A COHOMOLOGICAL OBSTRUCTION TO THE HASSE PRINCIPLE FOR HOMOGENEOUS SPACES

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Abstract. For a homogeneous space with connected or abelian stabilizer of a connected linear algebraic group defined over a number field, a cohomological obstruction to the Hasse principle is defined in terms of Galois hypercohomology with coefficients in a complex of two abelian algebraic groups. This obstruction is proved to be the only obstruction to the Hasse principle. It is proved that up to sign this cohomological obstruction coincides with the Brauer-Manin obstruction.

Introduction

In [Bo2] we proved that for a homogeneous space of a connected linear algebraic group with connected or abelian stabilizer, the Brauer-Manin obstruction to the Hasse principle is the only obstruction. This obstruction was not computed, however. Here we define an obstruction to the Hasse principle for homogeneous spaces in terms of Galois hypercohomology. We prove that this is the only obstruction to the Hasse principle. Then we prove that this obstruction up to sign is the Brauer-Manin obstruction.

In more detail, let X be a homogeneous space of a connected linear algebraic group G defined over a number field k. Let $\bar{x} \in X(\bar{k})$ be a point, and let $\bar{H} = \mathrm{Stab}(\bar{x})$ be the stabilizer of $\bar{x}$ in $G_{\bar{k}}$. In the Introduction we assume for simplicity that $\bar{H}$ is connected. We may assume that "the semisimple part" G^{ss} of G is simply connected, because a homogeneous space X of a connected group G with connected stabilizer is a homogeneous space with connected stabilizer of another connected group G' such that $(G')^{\mathrm{ss}}$ is simply connected, cf. [Bo2], 5.2.

The stabilizer $\bar{H}$ of $\bar{x}$ in general is not defined over k. Let $\bar{H}^{\mathrm{tor}}$ denote the maximal toric quotient of $\bar{H}$. Then $\bar{H}^{\mathrm{tor}}$ has a canonical k-form

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H^{m}, cf. [Bo2], 4.1. or 1.2 below. We show in 1.2 that the induced homomorphism $H^{\mathrm{m}}_{\bar{k}} = \bar{H}^{\mathrm{tor}} \to G^{\mathrm{tor}}_{\bar{k}}$ is defined over k.

Let $H^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}})$ denote the first Galois hypercohomology group of $\mathrm{Gal}(\bar{k}/k)$ with coefficients in the complex of abelian k-groups

$$0 \to H^{\mathrm{m}} \to G^{\mathrm{tor}} \to 0$$

where G^{tor} is in degree 0 and H^{m} is in degree -1. In Section 1 we construct an obstruction

$$\eta(X) \in H^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}})$$

to the existence of a k-point in X.

If $X(k_v) \neq \varnothing$ for every completion k_v of k, then the localization $\mathrm{loc}_v \eta(X)$ is zero for any place v of k. Hence $\eta(X) \in \ker^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}})$, where

$$\ker^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}}) := \ker \left[H^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}}) \to \prod_v H^1(k_v, H^{\mathrm{m}} \to G^{\mathrm{tor}}) \right],$$

v running over all places of k. We obtain an obstruction to the Hasse principle for X: if $\eta(X) \neq 0$ then $X(k) = \varnothing$, though $X(k_v) \neq \varnothing$ for every place v of k.

In Section 2 we prove that the obstruction $\eta(X)$ is the only obstruction to the Hasse principle for X. This means that if $X(k_v) \neq \varnothing$ for any place v of k and $\eta(X) = 0$, then $X(k) \neq \varnothing$.

In Section 3 we compute the group Б(X) related to the Brauer group of X, cf. Notation and Conventions. The (first) Brauer-Manin obstruction to the Hasse principle lives in the dual group Б$(X)^D$ to Б(X), cf. [Bo2].

In Section 4 a proposition generalizing Tate-Nakayama duality for tori is proved. Using this result, we identify $\ker^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}})$ and Б$(X)^D$, and prove that the Brauer-Manin obstruction to the Hasse principle for X up to sign equals $\eta(X)$. Thus we compute the Brauer-Manin obstruction in terms of Galois hypercohomology.

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Notation and Conventions

k is always a field of characteristic 0, and $\bar{k}$ is an algebraic closure of k. If not otherwise stated, k is assumed to be a number field. Then

$\mathcal{V} = \mathcal{V}(k)$ is the set of places of k; $\mathcal{V}_f$ and $\mathcal{V}_\infty$ are the sets of finite places and of infinite places of k, respectively; we write k_v for the completion of k at a place v.

If S is a finite set of places of k, then $k_S = \prod_{v\in S} k_v$; we write $k_\infty = k_{\mathcal{V}_\infty}$.

By a k-variety we mean a geometrically irreducible algebraic variety X over a field k; we assume X to be nonsingular. We set $U(X) = k[X]^\times/k^\times$. We write Pic X for the Picard group of X, and $\operatorname{Br} X$ for the cohomological Brauer group of X, $\operatorname{Br} X = H^2_{\text{ét}}(X, \mathbb{G}_m)$. We set $\operatorname{Br}_0 X = \operatorname{im}[\operatorname{Br} k \to \operatorname{Br} X]$; $\operatorname{Br}_1 X = \ker[\operatorname{Br} X \to \operatorname{Br} X_{\bar{k}}]$; $\operatorname{Br}_{\mathrm{a}} X = \operatorname{Br}_1 X/\operatorname{Br}_0 X$, the "algebraic" Brauer group of X.

If k is a number field and S a finite subset of $\mathcal{V}(k)$, then Б(X) is the subgroup of $\operatorname{Br}_{\mathrm{a}} X$ consisting of elements b whose localizations $\operatorname{loc}_v b$ in $\operatorname{Br}_{\mathrm{a}} X_{k_v}$ are trivial for all places v of k.

By a k-group we mean an affine algebraic group defined over k. If H is a k-group, then H° is the connected component of identity of H; H^{u} is the unipotent radical of H°; $H^{\mathrm{red}} = H^\circ/H^{\mathrm{u}}$ (it is a connected reductive group); H^{ss} is the derived group of H^{red} (it is semisimple); $H^{\mathrm{tor}} = H^{\mathrm{red}}/H^{\mathrm{ss}}$ (it is a torus); $H^{\mathrm{ssu}} = \ker[H^\circ \to H^{\mathrm{tor}}]$ (it is an extension of H^{ss} by H^{u}).

Recall that an algebraic group is of multiplicative type if it is abelian and its connected component is a torus. The group H/H^{ssu} is an extension of a finite group by a torus. We always assume that the group H/H^{ssu} is abelian, hence of multiplicative type, and set $H^{\mathrm{mult}} = H/H^{\mathrm{ssu}}$.

For any abelian group A we set $A^D = \operatorname{Hom}(A, \mathbf{Q}/\mathbf{Z})$ (the dual group of A).

1. Obstruction

1.1. Let X be a homogeneous space of a connected affine algebraic group G defined over a number field k. This means that there is a right action $X \times G \to X$ of G on X defined over k, which is transitive (i.e. $G(\bar{k})$ acts on $X(\bar{k})$ transitively). We do not assume that X has a k-point. Let $\bar{x} \in X(\bar{k})$ be a $\bar{k}$-point, and $\bar{H} = \operatorname{Stab}_{G_{\bar{k}}}(\bar{x})$ its stabilizer in $G_{\bar{k}}$. We will assume that

$$G^{\mathrm{ss}} \text{ is simply connected,} \tag{1.1.1}$$

$$\bar{H}/\bar{H}^{\mathrm{ssu}} \text{ is abelian, hence of multiplicative type} \tag{1.1.2}$$

(see Notation and Conventions), and write $\bar{H}^{\mathrm{mult}}$ for $\bar{H}/\bar{H}^{\mathrm{ssu}}$. Note that if G^{ss} is simply connected then G^{ssu} is simply connected.

1.2. Recall the definition of the k-form $H^{\rm m}$ of $\bar H^{\rm mult}$ depending only on G and X, cf. [Bo2], 4.1.

Chose $\bar x\in X(\bar k)$; we say that $\bar x$ is a base point. Let $\bar H$ be the stabilizer of $\bar x$ in $G_{\bar k}$. For $\sigma\in{\rm Gal}(\bar k/k)$, choose $g_\sigma\in G(\bar k)$ such that ${}^\sigma\bar x=\bar x\cdot g_\sigma$. For $h\in\bar H(\bar k)$ we have

$$\bar x\cdot h=\bar x,\quad \bar x\cdot g_\sigma{}^\sigma h=\bar x\cdot g_\sigma,\quad g_\sigma{}^\sigma\bar H g_\sigma^{-1}=\bar H$$

so we obtain a σ-semialgebraic automorphism (see [Bo1], 1.1 for a definition)

$$\nu_\sigma\colon \bar H\to\bar H,\quad h\mapsto g_\sigma{}^\sigma h g_\sigma^{-1}$$

of the algebraic group $\bar H$. Recall that if H is a k-form of $\bar H$, then a σ-semialgebraic automorphism of $\bar H$ is a composition of the Galois automorphism $\sigma_*\colon H_{\bar k}\to H_{\bar k}$ and an algebraic (usual) automorphism of $\bar H$; however the notion of a σ-semialgebraic automorphism is defined without assuming that $\bar H$ admits a k-form.

Clearly ν_σ induces a σ-semialgebraic automorphism $\nu_\sigma^{\rm m}$ of $\bar H^{\rm mult}$. If we choose another g_σ, say $g'_\sigma=h'g_\sigma$ with $h'\in\bar H(\bar k)$, then we obtain $\nu'_\sigma={\rm int}(h')\circ\nu_\sigma$. Since the inner automorphisms of $\bar H$ act trivially on $\bar H^{\rm mult}$, we see that the σ-semialgebraic automorphism $\nu_\sigma^{\rm m}$ of $\bar H^{\rm mult}$ does not depend on the choice of g_σ.

Let $g_\sigma,g_\tau\in G(\bar k)$ be such that ${}^\sigma\bar x=\bar x\cdot g_\sigma$, ${}^\tau\bar x=\bar x\cdot g_\tau$. Set $g_{\sigma\tau}=g_\sigma\cdot{}^\sigma g_\tau$; then ${}^{\sigma\tau}\bar x=\bar x\cdot g_{\sigma\tau}$. We use g_σ, g_τ and $g_{\sigma\tau}$ to define ν_σ, ν_τ and $\nu_{\sigma\tau}$, respectively. One can easily check that $\nu_{\sigma\tau}=\nu_\sigma\circ\nu_\tau$. It follows that $\nu_{\sigma\tau}^{\rm m}=\nu_\sigma^{\rm m}\circ\nu_\tau^{\rm m}$, hence the family $(\nu_\sigma^{\rm m})_{\sigma\in{\rm Gal}(\bar k/k)}$ is an action of the Galois group on $\bar H^{\rm mult}$, defining a k-form $H^{\rm m}$ of the $\bar k$-group $\bar H^{\rm mult}$. We have a canonical epimorphism $\mu\colon\bar H\to H^{\rm m}_{\bar k}$, and $\mu(g_\sigma{}^\sigma h g_\sigma^{-1})={}^\sigma(\mu(h))$.

Now let $\bar x_1\in X(\bar k)$ be another $\bar k$-point and $\bar H_1$ its stabilizer in $G_{\bar k}$. Let $H_1^{\rm m}$ be the corresponding k-form of $\bar H_1^{\rm mult}$. Choose $g\in G(\bar k)$ such that $\bar x_1=x\cdot g$. Consider the isomorphism

$$\lambda_g\colon\bar H\to\bar H_1,\quad \lambda_g(h)=g^{-1}hg\ .$$

One can easily check that the induced isomorphism $\lambda_g^{\rm m}\colon\bar H^{\rm mult}\to\bar H_1^{\rm mult}$ does not depend on the choice of g and defines a canonical isomorphism $H^{\rm m}\overset{\sim}{\to}H_1^{\rm m}$. We can therefore identify $H_1^{\rm m}$ with $H^{\rm m}$.

Consider the map

$$i_*\colon H^{\rm m}_{\bar k}=\bar H^{\rm mult}\to G^{\rm tor}_{\bar k}$$

induced by the inclusion map $i\colon\bar H\to G_{\bar k}$. Note that i_* is defined over k. Indeed, if $h\in\bar H(\bar k)$, $\quad\hat h=h \bmod \bar H^{\rm ssu}$, then

$$i_*({}^\sigma\hat h)=g_\sigma{}^\sigma h g_\sigma^{-1} \bmod G^{\rm ssu}\ ={}^\sigma h \bmod G^{\rm ssu}\ ={}^\sigma(h \bmod G^{\rm ssu})\ ={}^\sigma(i_*(\hat h)).$$

1.3. We define a cohomological obstruction to the existence of a k-point in X. For $\sigma,\tau\in \mathrm{Gal}(\bar k/k)$, we have

$$^{\sigma\tau}\bar x=\bar x\cdot g_{\sigma\tau},\quad {}^{\sigma}({}^{\tau}\bar x)={}^{\sigma}(\bar x\cdot g_\tau)=\bar x\cdot g_\sigma{}^{\sigma}g_\tau,\quad \bar x\cdot g_{\sigma\tau}(g_\sigma{}^{\sigma}g_\tau)^{-1}=\bar x.$$

Set $u_{\sigma,\tau}=g_{\sigma\tau}(g_\sigma{}^{\sigma}g_\tau)^{-1}$, then

$$u_{\sigma,\tau}\in \bar H(\bar k),\qquad u_{\sigma,\tau}g_\sigma{}^{\sigma}g_\tau=g_{\sigma\tau}.$$

For $\sigma,\tau,\upsilon\in\mathrm{Gal}(\bar k/k)$ one can easily check that

$$u_{\sigma,\tau\upsilon}g_\sigma{}^{\sigma}u_{\tau,\upsilon}g_\sigma^{-1}=u_{\sigma\tau,\upsilon}u_{\sigma,\tau}. \tag{1.3.1}$$

Set $\hat u_{\sigma,\tau}=u_{\sigma,\tau}\mod \bar H^{\mathrm{ssu}}\in H^{\mathrm{m}}(\bar k)$. Then the equality (1.3.1) yields

$$\hat u_{\sigma,\tau\upsilon}{}^{\sigma}\hat u_{\tau,\upsilon}=\hat u_{\sigma\tau,\upsilon}\hat u_{\sigma,\tau}. \tag{1.3.2}$$

Set $\hat g_\sigma=g_\sigma\mod G^{\mathrm{ssu}}\in G^{\mathrm{tor}}(\bar k)$; then we have

$$i_*(\hat u_{\sigma,\tau})\hat g_\sigma{}^{\sigma}\hat g_\tau=\hat g_{\sigma\tau} \tag{1.3.3}$$

(recall that $i\colon \bar H\to G$ is the inclusion map). The equalities (1.3.2) and (1.3.3) mean that $(\hat u,\hat g)$ is a hypercocycle,

$$(\hat u,\hat g)\in Z^1(k,H^{\mathrm{m}}\to G^{\mathrm{tor}}),$$

where G^{tor} is in degree 0 and H^{m} is in degree -1. Let $\eta(X)$ denote the hypercohomology class of $(\hat u,\hat g)$, $\eta(X)=\mathrm{Cl}(\hat u,\hat g)\in H^1(k,H^{\mathrm{m}}\to G^{\mathrm{tor}})$.

1.4. One can easily check that the hypercohomology class $\eta(X)$ is well-defined.

Let us take another system of (g_σ), say $g'_\sigma=h_\sigma g_\sigma$ with $h_\sigma\in\bar H(\bar k)$. Then we obtain another cocycle $(\hat u',\hat g')\in Z^1(k,H^{\mathrm{m}}\to G^{\mathrm{tor}})$, and an easy calculation shows that $(\hat u',\hat g')$ is cohomologous to $(\hat u,\hat g)$. Thus the hypercohomology class $\eta(X)$ does not depend on the choice of a system (g_σ).

Now let us take instead of $\bar x$ another base point, say $\bar x''=\bar x\cdot g$ where $g\in G(\bar k)$. Then we can identify $\bar H=\mathrm{Stab}(\bar x)$ with $\bar H''=\mathrm{Stab}(\bar x'')$ using

$$\lambda\colon \bar H\to\bar H'',\ \lambda(h)=g^{-1}hg,\ h\in\bar H(\bar k)\ .$$

We obtain an induced map $\lambda_*\colon H^{\mathrm{m}}\to(H'')^{\mathrm{m}}$. An easy calculation shows that under this identification the corresponding hypercocycle $(\hat u'',\hat g'')$ is cohomologous to $(\hat u,\hat g)$. Thus the hypercohomology class $\eta(X)$ does not depend on the choice of the base point $\bar x\in X(\bar k)$.

1.5. Now assume that $X(k)\neq\varnothing$. Take a k-point $x\in X(k)$ for the base point. We obtain $g_\sigma=1,\ u_{\sigma,\tau}=1$ for any $\sigma,\tau\in\mathrm{Gal}(\bar k/k)$, hence $\eta(X)=0$. Thus if $\eta(X)\neq 0$ then $X(k)=\varnothing$. We see that $\eta(X)$ is an obstruction to the existence of a k-point in X.

1.6. Note that the hypercohomology class $\eta(X)$ is functorial in (G, X). Namely, let $\psi\colon G_1 \to G_2$ be a homomorphism of k-groups and $\varphi\colon X_1 \to X_2$ be a ψ-equivariant morphism of homogeneous spaces, where (G_1, X_1) and G_2, X_2 are as in 1.1. Let $\bar{x}_1 \in X_1(\bar{k})$, $\bar{x}_2 = \varphi(\bar{x}_1) \in X_2(\bar{k})$, $\bar{H}_1 = \mathrm{Stab}(\bar{x}_1)$, $\bar{H}_2 = \mathrm{Stab}(\bar{x}_2)$. We define $\eta(X_1) \in H^1(k, H_1^{\mathrm{m}} \to G_1^{\mathrm{tor}})$, $\eta(X_2) \in H^1(k, H_2^{\mathrm{m}} \to G_2^{\mathrm{tor}})$ as in 1.3. We have a morphism of complexes $\psi_*\colon (H_1^{\mathrm{m}} \to G_1^{\mathrm{tor}}) \to (H_2^{\mathrm{m}} \to G_2^{\mathrm{tor}})$. Then one can easily see that $\eta(X_2) = \psi_*(\eta(X_1))$.

2. Case of a local field or a number field

Theorem 2.1. *Let k be a nonarchimedean local field. Let G, X be as in* 1.1. *If $\eta(X) = 0$ then $X(k) \neq \varnothing$.*

Proof. The idea of the proof is the following. Using the assumption $\eta(X) = 0$, we choose $\bar{x} \in X(\bar{k})$ and (g_σ) such that $g_\sigma \in G^{\mathrm{ssu}}(\bar{k})$ for all σ and $u_{\sigma,\tau} \in \bar{H}^{\mathrm{ssu}}(\bar{k})$ for all σ, τ. Then we use a result of [Bo1] asserting that every element of the second nonabelian Galois cohomology of k with coefficients in $\bar{H}^{\mathrm{ssu}}$ is neutral, and Kneser's theorem $H^1(k, G^{\mathrm{ssu}}) = 1$ (both are valid because k is a nonarchimedean local field). Using these results we find $x \in X(k)$.

We start proving the theorem. Let $\bar{x} \in X(\bar{k})$. Write ${}^\sigma\bar{x} = \bar{x} \cdot g_\sigma$, $u_{\sigma,\tau} g_\sigma {}^\sigma g_\tau = g_{\sigma\tau}$, where $g_\sigma \in G(\bar{k})$, $u_{\sigma,\tau} \in \bar{H}(\bar{k})$.

Now assume that $\eta(X) = 0$. We have $\eta(X) = \mathrm{Cl}(\hat{u}, \hat{g})$ where $\hat{u}, \hat{g}$ are defined in 1.3. The fact that the hypercocycle $(\hat{u}, \hat{g})$ is cohomologous to zero means that there exist $a\colon \mathrm{Gal}(\bar{k}/k) \to H^{\mathrm{m}}(\bar{k})$ and $s \in G^{\mathrm{tor}}(\bar{k})$ such that

$$\hat{g}_\sigma = s^{-1} i_*(a_\sigma) \cdot {}^\sigma s, \quad u_{\sigma,\tau} = a_\sigma {}^\sigma a_\tau a_{\sigma\tau}^{-1}$$

for all $\sigma, \tau \in \mathrm{Gal}(\bar{k}/k)$.

We choose $\tilde{a}_\sigma \in \bar{H}(\bar{k})$ representing a_σ ($\sigma \in \mathrm{Gal}(\bar{k}/k)$) and $\tilde{s} \in G(\bar{k})$ representing s. Set

$$\begin{aligned}
\bar{x}' &= \bar{x} \cdot \tilde{s}^{-1} \ \in X(\bar{k}),\\
\bar{H}' &= \mathrm{Stab}(\bar{x}'),\\
g'_\sigma &= \tilde{s}\tilde{a}_\sigma g_\sigma {}^\sigma\tilde{s}^{-1},\\
u'_{\sigma,\tau} &= g'_{\sigma\tau}(g'_\sigma {}^\sigma g'_\tau)^{-1}.
\end{aligned}$$

Easy calculations show that

$$\begin{gathered}
{}^\sigma\bar{x}' = \bar{x}' \cdot g'_\sigma\ , \quad u'_{\sigma,\tau} g'_\sigma {}^\sigma g'_\tau = g'_{\sigma\tau}\ ,\\
g'_\sigma \in G^{\mathrm{ssu}}(\bar{k}), \quad u'_{\sigma,\tau} \in (\bar{H}')^{\mathrm{ssu}}(\bar{k}) \subset \bar{H}'(\bar{k}).
\end{gathered}$$

We can therefore assume that $g_\sigma \in G^{\rm ssu}(\bar k)$, $u_{\sigma,\tau} \in \bar H^{\rm ssu}(\bar k)$. Set $f_\sigma(h) = g_\sigma {}^\sigma h g_\sigma^{-1}$; then $f_\sigma\colon \bar H^{\rm ssu} \to \bar H^{\rm ssu}$ is a σ-semialgebraic automorphism in the sense of [Bo1], 1.1, and

$$\mathrm{int}(u_{\sigma,\tau}) \circ f_\sigma \circ f_\tau = f_{\sigma\tau}, \qquad u_{\sigma,\tau\upsilon} f_\sigma(u_{\tau,\upsilon}) = u_{\sigma\tau,\upsilon} u_{\sigma,\tau}\ .$$

We see that (f,u) is a 2-dimensional nonabelian cocycle, cf. [Bo1], and it defines a nonabelian cohomology class in $H^2(k, \bar H^{\rm ssu}, \kappa)$ for suitable κ, cf. [Bo1], 1.3. Since k is a nonarchimedean local field and $\bar H^{\rm ssu}$ is connected with trivial toric part, by [Bo1], 6.3(ii) every cohomology class in $H^2(k, \bar H^{\rm ssu}, \kappa)$ is neutral. Hence the cohomology class of (f,u) is neutral. This means that there exists a family (c_σ), $c_\sigma \in \bar H^{\rm ssu}(\bar k)$ $(\sigma \in \mathrm{Gal}(\bar k/k))$ such that

$$c_{\sigma\tau} u_{\sigma,\tau} g_\sigma {}^\sigma c_\tau^{-1} g_\sigma^{-1} c_\sigma^{-1} = 1.$$

Set $g''_\sigma = c_\sigma g_\sigma$, $u''_{\sigma,\tau} = g''_{\sigma\tau} ({}^\sigma g''_\tau)^{-1} (g''_\sigma)^{-1}$. An easy calculation shows that $u''_{\sigma,\tau} = 1$. Hence $g''_{\sigma\tau} = g''_\sigma {}^\sigma g''_\tau$. We see that (g''_σ) is a 1-cocycle of $\mathrm{Gal}(\bar k/k)$ with values in $G^{\rm ssu}$. Since k is local nonarchimedean and $G^{\rm ssu}$ is simply connected, by Kneser's theorem (cf. [Kn]) $H^1(k, G^{\rm ssu}) = 1$, hence (g''_σ) is a coboundary. This means that there exists $g \in G^{\rm ssu}(\bar k)$ such that $g''_\sigma = g^{-1}\, {}^\sigma g$. Then $g g''_\sigma {}^\sigma g^{-1} = 1$.

Set $x = \bar x \cdot g^{-1}$. Then

$${}^\sigma x = {}^\sigma \bar x \cdot {}^\sigma g^{-1} = \bar x \cdot g''_\sigma {}^\sigma g^{-1} = x \cdot g g''_\sigma {}^\sigma g^{-1} = x$$

because $g g''_\sigma {}^\sigma g^{-1} = 1$. Thus ${}^\sigma x = x$, so $x \in X(k)$ and $X(k) \neq \varnothing$. □

Theorem 2.2. *Let k be a number field, and let G, X be as in* 1.1. *Assume that $X(k_v) \neq \varnothing$ for every place v of k and that $\eta(X) = 0$. Then $X(k) \neq \varnothing$.*

Proof. Choose an embedding $j\colon H^{\rm m} \to T$ of $H^{\rm m}$ into an induced torus T. Since $X(k_v) \neq \varnothing$ for any place v of k, by [Bo2], 4.3 there exists a torsor $(Y, \pi\colon Y \to X)$ under T, where Y is a homogeneous space of the group $F = G \times T$ and π is F-equivariant with respect to the projection $F \to G$. It suffices to prove that $Y(k) \neq \varnothing$.

We have $F^{\rm tor} = G^{\rm tor} \times T$, hence the map $H^{\rm m} \to F^{\rm tor}$ is injective. From the exact sequence of complexes

$$1 \to (1 \to T) \to (H^{\rm m} \to F^{\rm tor}) \to (H^{\rm m} \to G^{\rm tor}) \to 1$$

we obtain exact sequences

$$0 = H^1(k, T) \to H^1(k, H^{\rm m} \to F^{\rm tor}) \to H^1(k, H^{\rm m} \to G^{\rm tor}) \to H^2(k, T)$$

and

$$0 \to \ker^1(k, H^{\rm m} \to F^{\rm tor}) \to \ker^1(k, H^{\rm m} \to G^{\rm tor}) \to \ker^2(k, T) = 0\ ,$$

where $\ker^2(k,T) := \ker\,[H^2(k,T) \to \prod_v H^2(k_v,T)]$, and $\ker^2(k,T) = 0$ because T is an induced torus. We have $\pi_*\eta(Y) = \eta(X)$, hence $\eta(Y) = 0$. It suffices to prove our theorem for Y. We have therefore reduced the theorem to the case when the map $H^{\mathrm{m}} \to G^{\mathrm{tor}}$ is injective.

Now assume that the map $H^{\mathrm{m}} \to G^{\mathrm{tor}}$ is injective, i.e. $\bar{H} \cap G_{\bar{k}}{}^{\mathrm{ssu}} = \bar{H}^{\mathrm{ssu}}$. Set $Y = X/G^{\mathrm{ssu}}$; this quotient exists, cf. [Bo2], 3.1. Let $\varphi\colon X \to Y$ be the quotient map. We have $\eta(Y) = \varphi_*\eta(X)$, hence $\eta(Y) = 0$. The variety Y is a homogeneous space of G^{tor} with stabilizer H^{m}, hence a principal homogeneous space of $G^{\mathrm{tor}}/H^{\mathrm{m}}$. We see that

$$\eta(Y) \in H^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}}) = H^1(k, G^{\mathrm{tor}}/H^{\mathrm{m}})\ ,$$

and $\eta(Y) = 0$, hence the principal homogeneous space Y of the torus $G^{\mathrm{tor}}/H^{\mathrm{m}}$ has a k-point $y_0 \in Y(k)$.

The set $\varphi(X(k_\infty))$ is nonempty (because $X(k_v) \neq \varnothing$ for every v) and open in $Y(k_\infty)$. By the real approximation theorem for tori (Serre, cf. [Sa], 3.5(iii)), $Y(k)$ is dense in $Y(k_\infty)$. It follows that there exists $y \in Y(k) \cap \varphi(X(k_\infty))$.

Consider the fibre $X_y = \varphi^{-1}(y) \subset X$. We know that $X_y(k_\infty) \neq \varnothing$, because $y \in \varphi(X(k_\infty))$. Since X_y is a homogeneous space of a simply connected group G^{ssu} with stabilizer $\bar{H}^{\mathrm{ssu}}$ satisfying $(\bar{H}^{\mathrm{ssu}})^{\mathrm{m}} = 1$, by [Bo2], 3.4 $X_y(k) \neq \varnothing$. Hence $X(k) \neq \varnothing$. $\square$

Corollary 2.3. *Let k be a number field and let G, X be as in* 1.1. *Assume that $X(k_v) \neq \varnothing$ for every $v \in \mathcal{V}_\infty$ and that $\eta(X) = 0$. Then $X(k) \neq \varnothing$.*

Proof. Since $\eta(X) = 0$, we see that $\eta(X_{k_v}) = 0$ for every v, because $\eta(X_{k_v}) \in H^1(k_v, H^{\mathrm{m}} \to G^{\mathrm{tor}})$ is the localization of η at v. By Theorem 2.1 $X(k_v) \neq \varnothing$ for every $v \in \mathcal{V}_f$. By hypothesis $X(k_v) \neq \varnothing$ for every $v \in \mathcal{V}_\infty$. Thus $X(k_v) \neq \varnothing$ for every place v of k. By Theorem 2.2 $X(k) \neq \varnothing$. $\square$

2.4. We note that our cohomological obstruction $\eta(X)$ is the only obstruction to the Hasse principle for X. Indeed, let k be a number field, and assume that $X(k_v) \neq \varnothing$ for every place v of k. Then the localization $\mathrm{loc}_v\eta \in H^1(k_v, H^{\mathrm{m}} \to G^{\mathrm{tor}})$ is zero for every v. It follows that $\eta(X)$ is contained in the Tate-Shafarevich kernel

$$\ker^1(H^{\mathrm{m}} \to G^{\mathrm{tor}}) = \ker\left[H^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}}) \to \prod_v H^1(k_v, H^{\mathrm{m}} \to G^{\mathrm{tor}})\right].$$

We have seen in 1.5 that $\eta(X)$ is an obstruction to the existence of a k-point in X, thus $\eta(X)$ is an obstruction to the Hasse principle for X. By Theorem 2.2 if $\eta(X) = 0$ then $X(k) \neq \varnothing$. This proves that $\eta(X)$ is the only obstruction.

3. The group Б(X)

In this section we compute the group Б$(X) = \ker[\mathrm{Br_a}\, X \to \prod \mathrm{Br_a}\, X_{k_v}]$ for a homogeneous space X as in 1.1 over a number field k.

3.1. Let k be a field of characteristic 0. Let G, X, $\bar{x}$, $\bar{H}$ be as in 1.1. We write $\mathbf{X}(G)$ for the character group $\mathbf{X}(G_{\bar{k}})$ with the corresponding Galois action. We have $\mathbf{X}(G) = \mathbf{X}(G^{\mathrm{tor}})$. We write $\mathbf{X}(H^{\mathrm{m}})$ for the character group $\mathbf{X}(\bar{H}^{\mathrm{mult}})$ with the Galois action defined by the k-form H^{m} of $\bar{H}^{\mathrm{mult}}$. Consider the complex of finitely generated groups with Galois action $\mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}})$, where $\mathbf{X}(G^{\mathrm{tor}})$ is in degree 0 and $\mathbf{X}(H^{\mathrm{m}})$ is in degree 1.

Conjecture 3.2. *Let k be a local field or a number field, and let (G, X) be as in* 1.1*. Then*

$$\mathrm{Br_a}\,(X) = H^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}}))\ .$$

where $\mathbf{X}(G^{\mathrm{tor}})$ is in degree 0 *and $\mathbf{X}(H^{\mathrm{m}})$ is in degree* 1*.*

We will prove the following consequence of Conjecture 3.2.

Theorem 3.3. *Let k be a number field, and let (G, X) be as in* 1.1*. Assume that $X(k_v) \neq \varnothing$ for every place v of k. Then*

$$Б(X) = \ker^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}}))$$

To prove Theorem 3.3 we first consider the case when the map $H^{\mathrm{m}} \to G^{\mathrm{tor}}$ is injective, i.e. $\bar{H} \cap G_{\bar{k}}{}^{\mathrm{ssu}} = \bar{H}^{\mathrm{ssu}}$. In this case we can prove Conjecture 3.2.

Theorem 3.4. *Let k be a local field or a number field. Let (G, X) be as in* 1.1*. Assume that the map $H^{\mathrm{m}} \to G^{\mathrm{tor}}$ is injective. Then*

$$\mathrm{Br_a}\,(X) = H^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}}))\ .$$

Proof. Set $Y = X/G^{\mathrm{ssu}}$, this quotient exists by [Bo2], 3.1. We have maps

$$\varphi\colon X \to Y \qquad \varphi^*\colon \mathrm{Br}\, Y \to \mathrm{Br}\, X\ .$$

The variety Y is a principal homogeneous space of the torus coker $[H^{\mathrm{m}} \to G^{\mathrm{tor}}] = G^{\mathrm{tor}}/H^{\mathrm{m}}$.

Since k is a local field or a number field, by [Sa], 6.8 we have $\mathrm{Br_a}\, Y = \mathrm{Br_a}\,(G^{\mathrm{tor}}/H^{\mathrm{m}})$. By [Sa], 6.9(ii) $\mathrm{Br_a}\,(G^{\mathrm{tor}}/H^{\mathrm{m}}) = H^2(k, \mathbf{X}(G^{\mathrm{tor}}/H^{\mathrm{m}}))$.

The morphism of complexes

$$(\mathbf{X}(G^{\mathrm{tor}}/H^{\mathrm{m}}) \to 1) \to (\mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}}))$$

is a quasi-isomorphism, hence

$$\begin{aligned} H^2(k, \mathbf{X}(G^{\mathrm{tor}}/H^{\mathrm{m}})) &= H^2(k, \mathbf{X}(G^{\mathrm{tor}}/H^{\mathrm{m}}) \to 1) \\ &= H^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}}))\ . \end{aligned}$$

Thus

$$\mathrm{Br_a}\, Y = H^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}}))\ .$$

Now we prove that $\varphi *\colon \mathrm{Br_a}\, Y \to \mathrm{Br_a}\, X$ is an isomorphism. By [Sa], 6.3(iv) we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} (\mathrm{Pic}\ Y_{\bar k})^\Gamma & \longrightarrow & H^2(k, U(Y_{\bar k})) & \longrightarrow & \mathrm{Br_a}\, Y & \longrightarrow & H^1(k, \mathrm{Pic}\ Y_{\bar k}) & \longrightarrow & H^3(k, U(Y_{\bar k})) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\mathrm{Pic}\ X_{\bar k})^\Gamma & \longrightarrow & H^2(k, U(X_{\bar k})) & \longrightarrow & \mathrm{Br_a}\, X & \longrightarrow & H^1(k, \mathrm{Pic}\ X_{\bar k}) & \longrightarrow & H^3(k, U(X_{\bar k})) \end{array}$$

Clearly $U(Y_{\bar k}) = \mathbf{X}(G^{\mathrm{tor}}/H^{\mathrm{m}}) = U(X_{\bar k})$.

Consider the map $\varphi^*\colon \mathrm{Pic}\ Y_{\bar k} \to \mathrm{Pic}\ X_{\bar k}$. We have $X_{\bar k} = \bar H \backslash G_{\bar k}$, $Y_{\bar k} = H^{\mathrm{m}}_{\bar k} \backslash G^{\mathrm{tor}}_{\bar k}$. By [Sa], (6.10.1) we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbf{X}(G^{\mathrm{tor}}_{\bar k}) & \longrightarrow & \mathbf{X}(\bar H^{\mathrm{mult}}) & \longrightarrow & \mathrm{Pic}\ Y_{\bar k} & \longrightarrow & \mathrm{Pic}\ G^{\mathrm{tor}}_{\bar k} = 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{X}(G_{\bar k}) & \longrightarrow & \mathbf{X}(\bar H) & \longrightarrow & \mathrm{Pic}\ X_{\bar k} & \longrightarrow & \mathrm{Pic}\ G_{\bar k} = 0 \end{array}$$

where $\mathrm{Pic}\ G_{\bar k} = 0$ because G^{ss} is simply connected, cf. [Sa] 6.9(ii), (iii) and [Sa] 6.11. Since $\mathbf{X}(G^{\mathrm{tor}}_{\bar k}) = \mathbf{X}(G_{\bar k})$ and $\mathbf{X}(\bar H^{\mathrm{mult}}) = \mathbf{X}(\bar H)$, we conclude that $\mathrm{Pic}\ X_{\bar k} = \mathrm{Pic}\ Y_{\bar k}$.

Now in the first diagram four vertical arrows are isomorphisms, and by the five-lemma $\varphi^*\colon \mathrm{Br_a}\, Y \to \mathrm{Br_a}\, X$ is an isomorphism. Hence $\mathrm{Br_a}\, X = H^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}}))$. □

3.5. *Proof of Theorem* 3.3. First, assume that the map $\bar H^{\mathrm{tor}} \to G^{\mathrm{tor}}_{\bar k}$ is injective. Then by Theorem 3.4 $\mathrm{Br_a}\, X = H^2(k, \mathbf{X}(G) \to \mathbf{X}(\bar H))$ when k is a local field or a number field. Hence Б$(X) = \ker^2(k, \mathbf{X}(G) \to \mathbf{X}(\bar H))$.

Now consider the general case. Let us embed H^{m} into an induced torus T. We obtain an embedding $\bar H \to F_{\bar k}$ where $F = G \times T$. Since $X(k_v) \neq \varnothing$ for every place v of k, by [Bo2], 4.3 there exists a homogeneous space Y of F defined over k and an F-equivariant map $\pi\colon Y \to X$, and (Y, π) is a torsor over X under T. By [Bo2], 4.4, the induced map Б$(X) \to$ Б(Y) is an isomorphism.

We compare $\ker^2(k, \mathbf{X}(F) \to \mathbf{X}(\bar H))$ and $\ker^2(k, \mathbf{X}(G) \to \mathbf{X}(\bar H))$. We have an exact sequence

$$1 \to (\mathbf{X}(G) \to \mathbf{X}(\bar H)) \to (\mathbf{X}(F) \to \mathbf{X}(\bar H)) \to (\mathbf{X}(T) \to 1) \to 1,$$

whence the hypercohomology exact sequence

$$0 \to H^2(k, \mathbf{X}(G) \to \mathbf{X}(\bar{H})) \to H^2(k, \mathbf{X}(F) \to \mathbf{X}(\bar{H})) \to H^2(k, \mathbf{X}(T)) \ ,$$

because T is an induced torus and therefore $H^1(k, \mathbf{X}(T)) = 0$. We have similar exact sequences for all the completions k_v of k, whence the exact sequence

$$0 \to \ker^2(k, \mathbf{X}(G) \to \mathbf{X}(\bar{H})) \to \ker^2(k, \mathbf{X}(F) \to \mathbf{X}(\bar{H})) \to 0 \ ,$$

because T is an induced torus and therefore $\ker^2(k, \mathbf{X}(T)) = 0$, cf. e.g. [Sa], (1.9.1). Hence

$$\ker^2(k, \mathbf{X}(G) \to \mathbf{X}(\bar{H})) = \ker^2(k, \mathbf{X}(F) \to \mathbf{X}(\bar{H})) \ .$$

By what we have proved $\text{Б}(Y) = \ker^2(k, \mathbf{X}(F) \to \mathbf{X}(\bar{H}))$. Hence

$$\text{Б}(X) = \ker^2(k, \mathbf{X}(G) \to \mathbf{X}(\bar{H})) \ .$$

□

4. Duality

In this section we establish a relation between two obstructions to the Hasse principle: the cohomological obstruction $\eta(X)$ and the Brauer–Manin obstruction.

4.1. Let k be a number field. Consider a complex of k-groups $M \to T$ of length 2, where M is a group of multiplicative type and T is a torus. Here T is in degree 0 and M is in degree -1. Let $\mathbf{X}(T) \to \mathbf{X}(M)$ be the complex of character groups, where $\mathbf{X}(T)$ is in degree 0 and $\mathbf{X}(M)$ is in degree 1. We are interested in the Tate-Shafarevich kernels $\ker^1(k, M \to T)$ and $\ker^2(k, \mathbf{X}(T) \to \mathbf{X}(M))$.

Proposition 4.2. *There exists a functorial isomorphism of finite groups*

$$\omega\colon\ \ker^1(k, M \to T) \to \ker^2(k, \mathbf{X}(T) \to \mathbf{X}(M))^D \tag{4.2.1}$$

Proof. (Due to R.E. Kottwitz, private communication.) First we reduce the assertion to the case where both M and T are tori. Set $X = \mathbf{X}(M)$, $Y = \mathbf{X}(T)$; then X is a finitely generated abelian group, and Y is a free finitely generated abelian group. Consider the complex $Y \to X$ of Galois modules. Choose an epimorphism $X' \to X$ of Galois modules, where X' is a free finitely generated abelian group. Consider the Cartesian square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

where Y' is the fibre product of Y and X' over X. It is easy to see that Y' is a free finitely generated abelian group and that the morphism of complexes $(Y' \to X') \to (Y \to X)$ given by the commutative diagram is a quasi-isomorphism. Let M' and T' be k-tori such that $X' = \mathbf{X}(M')$, $Y' = \mathbf{X}(T')$. Then we have a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & T \\ \downarrow & & \downarrow \\ M' & \longrightarrow & T' \end{array}$$

which defines a quasi-isomorphism of complexes of abelian k-groups $(M \to T) \to (M' \to T')$. Since a quasi-isomorphism induces an isomorphism on cohomology, we have

$$\ker^i(k, \mathbf{X}(T) \to \mathbf{X}(M)) = \ker^i(k, \mathbf{X}(T') \to \mathbf{X}(M'))$$
$$\ker^i(k, M \to T) = \ker^i(k, M' \to T'))$$

for any i. Thus we have reduced our assertion to the case of a complex of tori.

Now we may and will assume that both M and T are tori. By [KS], Lemma C.3.B, the group $\ker^1(k, M \to T)$ is finite and its dual is $\ker^1(W_k, \hat{T} \to \hat{M})_{\mathrm{red}}$ with the notation of [KS], Appendix C. Note that the indexing of complexes of tori in [KS] is different from ours, and [KS] write $\ker^2(k, M \to T)$ for our $\ker^1(k, M \to T)$. By [KS], Lemma C.3.C $\ker^1(W_k, \hat{T} \to \hat{M})_{\mathrm{red}} = \ker^2(k, Y \to X)$. Thus the dual of $\ker^1(k, M \to T)$ is $\ker^2(k, Y \to X)$, where $X = \mathbf{X}(M)$, $Y = \mathbf{X}(T)$. □

Remark 4.3. Proposition 4.2 was proved earlier in the case when one of the groups M or T is trivial.

When $T = 1$, the proposition asserts that there exists a functorial isomorphism of finite groups

$$\omega\colon \ker^2(k, M) \to \ker^1(k, \mathbf{X}(M))^D.$$

This is proved in [Mi], I-4.20(a).

When $M = 1$, the proposition asserts that there exists a functorial isomorphism of finite groups

$$\omega\colon \ker^1(k, T) \to \ker^2(k, \mathbf{X}(T))^D.$$

This is proved in [KS], D.2.C.

4.4. Let (G, X) be as in 1.1. Assume that $X(k_v) \neq \varnothing$ for every place v of k. In Section 1 we constructed an obstruction $\eta(X) \in \ker^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}})$. We obtain

$$\omega(\eta(X)) \in \ker^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}}))^D .$$

On the other hand, in Section 3 we constructed an isomorphism

$$\alpha\colon \text{Б}(X) \to \ker^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}})).$$

In [Bo2] we described the (first) Brauer-Manin obstruction to the Hasse principle $m_X \in \text{Б}(X)^D$.

Theorem 4.5. *Up to sign* $m_X = (\alpha^{-1} \circ \omega)(\eta(X))$.

The theorem computes the Brauer-Manin obstruction $m(X)$ in terms of Galois hypercohomology.

Proof. First consider the case when $G = G^{\mathrm{tor}}$. Then X is a principal homogeneous space of the torus $T = G^{\mathrm{tor}}/H^{\mathrm{m}}$. We can identify

$$\ker^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}}) = \ker^1(k, T),$$
$$\ker^2(k, \mathbf{X}(G^{\mathrm{tor}}) \to \mathbf{X}(H^{\mathrm{m}})) = \ker^2(k, \mathbf{X}(T)) .$$

In this case the theorem was proved in [Sa], 8.4.

Now consider the case when the map $\bar{H}^{\mathrm{mult}} \to G^{\mathrm{tor}}_{\bar{k}}$ is injective. Set $Y = X/G^{\mathrm{ssu}}$ and let $\varphi\colon X \to Y$ be the canonical map. Then Y is a homogeneous space of G^{tor} with stabilizer H^{m}. The diagram

$$\begin{array}{ccc} \ker^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}}) & \xrightarrow{\omega_X} & \text{Б}(X)^D \\ \| & & \downarrow{\scriptstyle \varphi_*} \\ \ker^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}}) & \xrightarrow{\omega_Y} & \text{Б}(Y)^D \end{array}$$

commutes. We have seen in the proof of Theorem 3.4 that $\varphi^*\colon \mathrm{Br}_{\mathrm{a}}\, Y \to \mathrm{Br}_{\mathrm{a}}\, X$ is an isomorphism. Hence $\varphi_*\colon \text{Б}(X)^D \to \text{Б}(Y)^D$ is an isomorphism. Since up to sign $\omega_Y(\eta(Y)) = m_Y$ and $\varphi_*(m_X) = m_Y$, we conclude that up to sign $\omega_X(\eta(X)) = m_X$, which proves the theorem in this case.

We consider the general case. By assumption $X(k_v) \neq \varnothing$ for every v. Choose an embedding $H^{\mathrm{m}} \to T$ where T is an induced torus. It follows from [Bo2], 4.3 that there exists a torsor $(Y, \pi\colon Y \to X)$ over X under T which is a homogeneous space of the group $F = G \times T$ with stabilizer $\bar{H}$. Consider the commutative diagram

$$\begin{array}{ccc} \ker^1(k, H^{\mathrm{m}} \to F^{\mathrm{tor}}) & \xrightarrow{\omega_Y} & \text{Б}(Y)^D \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ \ker^1(k, H^{\mathrm{m}} \to G^{\mathrm{tor}}) & \xrightarrow{\omega_Y} & \text{Б}(X)^D \end{array} \qquad (4.1)$$

We have $m_X = \pi_* m_Y$, $\eta(X) = \pi_*\eta(Y)$. The map $\bar{H}^{\mathrm{mult}} \to F^{\mathrm{tor}}_{\bar{k}}$ is injective, and in this case we have proved that up to sign $m_Y = \omega_Y(\eta(Y))$. It follows that up to sign $m_X = \omega_X(\eta(X))$. ∎

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