Solution of Minisum and Minimax Location–Allocation Problems with Euclidean Distances

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A new method for the solution of minimax and minisum location–allocation problems with Euclidean distances is suggested. The method is based on providing differentiable approximations to the objective functions. Thus, if we would like to locate m service facilities with respect to n given demand points, we have to minimize a nonlinear unconstrained function in the $2m$ variables $x_1, y_1, \ldots, x_m, y_m$. This has been done very efficiently using a quasi-Newton method. Since both the original problems and their approximations are neither convex nor concave, the solutions attained may be only local minima. Quite surprisingly, for small problems of locating two or three service points, the global minimum was reached even when the initial position was far from the final result. In both the minisum and minimax cases, large problems of locating 10 service facilities among 100 demand points have been solved. The minima reached in these problems are only local, which is seen by having different solutions for different initial guesses. For practical purposes, one can take different initial positions and choose the final result with best values of the objective function. The likelihood of the best results obtained for these large problems to be close to the global minimum is discussed. We also discuss the possibility of extending the method to cases in which the costs are not necessarily proportional to the Euclidean distances but may be more general functions of the demand and service points coordinates. The method also can be extended easily to similar three-dimensional problems.

1. INTRODUCTION

The location–allocation problem, first mentioned by Miehle [23] and later accurately formulated by Cooper [4–6], is that of optimally locating a number (m) of identical service facilities among n demand points and simultaneously assigning each demand point to be served by the closest service facility. The problem as formulated and solved by Cooper and others (e.g., see [8] and [10]) is that of minimizing the sum of weighted Euclidean distances between the given demand points in R^2 and the service facilities. One can also consider optimizing the number of service facilities to be determined. We shall concentrate, however, on the problem with a given number of service facilities. The problem has usually been mathematically stated as

$$\begin{aligned} \text{minimize} \quad & \psi = \sum_{j=1}^{m} \sum_{i=1}^{n} Z_{ji} w_i [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}, \\ \text{subject to} \quad & \sum_{j=1}^{m} Z_{ji} = 1, \quad Z_{ji} = (0,1), \quad i = 1, \ldots, n, \\ & j = 1, \ldots, m, \end{aligned} \tag{1}$$

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where ψ is the total cost per unit time; (a_i,b_i), $i = 1, \ldots, n$, are the coordinates of the demand points; (x_j,y_j), $j = 1, \ldots, m$, are the coordinates of the service facilities (which are to be determined); w_i are the weights associated with the points (a_i,b_i); and Z_{ij} are the assignment variables which attain the value of 1 if point i is assigned to service location j, and 0 otherwise. The analogous problem of locating P service points on a network is usually termed the P-median problem, a term which is sometimes used for the Euclidean distance case as well [12]. A special case of this problem is that of locating optimally a single-service center by the same "minisum" criterion. This was first solved by Weiszfeld [30] and further investigated by numerous workers (e.g., see [15,17,18]). The main feature of the single-facility location with Euclidean distances is that the problem is convex and therefore, if a local solution is found, it is known to be global. The cost function is differentiable everywhere except at the demand points. Kuhn [18] has shown how to identify directly cases where the solution coincides with a demand point and how to bypass the (rare) situation where, although the solution does not coincide with a demand point, it may pass through one during the iterative process. The additional difficulty in solving the location–allocation problem as compared with the single-facility one is due not only to the fact that this is a problem in $2m$ variables rather than merely two, but mainly to two additional factors. The main additional difficulty is related to the fact, shown by Cooper [4,5] and others, that the problem is neither convex nor concave, and, therefore, any solution that one may find while using any nonlinear programming method is usually only a local minimum. The other difficulty is associated with the fact that the cost function is not differentiable in additional points to the demand points. This is related to an abrupt possible change of the assignment variables Z_{ij} from zero to unity or vice versa.

Concerning the nonconvexity of the problem, Cooper [5] pointed out the possibility of considering all the possible assignments of demand points to service facilities. This number is given by

$$S(n,m) = \frac{1}{m!} \sum_{K=0}^{m} \binom{m}{K} (-1)^K (m - K)^n, \tag{2}$$

the Stirling number of second kind. For large n these are extremely large numbers and, therefore, the possibility of considering all the assignments is feasible only for very small problems. A number of heuristic methods that can be used for solving larger problems have been developed. Cooper [5] suggested arbitrarily choosing a set of m initial positions for the service facilities, assigning each demand point to the closest service facility, locating each service facility as a single facility with respect to the demand points currently assigned to it, and checking at the end of such cycle if each demand point is still assigned to the closest service facility and, if not, reassigning it appropriately. The process is repeated until a certain termination criterion is reached. The solution thus obtained is a local, not necessarily global, solution. Eilon, Watson-Gandy, and Christofides [8] improved this method and made it a true iterative decision process in that first a reallocation and then a relocation decision is made at *each iteration*, whereas Cooper's method optimally locates facilities before testing the allocation. Eilon et al. solved a problem with 50 demand points and 2,3,4, and 5 service centers, each with 20 different sets of initial positions. As could be expected, the final results indeed depended on the initial guesses; no simple correlation was found between apparently "intelligent" guesses and good final results.

Optimal attempts to solve the location–allocation problem have been made by developing branch-and-bound methods [9,16]. Their use has been limited, however, to relatively small problems. More works on various versions of the heuristic solution of the location–allocation problem are found in the literature [2,13,19,21,24,25,27,29]. In a recent work, Juel [14] developed a family of lower bounds on the objective function value of the location–allocation problem. There are also a number of articles dealing with location–allocation problems with rectilinear distances [20,21,22,28]. The state of the art at the moment seems to be that whereas small minisum location–allocation problems can be solved optimally, larger problems are being solved using heuristic methods which yield local minima that may or may not be global.

The essence of the present work is that we write a differentiable approximation to the location–allocation objective function and solve it using efficient nonlinear programming methods. Despite some advantages that will be discussed below, the main difficulty—that the problem remains neither convex nor concave—is still there, and, therefore, the solution is not necessarily global. The main novelty of this article is that a similar approximation is being used for solving the minimax location–allocation problem with Euclidean distances (analogous to the P-center problem in networks). The solution of this problem has not been reported in the literature and, although the solution here is also not necessarily global, it may be of use for practical problems such as the location of several emergency facilities or a number of broadcasting stations that should cover a given set of demand points. A discussion is given concerning a heuristic way to evaluate how close a local minimum is to the global minimum.

2. METHOD FOR SOLVING MINISUM LOCATION–ALLOCATION PROBLEMS

The problem as formulated in Equation (1) can also be written as

$$\min_{x_J, y_J} \sum_{i=1}^{n} w_i \min_j \, [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}, \qquad J = 1, \ldots, m. \tag{3}$$

The meaning of this formulation is that $\min_j$ selects for each demand point the closest service facility, $\Sigma_{i=1}^{n}$ sums all the weighted Euclidean distances, and the minimization is over the $2m$ variables $x_1, y_1, \ldots, x_m, y_m$.

Although we choose the $\min_j$ of the terms $[(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}$, the index j remains since we can assume that at least one demand point is assigned to each service point. In order to reduce possible confusion, this index is denoted by J after the $\min_j$ is taken. Thus, the minimization is over $x_J, y_J, J = 1, \ldots, m$. The terms $r_{ij} = [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}$ are all positive except for the possible coincidence of $(a_i, b_i) = (x_j, y_j)$ which, as mentioned above, must be avoided anyway. Following Charalambous and Bandler [3], we argue that for a given set of positive numbers $C_1, \ldots, C_m$ we have

$$\min\{C_1, \ldots, C_m\} = \lim_{N \to \infty} \left\{ \sum_{j=1}^{m} C_j^{-N} \right\}^{-1/N}. \tag{4}$$

Thus, a good approximation for the solution of (3) can be found by choosing a large enough value of N and solving

$$f(x,y) = \min_{x_J,y_J} \sum_{i=1}^{m} w_i \left[\sum_{j=1}^{m} r_{ij}^{-N} \right]^{-1/N}, \qquad J = 1, \ldots, m. \tag{5}$$

It has been found that setting $N = 100$ made the approximation very good.

The problem in $2m$ variables can be solved using standard nonlinear programming methods. Since the quasi-Newton methods are considered very efficient, one of them—the Broyden–Fletcher–Shano (BFS)—has been chosen (e.g., see, [1], pp. 332, 350). An important feature of the quasi-Newton methods is that only first, not second, derivatives are needed. For each J we can find the first derivatives $\partial f/\partial x_J$ and $\partial f/\partial y_J$, $J = 1, \ldots, m$.

A difficulty may occur if r_{ij} is very small since r_{ij}^{-N} may be too large. This difficulty has been overcome by choosing a small enough constant η such that if $r_{ij} \leq \eta$, the demand point (a_i,b_i) is assigned to (x_j,y_j) without using the approximation implied in (4). This has been incorporated appropriately into the developed program by directly adding in this case $w_i r_{ij}$ to the objective function and, accordingly, to the derivatives. It seems as if the possibility of exact coincidence $(a_i,b_i) = (x_j,y_j)$ should be of concern. However, it has been pointed out by Overton [26] that the ill-conditioning of the Hessian (in a multidimensional, single-facility location) is in precisely the desired direction and actually results in quadratic convergence. It seems that the same situation occurs here where an updated approximation to the Hessian is calculated in every iteration. Furthermore, in some of the problems tested, such coincidence (up to the chosen termination parameter ϵ) did occur without damaging the performance of the algorithm.

First, in the examples given by Cooper [4], eight problems—each having seven demand points and two service facilities—have been solved. All the results coincided with those of Cooper with the exception of his case 6 where he apparently had an error (see also [16]). These results, which Cooper obtained by checking all the possible assignments, were found in the present work irrespective of the initial guess taken for starting the iterative process. The computer used was the CDC 6600 and the average CPU time needed was 0.2 sec.

Another problem tested by Cooper was one of 15 demand points and three service facilities. The data are summed in Table 1. Cooper's best result was $(x_1,y_1) = (8.888,14.466)$, $(x_2,y_2) = (20.997,44.998)$, $(x_3,y_3) = (40.361,17.968)$, and the value of the objective function at the minimum was $f_{\min} = 143.209$. Starting from the following first guess—$(x_{01},y_{01}) = (5,10)$, $(x_{02},y_{02}) = (12,30)$, $(x_{03},y_{03}) = (30,25)$—the solution obtained using the present method after 1.0 sec CPU time was $(x_1,y_1) = (8.947,14.639)$ with points 1,2,3,4,5 assigned to the center, $(x_2,y_2) = (21.000,45.000)$ with points (3,6,7,8,9) assigned to the center, and $(x_3,y_3) = (40.053,17.509)$ with points 10,11,12,13,14,15 assigned to the center. The value of the objective function was $f_{\min} = 143.1962$, slightly better than Cooper's. The same solution has been found

Table 1. Cooper's data for a problem with 15 demand points and 3 service centers.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_i	5	5	5	13	12	13	28	21	25	31	39	39	45	41	49
b_i	9	25	48	4	19	39	37	45	50	9	2	16	22	30	31

while starting from different initial guesses. The example given here is the one in which the initial positions differed most from the final solution. The point mentioned above is seen here, namely, that a service facility (x_2, y_2) can coincide with a demand point (No. 8).

Of course, it cannot be argued that the problem that is neither concave nor convex turned convex using the present procedure. The present algorithm seems to be a powerful method to achieve local minima. It is the structure of the problem that makes the probability of a local minimum global, quite large at least for small problems [4–6,16].

In order to check the solution of larger problems, we took that of locating 2,3,4, and 5 service facilities among 50 demand points given by Eilon, Watson-Gandy, and Christofides [8]. The comparison of our best results and Eilon's are summarized only briefly. In locating two facilities, Eilon's result was now found in ca. 0.7 CPU sec irrespective of the initial position of the facility points (four trials). In locating three centers, their best result (out of 20 trials) was now found in one out of three trials. The execution CPU time was ca. 1.2 sec. In locating four centers, all four trials resulted in solutions within 0.1% from the best given by Eilon et al., the average computation time being ca. 4 CPU sec. The best of the four was less than 0.01% worse than the best reported by Eilon. In locating five centers, all three trials resulted in the same result which was slightly better than the best reported by Eilon et al.; the average time was ca. 6 CPU sec.

The problem has also been solved with the same demand points to 6,7,8,9, and 10 facilities. In the ten-facility case, an average time of 20 CPU sec was needed, and the variation between the different results was larger. In five solutions found in the ten-facility problem, the worst differed from the best by 7.5%.

Similar to other methods reported so far (e.g., [2,8,25]), the present method is a strong heuristics for solving location–allocation problems with the advantage that the search method used, i.e., the quasi-Newton method, is rather powerful. The main point, however, about the present method is that it can be extended rather easily to solve the minimax location–allocation problem, the solution of which has not been reported so far.

3. SOLUTION OF THE MINIMAX LOCATION–ALLOCATION PROBLEM

In analogy to the formulation of the minisum location–allocation problem as min Σ min [Equation (3)], the minimax location–allocation problem can be written as

$$\min_{x_J, y_J} \max_i w_i \{\min_j [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2}\}, \; J = 1, \ldots, m. \tag{6}$$

Here, again, $\min_j$ selects for each demand point its closest facility and the $\min_{x_J, y_J} \max_i$ operations are to be performed. The procedure of finding the smallest of m positive numbers, given by Equation (4), will be used here. We shall also need a differentiable expression for selecting the largest of n given positive magnitudes. As shown in the literature [7,11], a possible approximation that can be used is

$$\max\{C_1, \ldots, C_n\} = \lim_{M \to \infty} \left\{ \sum_{i=1}^{n} C_i^M \right\}^{1/M}. \tag{7}$$

In order to approximate expression (6) by a differentiable function, we have to use both (4) and (7), along with two large parameters N and M; for the sake of simplicity we take $N = M$. Usually, $N = M = 100$ was found to suffice. Using (7), we can now write

$$f_i(x,y) = w_i \min_j [(a_i - x_j)^2 + (b_i - y_j)^2]^{1/2} \simeq w_i \left\{ \sum_{j=1}^{m} r_{ij}^{-N/2} \right\}^{-1/N}. \tag{8}$$

We are now interested in $\min_{x,y} \max_i f_i(x,y)$ which can be approximated as

$$\min_{x,y} \max_i f_i(x,y) \simeq \min_{x,y} \left\{ \sum_{i=1}^{n} [f_i(x,y)]^N \right\}^{1/N}. \tag{9}$$

Substituting expression (8) into (9) yields the differentiable approximation to (6), namely,

$$\min_{x,y} \left(\sum_{i=1}^{n} w_i \left\{ \sum_{j=1}^{m} [(a_i - x_j)^2 + (b_i - y_j)^2]^{-N/2} \right\}^{-1} \right)^{1/N}. \tag{10}$$

For any finite value of N, large as it may be, raising the objective function to the Nth power should yield a problem that minimizes at the same points. Thus, we can solve the equivalent problem

$$\min_{x,y} \sum_{i=1}^{n} w_i \left\{ \sum_{j=1}^{m} [(a_i - x_j)^2 + (b_i - y_j)^2]^{-N/2} \right\}^{-1}. \tag{11}$$

Similar to problem (5), this is an unconstrained minimization problem of a differentiable nonlinear function in $2m$ variables, $x_1, y_1, \ldots, x_m, y_m$. Again, the function is neither convex nor concave and, therefore, nonlinear programming methods would yield local minima which may depend on the initial guess taken for starting the iterations. The problem has been solved by using the same quasi-Newton method mentioned above. Here, also, only first derivatives are needed. These are directly found by differentiating expression (11) with respect to the $2m$ variables. The convergence was quite rapid. Again, for small problems it was found that chances are good to get the global minimum, whereas for larger problems the final result did depend on the starting point of the iteration. In order to check the method, a problem with 20 points, as shown in Figure 1, has been solved. The demand points are within the area of two overlapping circles, one centered at (4,4) having a radius of 3 and the other centered at (10,4) with a radius of 4. Three of the demand points are located on the circumference of each of the circles. One would expect the value of the objective function at the solution to be equal to the radius of the larger circle in this equiweighted problem; $N = 200$ has been chosen. A rather "bad" initial guess has been chosen, namely, centers at (3,1) and (1.5,1.5). After a computation time of 0.3 CPU sec, the solution was found at (10.032,4),(4.5,4.3) and the radius of the larger circle was 4.000, as expected. It should be noted that a different (better) solution would have been found had we "moved" the two demand points at (10,0), (10,8) on the circumference much closer to the smaller circle. At a certain point, it would be "profitable" for one or both of these points to be served by the other center, increasing the radius of the smaller covering circle, but decreasing the radius of the larger (critical) one, thus improving

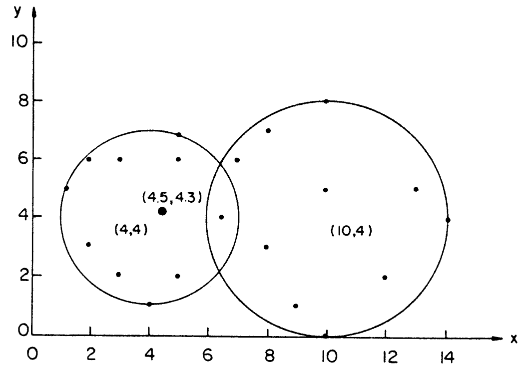

Figure 1. Twenty points in the plane used as demand points for the two-center location–allocation problem.

the solution. The fact that the second point found was (4.5,4.3), which does not coincide with the center of the second circle (4,4), is related directly to the fact that this circle is not critical. Furthermore, the distance of the farthest point assigned to this center is smaller than the "value" of the solution, namely 4.000. This points out the freedom in choosing the location of all but the critical center. Once the location of the critical center and the allocation of demand points to centers are established, another criterion may be used for the exact location of the $m - 1$ centers, for example, by relocation of each of them as a one-center problem.

It has been mentioned that it is advantageous to transfer demand points from assignment to one center to assignment to another and thus reducing the largest circle though increasing another, provided that the increased one does not exceed the magnitude of the largest. Thus, it can be argued heuristically that a property of a "good" solution is that the radii of the different covering circles are as close as possible to one another, with this being the main criterion for examining the final solution in more complicated cases.

Though different size problems have been solved, only the largest one will be reported here, namely, a problem with 100 demand points and 10 service facilities. The demand points were generated using random numbers; they were equiweighted and distributed over a square of 100 × 100. The problem was solved several times starting from different sets of initial values. The worst and best results obtained are given in Tables 2 and 3, respectively. The first and second columns are the x and y coordinates of the demand points and the third one gives the distance from the center to which it is assigned. In Table 2, the critical center is No. 7 and the value of the minimax solution is 28.329. In Table 3, the critical point is No. 3 and the value of the solution is 21.11. The computation time was about 20 CPU sec. It is certainly not claimed that this is the global minimum. Neither is it possible at this stage to check how far this solution is from the global optimum. Some heuristic considerations can be made as to the goodness of this solution—apart from the fact that this is the best one found in a number of trials. One feature of the "good" result is that there is a large number of nearly critical points. The distance of ten demand points from their respective centers was over 20 and 11 additional points had distances between 19 and 20. Thus, 21 demand points were within 10% of the value of 21.11. In the "bad" solution, on the other hand, only five points were in the range of 10%, namely, above 25.

Table 2. Minimax location of 10 service points and allocation of 100 demand points: worst case.

Initial values: j	1	2	3	4	5	6	7	8	9	10
x_{j0}	25.	80.	30.	75.	70.	30.	20.	60.	99.	10.
y_{j0}	75.	30.	80.	70.	20.	30.	20.	60.	0.	10.

	a_i	b_i	r_{ij}
Center No. 1	16.	72.	8.202
at (16.828,80.161),	8.	60.	22.009
points allocated	4.	65.	19.859
to center 1	14.	77.	4.241
	2.	98.	23.197
	23.	59.	22.042
	15.	78.	2.830
	11.	77.	6.629
	13.	59.	21.504
	5.	89.	14.766
	11.	54.	26.802
	21.	60.	20.588
	7.	79.	9.896
Center No. 2	89.	25.	10.296
at (80.,30.),	85.	46.	16.763
points allocated	74.	33.	6.708
to center 2	81.	25.	5.099
	70.	36.	11.662
	84.	43.	13.601
	83.	18.	12.369
	83.	31.	3.162
	83.	51.	21.213
	79.	40.	10.050
	87.	34.	8.062
Center No. 3	43.	88.	15.463
at (29.625,80.241),	40.	89.	13.578
points allocated	38.	81.	8.409
to center 3	27.	75.	5.862
	25.	61.	19.789
	39.	63.	19.625
	36.	85.	7.955
	26.	63.	17.618
	53.	86.	24.074
Center No. 4	88.	85.	4.684
at (84.506,81.881),	92.	89.	10.336
points allocated	96.	96.	18.206
to center 4	93.	97.	17.342
	87.	84.	3.273
	77.	66.	17.566
	72.	97.	19.621
	78.	88.	8.931
	96.	63.	22.105
	82.	83.	2.744
	85.	94.	12.129
	99.	89.	16.148
	98.	61.	24.862
	94.	92.	13.878
	95.	89.	12.681
	90.	89.	8.992

	a_i	b_i	r_{ij}
Center No. 5	47.	9.	24.427
at (69.210,19.169),	77.	8.	13.617
points allocated	74.	4.	15.907
to center 5	59.	19.	9.792
	70.	4.	15.189
	51.	0.	26.439
	63.	20.	6.265
Center No. 6	28.	29.	2.698
at (27.717,31.683),	15.	35.	13.142
points allocated	30.	25.	7.063
to center 6	24.	43.	11.911
	44.	35.	16.618
	18.	47.	18.139
	28.	37.	5.324
	20.	47.	17.151
	7.	49.	27.001
	18.	36.	10.632
	37.	53.	23.250
Center No. 7	22.	20.	3.151
at (21.002,22.989),	7.	25.	14.146
points allocated	26.	13.	11.170
to center 7	42.	4.	28.311
	0.	42.	28.329
	24.	14.	9.476
	21.	17.	5.989
Center No. 8	41.	50.	21.471
at (60.,60.),	68.	44.	17.889
points allocated	48.	68.	14.422
to center 8	66.	62.	6.325
	52.	60.	8.000
	68.	52.	11.314
	66.	70.	11.662
	54.	77.	18.028
	74.	59.	14.036
	75.	62.	15.133
	54.	72.	13.416
	62.	71.	11.180
	60.	46.	14.000
	55.	38.	22.561
	49.	61.	11.045
	65.	75.	15.811
	73.	56.	13.601
Center No. 9	91.	10.	12.806
at (99.,0.),	99.	5.	5.000
points allocated	98.	17.	17.029
to center 9			
Center No. 10	12.	3.	7.280
at (10.,10.),	1.	9.	9.055
points allocated	19.	1.	12.728
to center 10	9.	19.	9.055
	11.	12.	2.236
	28.	0.	20.591

Table 3. Minimax location of 10 service points and allocation of 100 demand points: best case.

Initial values: j	1	2	3	4	5	6	7	8	9	10
x_{j0}	0.	99.	30.	75.	70.	50.	0.	99.	99.	10.
y_{j0}	50.	50.	80.	70.	20.	50.	0.	99.	0.	10.

	a_i	b_i	r_{ij}
Center No. 1 at (11.764,57.246), points allocated to center 1	16.	72.	15.350
	8.	60.	4.664
	24.	43.	18.780
	4.	65.	10.973
	23.	59.	11.372
	25.	61.	13.758
	13.	59.	2.146
	18.	47.	11.995
	11.	54.	3.334
	0.	42.	19.257
	20.	47.	13.146
	7.	49.	9.523
	21.	60.	9.638
	26.	63.	15.355
Center No. 2 at (94.745,43.476), points allocated to center 2	85.	46.	10.067
	96.	63.	19.565
	84.	43.	10.756
	83.	31.	17.134
	98.	61.	17.824
	83.	51.	13.944
	79.	40.	16.124
	87.	34.	12.238
Center No. 3 at (22.385,92.542), points allocated to center 3	43.	88.	21.110
	14.	77.	17.659
	40.	89.	17.968
	2.	98.	21.103
	38.	81.	19.418
	27.	75.	18.139
	15.	78.	16.309
	11.	77.	19.265
	5.	89.	17.742
	36.	85.	15.564
	7.	79.	20.495
Center No. 4 at (69.951,77.942), points allocated to center 4	87.	84.	18.093
	77.	66.	13.867
	66.	70.	8.870
	54.	77.	15.979
	74.	59.	19.370
	75.	62.	16.722
	54.	72.	17.022
	72.	97.	19.167
	62.	71.	10.555
	78.	88.	12.882
	82.	83.	13.068
	65.	75.	5.759
	53.	86.	18.769
Center No. 5 at (60.826,17.862), points allocated to center 5	47.	9.	16.423
	74.	33.	20.068
	74.	4.	19.123
	70.	36.	20.326
	51.	0.	20.386
	63.	20.	3.049
	70.	4.	16.622
Center No. 6 at (54.070,50.950), points allocated to center 6	41.	50.	13.104
	68.	44.	15.568
	48.	68.	18.098
	66.	62.	16.261
	52.	60.	9.284
	68.	52.	13.970
	44.	35.	18.863
	39.	63.	19.295
	60.	46.	7.725
	55.	38.	12.984
	49.	61.	11.256
	37.	53.	17.193
	73.	56.	19.592
Center No. 7 at (0.387,1.004), points allocated to center 7	12.	3.	11.784
	1.	9.	8.020
	11.	12.	15.283
Center No. 8 at (98.391,98.955), points allocated to center 8	88.	85.	17.399
	92.	89.	11.830
	96.	96.	3.801
	93.	97.	5.735
	85.	94.	14.279
	99.	89.	9.974
	94.	92.	8.225
	95.	89.	10.517
	90.	89.	13.020
Center No. 9 at (94.374,10.895), points allocated to center 9	89.	25.	15.094
	91.	10.	3.490
	77.	8.	17.613
	81.	25.	19.437
	99.	5.	7.494
	83.	18.	13.411
	79.	19.	17.379
	98.	17.	7.101
Center No. 10 at (26.314,17.681), points allocated to center 10	28.	29.	11.444
	15.	35.	20.687
	30.	25.	8.195
	22.	20.	4.898
	7.	25.	20.655
	19.	1.	18.214
	26.	13.	4.691
	9.	19.	17.364
	42.	4.	20.814
	28.	37.	19.393
	24.	14.	4.348
	28.	0.	17.761
	18.	36.	20.118
	21.	17.	5.358

Another heuristic criterion for the goodness of the solution is related to the fact that in some cases some of the service points do not move from their initial positions. From both this and other examples it can be concluded that a characteristic of a good solution is that none of its service points remains in the starting location. In the present example, in the worst solution, four points do not move with the iterations whereas, in the best one, all ten change position during the process.

4. DISCUSSION AND CONCLUSIONS

An algorithm for the solution of minisum and minimax location–allocation problems in Euclidean space is proposed. The method is based on writing differentiable approximations to the problems and solving them using a quasi-Newton method. As compared with other solutions of the minisum problem, the main advantage is in using a very powerful nonlinear programming method as compared with previous methods (e.g., [4,8]) which repeatedly use the Weiszfeld [30] algorithm which is a steepest-descent method with fixed step size. As for the minimax case, the solution of this problem has not been given in the literature so far to the best of the author's knowledge.

An important feature of the present method is that it is amenable to an easy extension, namely, to solve problems with more complicated cost functions of the Euclidean coordinates. Although the examples given are for equiweighted problems, the problems as formulated in Sections 2 and 3 include possible weights w_i. Moreover, the extension to more general functions of the Euclidean distances $f_i(r_i)$ or other norms such as the L_P norm is straightforward. Also, the extension to three-dimensional problems can be accomplished without difficulty, except that more variables (x_j,y_j,z_j) $j = 1, \ldots, m$ are to be included.

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