ON WEAK APPROXIMATION IN HOMOGENEOUS SPACES OF SIMPLY CONNECTED ALGEBRAIC GROUPS

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Introduction. Let K be an algebraic number field and let S be a finite set of its places. An algebraic variety X over K is said to satisfy *the condition of weak approximation with respect to* S if

$$(WA_S) \qquad X(K) \text{ is dense in } \prod_{v\in S} X(K_v) ,$$

K_v denoting the completion of K at v. We say that X satisfies *the condition of weak approximation* if

$$(WA) \qquad X \text{ satisfies } (WA_S) \text{ for any finite } S .$$

Set $K_S = \prod_{v\in S} K_v$; then $X(K_S) = \prod_{v\in S} X(K_v)$. Let $X(K)\hat{}_S$ denote the closure of $X(K)$ in $X(K_S)$.

Let T be an algebraic K–torus. Set $A_S(T) = T(K_S)/T(K)\hat{}_S$. The finite abelian group $A_S(T)$ is the defect of (WA_S) for T ; in other words, it is the measure of failure of (WA_S) for T. In particular, T satisfies (WA_S) if and only if $A_S(T) = 0$. The group $A_S(T)$ was studied by Voskresenskii (cf. [Vo1], [Vo2]). He related $A_S(T)$ to a certain group $H^1(K,\mathrm{Pic}\,\bar{V}(T))$. This group is a biratonal invariant of T ; as Sansuc later showed, it can be computed in terms of the arithmetic Brauer group $\mathrm{Br}_a T$ (cf. [Sa]).

For any connected algebraic K–group G one can define the set

$A_S(G) := G(K_S)/\widehat{G(K)}_S$. As above, $A_S(G)$ is the defect of (WA_S) for G . Sansuc [Sa] has generalized Voskresenskii's results. He proved that the subgroup $\widehat{G(K)}_S$ is normal in $G(K_S)$ and that the quotient group $A_S(G)$ is finite and abelian. He showed also that it is possible to compute $A_S(G)$ in terms of the arithmetic Brauer group $Br_a G$ (or in terms of $H^1(K, \text{Pic}\, \overline{V}(G))$, cf. [Sa], Propositions 8.9 and 9.8).

In this paper we consider the case of a homogeneous space. Let G be a simply connected algebraic K–group and H any connected K–subgroup of G . Set $X = H\backslash G$. We define a finite abelian group which we call the defect of (WA_S) for X . This group depends on H only; we denote it by $A_S(X) = \text{Ч}_S(H)$. The group $\text{Ч}_S(H)$ can be computed from the Picard group H . We can also compute $A_S(X)$ from $Br_a X$ in the same way as it was done in [Sa] for algebraic groups. Our obstruction to (WA_S) is related to the Brauer–Manin obstruction that has been constructed by Colliot–Thélène and Sansuc [CT–Sa] (see also [Sa], 8.13).

Remark 1. The author has recently obtained certain sufficient conditions for *strong* approximation for $X = H\backslash G$ (cf. [Brv1]).

Remark 2. A slightly more general case of $X = H\backslash G$ with G not necessarily simply connected (e.g. with G adjoint) is considered in [Brv3].

Remark 3. Some results concerning *the Hasse principle* for homogeneous spaces were obtained by Rapinchuk [Ra]. Note that this problem is more difficult than that of weak approximation.

Our constructions and proofs are based on the results of Kottwitz [Ko1], [Ko2], who generalized the duality theory of Tate and Nakayama from the case of tori to that of all connected reductive K–groups. We use the weak approximation theorem for simply connected groups which is due to Kneser [Kn1], [Kn2] in all the cases except E_8 and to Harder [Ha3] in the E_8 case; see Platonov [Pl1], [Pl2] for a uniform proof. We use the Hasse principle for simply connected groups which is due to Kneser [Kn3] (see also [Kn4]) in the classical cases and to Harder [Ha1] (see also [Ha2]) in exceptional ones, except E_8; the proof for the E_8 case, initiated by Harder [Ha1] in 1966, has been recently completed by V.I. Chernousov [Ch]. The idea to relate weak approximation to the Brauer group is inspired by the ideas and results of Manín, Voskresenskii, Colliot–Thélène and Sansuc.

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Notation.

K is an algebraic number field, $\bar{K}$ is an algebraic closure of K. $\mathscr{V}$ (resp. $\mathscr{V}_\infty$, $\mathscr{V}_f$) is the set of places (resp. of infinite places, of finite places) of K. For any finite $S \subset \mathscr{V}$ we set $K_S = \prod_{v \in S} K_v$. We let A denote the adele ring of K. For an algebraic K–group H we set

$$H^1(K_S,H) = \prod_{v \in S} H^1(K_v,H), \qquad H^1(A,H) = \bigoplus_{v \in \mathscr{V}} H^1(K_v,H) ,$$

where $\oplus$ denotes the subset of the direct product, consisting of the families (ξ_v) (where $\xi_v \in H^1(K_v,H)$) such that $\xi_v = 0$ for all v outside some finite set. We sometimes use the additive notation instead of the multiplicative one and write 0 instead of 1.

For a finite abelian group A we denote by $A^\sim$ the dual group $\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})$.

For a connected algebraic K–group H let H^u denote its unipotent radical. Set

$H^{red} = H/H^u$; it is a reductive group. Note that H is simply connected if and only if H^{red} is a simply connected semisimple group. We let H^{ss} denote the derived group of H^{red}, and set $H^{tor} = H^{red}/H^{ss}$. Let H^{sc} denote the universal covering of H^{ss}. We have a canonical homomorphism $\rho : H^{sc} \longrightarrow H^{ss} \longrightarrow H^{red}$.

For any K–variety we write $\bar{X}$ for $X_{\bar{K}}$. For a K–torus T we denote by $X_*(\bar{T})$ the cocharacter group $\mathrm{Hom}(\mathbb{G}_{m\bar{K}}, \bar{T})$.

1. Main results

We are going to construct the finite abelian group $Ч_S(H)$. Set $B(H) = (\mathrm{Pic}\ \bar{H})^{\sim}$. For $v \in \mathcal{V}$ set $B_v(H) = B(H_{K_v})$. The canonical morphism $H_{K_v} \longrightarrow H$ induces a homomorphism $\lambda_v : B_v(H) \longrightarrow B(H)$. Let $B^{(S)}(H)$ denote the subgroup of $B(H)$ generated by the groups $\mathrm{im}\ \lambda_v$ for $v \notin S$, and set $B'(H) = B^{(\emptyset)}(H)$. Now we set $Ч_S(H) = B'(H)/B^{(S)}(H)$.

Theorem 1.1. Let G be a simply connected group over a number field K, and let H be a connected K–subgroup of G. Set $X = H\backslash G$. Then there exists a canonical surjective map $X(K_S) \longrightarrow Ч_S(H)$ whose kernel is $\widehat{X(K)_S}$.

We see that the condition (WA_S) for X is equivalent to the condition $Ч_S(H) = 0$. In a sense $Ч_S(H)$ is the defect of (WA_S) for $X = H\backslash G$.

Theorem 1.2. There is a canonical exact sequence

$$H^1(K,H) \xrightarrow{j_S} \prod_{v\in S} H^1(K_v,H) \xrightarrow{\chi_S} Ч_S(H) \longrightarrow 0 ,$$

where j_S is the localization map.

Remark 1.3. One can show that the above exact sequence is functorial in H.

Theorem 1.4. $\text{Ш}_S(H) = \text{Ш}_S(H^{tor})$.

Now set $B^{(\omega)}(H) = \bigcap_S B^{(S)}(H)$, where S runs over all the finite subsets of $\mathcal{V}$. We set $\text{Ш}_\omega(H) = B'(H)/B^{(\omega)}(H)$. It is clear that $\text{Ш}_\omega(H) = 0$ if and only if $\text{Ш}_S(H) = 0$ for any finite $S \subset \mathcal{V}$.

Corollary 1.5. $H\backslash G$ satisfies (WA) if and only if $\text{Ш}_\omega(H) = 0$.

Corollary 1.6. Set $T = H^{tor}$. Let L/K be a Galois extension splitting T. Let $S_0 \subset \mathcal{V}$ be the finite set of places (ramified in L) with non–cyclic decomposition groups. Then $\text{Ш}_S(H) = \text{Ш}_{S \cap S_0}(H)$. In particular, if $S \cap S_0 = \emptyset$ then $H\backslash G$ satisfies (WA_S).

Corollary 1.7. If H^{tor} splits over a cyclic extension of K (in particular, if $H^{tor} = 1$, i.e. H is semisimple), then $H\backslash G$ satisfies (WA_S).

Corollary 1.8. (Real approximation). If $S \subset \mathcal{V}_\infty$ then $H\backslash G$ satisfies (WA_S) for any connected K–subgroup $H \subset G$.

Corollary 1.5 follows immediately from Theorem 1.1. Corollaries 1.7 and 1.8 follows from Corollary 1.6. To prove Corollary 1.6 note that $\text{Ш}_S(H) = \text{Ш}_S(T)$ and $\text{Ш}_{S \cap S_0}(H) = \text{Ш}_{S \cap S_0}(T)$ because of Theorem 1.4. By Theorem 1.2

$$\text{Ш}_S(T) = \operatorname{coker}[j_S : H^1(K,T) \longrightarrow \prod_{v \in S} H^1(K_v,T)] .$$

This cokernel is investigated in [Sa], where the notation $Ч_S$ is introduced. By Lemmas 1.5 and 1.8 of [Sa] $Ч_S(T) = Ч_{S\cap S_0}(T)$. Hence $Ч_S(H) = Ч_{S\cap S_0}(H)$. q.e.d.

Consider the arithmetic Brauer group $Br_a X$ (see [Sa]). Set $Б_S(X) = \ker [Br_a X \longrightarrow \prod_{v\notin S} Br_a(X_{K_v})]$, $Б(X) = Б_\emptyset(X)$.

Theorem 1.9. $Ч_S(H) = (Б_S(X)/Б(X))^\sim$.

Remark 1.10. One can show (cf. [Brv3]) that the group $A_S(X) := (Б_S(X)/Б(X))^\sim$ is the defect of (WA_S) for $X = H\backslash G$ for any pair of connected K–groups $H \subset G$ such that $A_S(G) = Ш(G) = 0$, where Ш denotes the Shafarevich–Tate group.

2. Results of Kottwitz

The group B(H) is computed in [Ko1] in terms of a connected Langlands dual group for H. We prefer to describe B(H) in terms of the algebraic fundamental group $\pi_1(H)$ (cf. [Brv2], [Brv3]).

Let temporarily K be any field of characteristic 0, and let H be a connected K–group. Choose a maximal torus $T \subset H^{red}$ (defined over K). We write $T^{(sc)}$ for $\rho^{-1}(T) \subset H^{sc}$ (see Notation).

Definition 2.1. $\pi_1(H) = X_*(T)/\rho_* X_*(T^{(sc)})$. If $T' \subset H^{red}$ is another maximal torus, then there is a canonical isomorphism

$$X_*(T')/\rho_* X_*(T'^{(sc)}) \xrightarrow{\sim} X_*(T)/\rho_* X_*(T^{(sc)})$$

of Galois modules (cf. [Brv2]). Thus the definition of the algebraic fundamental group $\pi_1(\bar{H})$ is correct.

One can easily see that π_1 is an exact functor from the category of connected K–groups to the category of $\mathrm{Gal}(\bar{K}/K)$–modules finitely generated over $\mathbb{Z}$. If $K = \mathbb{C}$ then $\pi_1(H)$ is the usual topological fundamental group $\pi_1^{top}(G(\mathbb{C}))$ (cf. [Brv2]); this justifies the term. For any K there is a canonical isomorphism of Galois modules $\pi_1(\bar{H}) = X^*(Z(\hat{H}))$; where $Z(\hat{H})$ is the center of a connected Langlands dual group for H and $X^*(Z(\hat{H}))$ is the character group of $Z(\hat{H})$ (cf. [Brv2], [Brv3]).

We write Γ for $\mathrm{Gal}(\bar{K}/K)$. Consider the torsion subgroup $(\pi_1(\bar{H})_\Gamma)_{tors}$ of the group of coinvariants $\pi_1(\bar{H})_\Gamma$.

Proposition 2.2. ([Kol] 2.4.1). $B(H) = (\pi_1(\bar{H})_\Gamma)_{tors}$.

Remark 2.2.1. Kottwitz states and proves Proposition 2.2 in terms of $Z(\hat{H})$.

Hereafter K is again a number field. Consider the homomorphism $\lambda_v : B_v(H) \longrightarrow B(H)$. Set $M = \pi_1(\bar{H})$. One can show (cf. [Ko2], 2.5, or [Brv2], Sect. 5) that in terms of $\pi_1(\bar{H})$ the homomorphism λ_v is the obvious map $(M_{\Gamma_v})_{tors} \longrightarrow (M_\Gamma)_{tors}$, where Γ_v is a decomposition group of v in $\bar{K}$ (defined up to conjugation). Let Δ be the image of Γ in Aut $\pi_1(\bar{H})$, and let L/K be the Galois extension of K in $\bar{K}$ corresponding to $\ker[\Gamma \longrightarrow \Delta]$. Let Δ_v be the image of Γ_v in Δ. Then $\mathrm{Gal}(L/K) = \Delta$, and Δ_v is a decomposition group of v in L.

Lemma 2.3. The subgroup $\mathrm{im}\,\lambda_v \subset B(H)$ depends only on the conjugacy class of a decomposition group of v in L.

Proof. The homomorphism λ_v is the corestriction (i.e. obvious map)

$$(M_{\Gamma_v})_{tors} = (M_{\Delta_v})_{tors} \longrightarrow (M_\Delta)_{tors} = (M_\Gamma)_{tors}$$

where $M = \pi_1(\bar{H})$. Thus if $v,w \in \mathscr{V}$ and the groups Δ_v and Δ_w are equal (up to conjugation), then $\operatorname{im} \lambda_v = \operatorname{im} \lambda_w$. q.e.d.

Lemma 2.4. If $H^{tor} = 1$, then $\lambda_v : B_v(H) \longrightarrow B(H)$ is surjective for any $v \in \mathscr{V}$.

Proof. Set $M = \pi_1(\bar{H})$. If $H^{tor} = 1$, then H^{red} is semisimple and M is finite. Hence the homomorphim

$$\lambda_v : B_v(H) = (M_{\Gamma_v})_{tors} = M_{\Gamma_v} \longrightarrow M_\Gamma = (M_\Gamma)_{tors} = B(H)$$

is surjective. q.e.d.

In [Ko2], Theorem 1.2, Kottwitz constructs canonical maps $\beta_v : H^1(K_v,H) \longrightarrow B_v(H)$ for $v \in \mathscr{V}$. He proves

Proposition 2.5. (Local non–archimedian Kottwitz theorem, [Ko1], 6.4, [Ko2], 1.2). If $v \in \mathscr{V}_f$ then β_v is bijective.

Set $\mu_v = \lambda_v \circ \beta_v : H^1(K_v,H) \longrightarrow B_v(H) \longrightarrow B(H)$. Define

$$\mu = \sum \mu_v : H^1(A,H) = \oplus H^1(K_v,H) \longrightarrow B(H)$$
$$\mu((\xi_v)) = \sum_v \mu_v(\xi_v) \ .$$

Consider the localization map $H^1(K,H) \longrightarrow \prod_{v \in \mathcal{V}} H^1(K_v,H)$. Since H is connected, one can easily show that the image of this map lies in $H^1(A,H) = \oplus H^1(K_v,H)$ (one can use Lang's theorem and Hensel's lemma). We denote the map $H^1(K,H) \longrightarrow H^1(A,H)$ by j.

Proposition 2.6. (Global Kottwitz theorem, [Ko2], 2.5, 2.6). $\ker \mu = \operatorname{im} j$.

For future needs we set $\mu_S = \sum \mu_v : H^1(K_S,H) \longrightarrow B(H)$. The image of μ_S is contained in $B'(H)$. Set

$$\chi_S = \mu_S \bmod B^{(S)} : H^1(K_S,H) \longrightarrow Ч_S(H) \ .$$

3. The group $Ч_S(H)$ and the Brauer group.

Proof of Theorem 1.4. First suppose that $H^{tor} = 1$. Then by Lemma 2.4 the homomorphism $\lambda_v : B_v(H) \longrightarrow B(H)$ is surjective for any $v \in \mathcal{V}$. Hence $B(H) = B'(H) = B^{(S)}(H)$, and therefore $Ч_S(H) = 0$.

In the general case let H^{nt} be the kernel of $H \longrightarrow H^{tor}$. We have the exact sequence

$$X(H^{nt}) \longrightarrow \mathrm{Pic}\, H^{tor} \longrightarrow \mathrm{Pic}\, H \longrightarrow \mathrm{Pic}\, H^{nt}$$

(cf. [Sa], 6.11), where the character group $X(H^{nt})$ is trivial. We obtain the exact sequence

$$B(H^{nt}) \longrightarrow B(H) \longrightarrow B(H^{tor}) \longrightarrow 0$$

and similar exact sequences for the groups B_v. We see that in the commutative diagram

$$\begin{array}{ccccccc} B(H^{nt}) = B^{(S)}(H^{nt}) & \longrightarrow & B^{(S)}(H) & \longrightarrow & B^{(S)}(H^{tor}) & \longrightarrow & 0 \\ \| & & \cap & & \cap & & \\ B(H^{nt}) = B'(H^{nt}) & \longrightarrow & B'(H) & \longrightarrow & B'(H^{tor}) & \longrightarrow & 0 \end{array}$$

the rows are exact, whence we derive the desired assertion.

To prove Theorem 1.9. we need to recall the definition of the arithmetic Brauer group (cf. [Sa]). Let $\mathrm{Br}\, X$ denote the Brauer group of X, formed of the equivalence classes of Azumaya algebras on X. Set $\mathrm{Br}_1 X = \ker[\mathrm{Br}\, X \longrightarrow \mathrm{Br}\, \overline{X}]$, $\mathrm{Br}_a X = \mathrm{Br}_1 X/\mathrm{im}[\mathrm{Br}\, K \longrightarrow \mathrm{Br}\, X]$, where $\overline{X} = X_{\overline{K}}$. For $x \in X(K)$ set $\mathrm{Br}_x X = \ker\, [x^* : \mathrm{Br}_1 X \longrightarrow \mathrm{Br}\, K]$. We have the canonical splitting $\mathrm{Br}_1 X = \mathrm{Br}_x X \oplus \mathrm{Br}\, K$, whence $\mathrm{Br}_x X \simeq \mathrm{Br}_a X$. Now let $\mathrm{Br}' X$ be the cohomological Brauer group, i.e. $\mathrm{Br}' X = H^2_{et}(X, \mathbb{G}_m)$. Set $\mathrm{Br}'_1 X = \ker[\mathrm{Br}' X \longrightarrow \mathrm{Br}' \overline{X}]$. Then $\mathrm{Br}'_1 X = \mathrm{Br}_1 X$ (cf. [Sa], (6.1.1)).

For a morphism $f : X \longrightarrow Y$ let

$$\mathrm{Br}_1 f : \mathrm{Br}_1 Y \longrightarrow \mathrm{Br}_1 X, \qquad \mathrm{Br}_x f : \mathrm{Br}_{f(x)} Y \longrightarrow \mathrm{Br}_x X$$

be the canonical homomorphisms. Then

$$\mathrm{Br}_1 f = \mathrm{Br}_x f \oplus 1 : \mathrm{Br}_{f(x)} Y \oplus \mathrm{Br}\, K \longrightarrow \mathrm{Br}_x X \oplus \mathrm{Br}\, K \ .$$

We set $\mathrm{Pic}_1 X = \ker\, [\mathrm{Pic}\, X \longrightarrow \mathrm{Pic}\, \overline{X}]$. For a connected K–group H set $B_1(H) = (\mathrm{Pic}_1 H)^{\sim}$ and define $B_{1v}(H)$, $B_{1v}^{(S)}(H)$, $B_1'(H)$ and $Ш_{1S}(H)$ like $B_v(H) \ldots Ш_S(H)$ but with $B_1(H)$ instead of $B(H)$. Let $\bar{\lambda} : B(\overline{H}) \longrightarrow B(H)$ be the homomorphism induced by the canonical morphism $\overline{H} \longrightarrow H$.

Lemma 3.1. $Ш_{1S}(H) = Ш_S(H)$.

Proof. Consider the morphisms $\overline{H} \longrightarrow H_{K_v} \longrightarrow H$. We see that $\mathrm{im}\, [\lambda_v : B_v(H) \longrightarrow B(H)] \supset \mathrm{im}\, \bar{\lambda}$. Hence $B^{(S)}(H) \supset \mathrm{im}\, \bar{\lambda}$ and $B'(H) \supset \mathrm{im}\, \bar{\lambda}$. We have $B_1(H) = B(H)/\mathrm{im}\, \bar{\lambda}$. Hence $Ш_{1S}(H) = B_1'(H)/B_1^{(S)}(H) \cong B'(H)/B^{(S)}(H) = Ш_S(H)$.

q.e.d.

Proof of Theorem 1.9. The torsor $f : G \longrightarrow X$ under H gives rise to the exact sequence

$$\mathrm{Pic}\, G \longrightarrow \mathrm{Pic}\, H \longrightarrow \mathrm{Br}' X \longrightarrow \mathrm{Br}' G$$

(cf. [Sa], (6.10.1). Since G is simply connected, Pic G = 0 (cf. [Sa], 6.9). From the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Pic}\, H & \longrightarrow & \mathrm{Br}' X & \longrightarrow & \mathrm{Br}' G \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathrm{Pic}\, \overline{H} & \longrightarrow & \mathrm{Br}' \overline{X} & \longrightarrow & \mathrm{Br}' \overline{G}
\end{array}$$

with exact rows, we get the exact sequence

$$0 \longrightarrow \mathrm{Pic}_1 H \longrightarrow \mathrm{Br}_1' X \xrightarrow{\mathrm{Br}_1 f} \mathrm{Br}_1 G \; .$$

Let $x \in X(K)$ be the image of the neutral element $e \in G(K)$. Then

$$\mathrm{Br}_1 f = \mathrm{Br}_e f \oplus 1 : \mathrm{Br}_x X \oplus \mathrm{Br}\, K \longrightarrow \mathrm{Br}_e G \oplus \mathrm{Br}\, K \; .$$

Thus the sequence

$$0 \longrightarrow \mathrm{Pic}_1 H \longrightarrow \mathrm{Br}_x X \xrightarrow{\mathrm{Br}_e f} \mathrm{Br}_e G$$

is exact. But $\mathrm{Br}_e G = \mathrm{Br}_a G = 0$ (cf. [Sa], 6.9), hence $\mathrm{Br}_a X = \mathrm{Br}_x X = \mathrm{Pic}_1 H$.

Now we see that

$$\begin{aligned} B_1^{(S)}(H) &= \mathrm{coker}\,[\bigoplus_{v \notin S} B_{1v}(H) \longrightarrow B_1(H)] \\ &= \mathrm{coker}\,[\bigoplus_{v \notin S} (\mathrm{Pic}_1 H_{K_v})^{\sim} \longrightarrow (\mathrm{Pic}_1 H)^{\sim}] \\ &= \ker\,[\mathrm{Pic}_1 H \longrightarrow \prod_{v \notin S} \mathrm{Pic}_1 H_{K_v}]^{\sim} \\ &= \ker\,[\mathrm{Br}_a X \longrightarrow \prod_{v \notin S} \mathrm{Br}_a X_{K_v}]^{\sim} \\ &= Б_S(X)^{\sim} \end{aligned}$$

Since $B_1'(H) = B_1^{(\emptyset)}(H)$ and $Б(X) = Б_{\emptyset}(X)$, we have $B'(H) = Б(X)^{\sim}$. By definition $Ш_{1S}(H) = B_1'(H)/B_1^{(S)}(H)$, hence

$$\Psi_{1S}(H) = Б(X)^{\sim}/Б_S(X)^{\sim} = (Б_S(X)/Б(X))^{\sim} .$$

By Lemma 3.1 $\Psi_{1S}(H) = \Psi_S(H)$. Thus $\Psi_S(H) = (Б_S(X)/Б(X))^{\sim}$.

q.e.d.

4. Cohomological results

To prove Theorem 1.2 we need

Lemma 4.1. (i) For $v \in \mathscr{V}_f$ we have $\operatorname{im} \lambda_v = \operatorname{im} \mu_v$.

(ii) For any finite $S \subset \mathscr{V}$ and any $v \in \mathscr{V}_\infty$ there exists $w \in \mathscr{V}_f - S$ such that $\operatorname{im} \mu_w = \operatorname{im} \lambda_w = \operatorname{im} \lambda_v \supset \operatorname{im} \mu_v$.

(iii) $B^{(S)}(H) = B^{(S \cap \mathscr{V}_f)}(H) = B^{(S \cup \mathscr{V}_\infty)}(H)$; in particular $B'(H) = B^{(\mathscr{V}_\infty)}(H)$.

(iv) $\Psi_S(H) = \Psi_{S \cap \mathscr{V}_f}(H)$.

Proof. By proposition 2.5 the map β_v is bijective for $v \in \mathscr{V}_f$, whence (i). By Lemma 2.3 the group $\operatorname{im} \lambda_v$ depends only on the conjugacy class of a decomposition group Δ_v of v in L, where L is defined in Section 2. For $v \in \mathscr{V}_\infty$ the group Δ_v is cyclic. Chebotarev's density theorem implies the existence of $w \in \mathscr{V}_f - S$ such that $\Delta_w = \Delta_v$ up to conjugation. Then $\operatorname{im} \lambda_w = \operatorname{im} \lambda_v$ by Lemma 2.3. We have $\operatorname{im} \lambda_v \supset \operatorname{im} \mu_v$; by the assertion (i) $\operatorname{im} \lambda_w = \operatorname{im} \mu_w$. The assertion (ii) is proved. It is clear that (ii) implies (iii) and (iv).

Proof of Theorem 1.2. Consider the sequence

$$H^1(K,H) \xrightarrow{\ j_S\ } H^1(K_S,H) \xrightarrow{\ \chi_S\ } Ч_S(H) \longrightarrow 0$$

where χ_S is defined at the end of Section 2. First we prove that χ_S is surjective. By definition the group $B'(H)$ is generated by the groups $\lambda_v(B_v(H))$ for all $v \in \mathscr{V}$. However, by Lemma 4.1 (iii) $B'(H) = B^{(\mathscr{V}_\infty)}(H)$, i.e. $B'(H)$ is generated by the groups $\mathrm{im}\,\lambda_v = \mathrm{im}\,\mu_v$ for $v \in \mathscr{V}_f = (S \cap \mathscr{V}_f) \cup (\mathscr{V}_f - S)$. We see that

$$\mathrm{im}\,[\mu_S : H^1(K_S,H) \longrightarrow B(H)] + B^{(S)}(H) \supset \mathrm{im}\,\mu_{S\cap \mathscr{V}_f} + B^{(S)}(H) = B'(H)\ ,$$

and the desired surjectivity follows.

Proposition 2.6 implies that $\mathrm{im}(\mu_S \circ j_S) \subset B^{(S)}(H)$, hence $\chi_S \circ j_S = 0$. We must show that $\ker \chi_S = \mathrm{im}\, j_S$. Suppose that $\xi_S \in H^1(K_S,H)$, $\chi_S(\xi_S) = 0$. Then $\mu_S(\xi_S) \in B^{(S)}(H)$. By Lemma 4.1 (iii) $\mu_S(\xi_S) \in B^{(S \cup \mathscr{V}_\infty)}(H)$. Hence there are a finite set $\Sigma \subset \mathscr{V}_f - S$ and an element $\xi_\Sigma \in H^1(K_\Sigma,H)$ such that $\mu_S(\xi_S) + \mu_\Sigma(\xi_\Sigma) = 0$. Set $\xi_A = \xi_S \times \xi_\Sigma \times 0 \in H^1(A,H)$. Then $\mu(\xi_A) = 0$. By Proposition 2.6 $\xi_A = j(\xi)$ for some $\xi \in H^1(K,H)$. Clearly $\xi_S = j_S(\xi)$. q.e.d.

5. Weak approximation

To prove Theorem 1.1 consider the orbit spaces: the set of orbits $\mathcal{O}(X,G,K)$ of $G(K)$ in $X(K)$ and the set of orbits $\mathcal{O}(X,G,K_S)$ of $G(K_S)$ in $X(K_S) = \prod_{v \in S} X(K_v)$. Any orbit of $G(K_S)$ in $X(K_S)$ is open, hence any orbit is closed. Since G is simply connected, G satisfies (WA_S) for any finite S (cf. [Pl1], or [Pl2], § 4). Thus $\widehat{(x \cdot G(K))}_S = x \cdot \widehat{G(K)}_S = x \cdot G(K_S)$ for any $x \in X(K)$. Consider the map

$i_S : \mathcal{O}(X,G,K) \longrightarrow \mathcal{O}(X,G,K_S)$ induced by the embedding $X(K) \longrightarrow X(K_S)$. We see that $\hat{X(K)}_S = \bigcup o$ where o runs over $\operatorname{im} i_S$. Thus (WA_S) for X is equivalent to the surjectivity of i_S.

The orbit spaces can be described in cohomological terms. Set

$$k = \ker[H^1(K,H) \longrightarrow H^1(K,G)] ,$$
$$k_S = \ker[H^1(K_S,H) \longrightarrow H^1(K_S,G)] .$$

Using the exact cohomology sequences associated with the subgroup H of G (see [Se], Ch.1, § 5.4, Cor. 1 of Prop. 36), we can identify

$$\mathcal{O}(X,G,K) = k , \quad \mathcal{O}(X,G,K_S) = k_S .$$

In these terms the map $i_S : k \longrightarrow k_S$ is the restriction of the localization map $j_S : H^1(K,H) \longrightarrow H^1(K_S,H)$ to k. Now it is clear that Theorem 1.1 follows from

Proposition 5.1. Let ν_S be the restriction of $\chi_S : H^1(K_S,H) \longrightarrow {}_S(H)$ to $k_S \subset H^1(K_S,H)$. Then the sequence

$$k \xrightarrow{i_S} k_S \xrightarrow{\nu_S} Ч_S(H) \longrightarrow 0$$

is exact.

Proof of Proposition 5.1. First we prove the surjectivity of ν_S. Set $\Sigma = S \cap \mathcal{V}_f$. By Theorem 1.2 χ_Σ is surjective, i.e. $B^{(\Sigma)}(H) + \operatorname{im} \mu_\Sigma = B'(H)$. Since $H^1(K_\Sigma,G) = 0$ by Kneser's theorem, $k_\Sigma = H^1(K_\Sigma,H)$. Hence $\mu_\Sigma(k_\Sigma) + B^{(\Sigma)}(H) = B'(H)$. By Lemma 4.1

(iii) $B^{(\Sigma)}(H) = B^{(S)}(H)$. Since $\mu_\Sigma(k_\Sigma) \subset \mu_S(k_S)$, we see that $\mu_S(k_S) + B^{(S)}(H) = B'(H)$, i.e. $\nu_S(k_S) = Ч_S(H)$.

By Theorem 1.2 $\chi_S \circ j_S = 0$, hence $\nu_S \circ i_S = 0$. We must show that $\ker \nu_S = \operatorname{im} i_S$. Assume that $\xi_S \in k_S$ and $\nu_S(\xi_S) = 0$ (i.e. $\chi_S(\xi_S) = 0$). We construct an element $\xi \in H^1(K,H)$ as in the proof of Theorem 1.2. Then $j_S(\xi) = \xi_S$. We wish to prove that $\xi \in k$. By construction we have $j_S(\xi) = \xi_A = (\xi_v^*)_{v \in \mathscr{V}} \in H^1(A,H)$, where $\xi_v^* = \xi_v$ for $v \in S$ and $\xi_v^* = 0$ for $v \in \mathscr{V}_\infty - S$. Hence $\xi_v^* \in k_v = \ker[H^1(K_v,H) \longrightarrow H^1(K_v,G)]$ for any $v \in \mathscr{V}_\infty$. By the Hasse principle for G , we have $\xi \in k$. Proposition 5.1 is proved, and so is Theorem 1.1.

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