-dimensional representation of two successive Lorentz transformations is effected naturally via three-space tensor algebra. An alternative derivation is afforded by means of Gibbs's addition law of spin vectors. The explicit dependence of the Wigner rotation operator on the intrinsic parameters of the two partaking boosts is derived. Applications are given to the Thomas precession and the Lorentz-invariant helicity of a massless particle.

I. INTRODUCTION

It is well known that a general orthogonal transformation in four dimensions is specified by six independent parameters. A pure Lorentz transformation (known as "boost") is characterized by three parameters (components of the translation velocity vector $\mathbf{v}$) and a pure rotation is determined by three parameters (e.g., the Euler angles). As might be expected from náive counting, a combination of two successive boosts which depend on six independent parameters (components of $\mathbf{v}_1$ and $\mathbf{v}_2$) cannot in general result in a pure boost but will render an additional pure rotation, known as "*Wigner's rotation*."[1,2] Physically, this spatial rotation is a relativistic kinematic effect that stems from the fact that the matrix of the resultant transformation includes antisymmetric contributions. Due to the said rotation, a spinning mass moving with a relativistic velocity will exhibit the known *Thomas precession*.[3]

The derivation of the rotation parameters was considered by several authors,[4–10] most of which are content with the limiting case of infinitesimal rotation. We intend to show that the problem lends itself to a rather compact solution which renders the rotation parameters in close form by means of tensor algebra[11] in three dimensions. Our results have interesting physical applications.

II. THE COMPOUND GENERAL LORENTZ TRANSFORMATION

Consider three frames of reference K, K', and K'' with parallel respective axes. The frame K'' moves with uniform velocity $\mathbf{v}_2$ with respect to K', which in turn moves with

