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Non-abelian hypercohomology of a group with coefficients in a crossed module, and Galois cohomology

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Introduction

Here we develop a hypercohomology theory of a group with coefficients in a crossed module, and apply it to define abelianization maps for Galois cohomology of reductive algebraic groups.

Let Γ be a group and let

$$1 \to \overset{-1}{F} \xrightarrow{\alpha} \overset{0}{G} \to 1$$

be a short complex (a complex of length 2) of (in general non-abelian) groups, where the numbers -1 and 0 over the letters denote the degrees: F is in degree -1 and G is in degree 0. We assume that the group Γ acts on F and G, and that α is a homomorphism of Γ-groups.

For applications to the Galois cohomology of connected algebraic groups, we would like to be able to define the first hypercohomology set $\mathbf{H}^1(\Gamma, F \to G)$ in a functorial way. In general this is not likely to be possible. Indeed, if we take G to be $\{1\}$, then we must have $\mathbf{H}^1(\Gamma, F \to 1) = H^2(\Gamma, F)$. However, as far as I know, there is no functorial definition of second cohomology in the non-abelian case (the second cohomology theory of Springer [Sp] and Giraud [Gi] is obviously non-functorial).

Fortunately it is possible to define the first hypercohomology in a functorial way when $F \to G$ is *a crossed module.* A crossed module is a group homomorphism $F \xrightarrow{\alpha} G$ with an action of G on F satisfying certain natural conditions (see 2.1 for the precise definition and [BHu] for a survey). The notion of a crossed module was introduced in 1946 by J.H.C.Whitehead [W1], [W2], who was motivated by topological problems.

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To write down hypercohomology exact sequences, what we need is not only $\mathbf{H}^1$, but also $\mathbf{H}^{-1}$ and $\mathbf{H}^0$. In Section 1 for any short complex of Γ-groups $F \to G$ (not necessarily a crossed module), we define, in terms of cocycles, an abelian group $\mathbf{H}^{-1}(F \to G)$ and pointed set $\mathbf{H}^0(F \to G)$, where we write $\mathbf{H}^i(F \to G)$ for $\mathbf{H}^i(\Gamma, F \to G)$. These definitions were earlier given by Deligne [Del], 2.4.3, in terms of torsors. (Deligne writes $\mathbf{H}^0$ for our $\mathbf{H}^{-1}$, and $\mathbf{H}^1$ for our $\mathbf{H}^0$.)

For a crossed module $F \to G$ with Γ-action, we define in Section 2 a group structure on $\mathbf{H}^0(\Gamma, F \to G)$. Then we define, again in cocyclic form, the first hypercohomology set $\mathbf{H}^1(\Gamma, F \to G)$. We follow Dedecker [Ded2],[Ded3], who defined $\mathbf{H}^1(\Gamma, F \to G)$ for a crossed module $F \to G$ with trivial Γ-action; the generalization to the case of non-trivial Γ-action is obvious. Note that Dedecker regards $\mathbf{H}^1(\Gamma, F \to G)$ not as hypercohomology of a complex, but as a nice, functorial definition of the second cohomology $H^2(\Gamma, F)$, so the group G and its action on F are for him just auxiliary structures necessary to define $H^2(\Gamma, F)$. Dedecker writes $H^2(\Gamma, F \to G)$ for our $\mathbf{H}^1(\Gamma, F \to G)$. We regard $\mathbf{H}^1(\Gamma, F \to G)$ as hypercohomology of a complex, and write down the hypercohomology exact sequence associated to a short exact sequence of crossed modules.

In Section 3 we use hypercohomology exact sequences to prove

Theorem (Theorem 3.3) *Let $(F_1 \to G_1) \to (F_2 \to G_2)$ be a quasi-isomoprhism of crossed modules with Γ-action. Then the induced maps $\mathbf{H}^i(F_1 \to G_1) \to \mathbf{H}^i(F_2 \to G_2)$ $(i = -1, 0, 1)$ are bijections.*

In Section 4 we apply results of Sections 1–3 to the crossed module of algebraic groups $G^{\mathrm{sc}} \xrightarrow{\rho} G$, introduced by Deligne ([Del], 2.4.7). Here G is a connected reductive algebraic group over a field K of characteristic 0, G^{sc} is the universal covering of the derived group G^{ss} of G, the homomorphism ρ is the composition $G^{\mathrm{sc}} \to G^{\mathrm{ss}} \to G$, and G acts on G^{sc} in the obvious way. Let Z be the center of G and $Z^{(\mathrm{sc})}$ the center of G^{sc}. Let $H^0(K, G)$ and $H^1(K, G)$ denote the 0-dimensional and 1-dimensional Galois cohomology of G. We define the abelian Galois cohomology groups of G by

$$H^i_{\mathrm{ab}}(K, G) = \mathbf{H}^i(K, Z^{(\mathrm{sc})} \to Z) \qquad (i \geq -1).$$

Using the morphism $(1 \to G) \to (G^{\mathrm{sc}} \to G)$ and the quasi-isomorphism $(Z^{(\mathrm{sc})} \to Z) \to (G^{\mathrm{sc}} \to G)$ of crossed modules of algebraic groups, we define for $i = 0, 1$ the abelianization maps

$$\mathrm{ab}^i \colon H^i(K, G) \to \mathbf{H}^i(G^{\mathrm{sc}} \to G) \xrightarrow{\sim} \mathbf{H}^i(Z^{(\mathrm{sc})} \to Z) = H^i_{\mathrm{ab}}(K, G).$$

The abelianization map ab^0 was first defined by Deligne [Del]. The map ab^1 generalizes a map of Kottwitz ([Ko2], Thm. 1.2), which he defined and extensively used in the case when K is a local field. Kottwitz defined the abelianization map with the help of a rather complicated method of z-extensions of reductive groups. The hypercohomology with coefficients in a crossed module permits us to define the maps ab^0 and ab^1 explicitly, in particular in terms of cocycles (Propsitions 4.3.1 and 4.3.2).

Constructions of Section 4 are used in our forthcoming paper [Bo3] (cf. also [Bo1]), where we describe "explicitly" the first Galois cohomology of a connected reductive group over a number field. Such constructions are are useful in cohomological calculations related to Shimura varieties, cf. [Mi].

Note that it is also possible to define the abelianization map $\mathrm{ab}^2\colon H^2(K,G) \to H^2_{\mathrm{ab}}(K,G)$ (cf. [Bo2]), where $H^2(K,G)$ is the second non-abelian Galois cohomology set of Springer [Sp] and Giraud [Gi]. If K is a local field or a number field and $\eta \in H^2(K,G)$, then $\mathrm{ab}^2(\eta) = 0$ if and only if η is a neutral class, i.e. it corresponds to a split extension.

Remarks (1) The cocyclic constructions of Sections 1–3 go through in the more general case of hypercohomology of a simplicial set with coefficients in a family of crossed modules.

(2) In [Br1] Breen defines $\mathbf{H}^{-1}$, $\mathbf{H}^0$ and $\mathbf{H}^1$ in a uniform way for a sheaf of crossed modules $F \to G$ on a site, and constructs the hypercohomology exact sequence (0.1.1). Breen uses the machinery of homotopical algebra. In the particular case of the site of Γ-sets his definitions appear to be equivalent to ours. Our resuls were obtained independently (before the paper [Br1] appeared).

(3) Breen ([Br1], 6.2) proves that $\mathbf{H}^1(F \to G)$ can be identified with the set of equivalence classes of torsors under the Picard category associated to the crossed module $F \to G$. Deligne noticed (private communication) that our Theorem 3.3 follows from this description of $\mathbf{H}^1(F \to G)$, because quasi-isomorphic crossed modules define equivalent Picard categories.

(4) We claim no originality. Most of the results of Sections 1–3 are known (except the construction of the connecting map $H^1 \to H^2$ in 2.17 – 2.22), see Remarks (2–3) above. We need however this cocyclic exposition for Section 4, where we write down explicit cocyclic formulas for ab^0 and ab^1.

1 Hypercohomology in degrees -1 and 0

1.1 *Short complexes of groups.* Let Γ be a pro-finite group. A dicrete Γ-group is a group G endowed with a left action of Γ which is continuous with respect to the discrete topology on G. Here "continuous" means that the stabilizer of any element $g \in G$ is open in Γ. From now on, by a Γ-group we mean a dicrete Γ-group.

Let $\alpha\colon F \to G$ be a morphism of Γ-groups, i.e. a group homomorphism respecting the action of Γ. We consider $F \to G$ as a *short complex*

$$1 \to \overset{-1}{F} \to \overset{0}{G} \to 1$$

where F is in degree -1 and G is in degree 0.

1.2 *Hypercohomology.* We define hypercohomology in degree -1. We set

$$\mathbf{H}^{-1}(F \xrightarrow{\alpha} G) = (\ker\, \alpha)^\Gamma$$

where $(\ )^\Gamma$ means (the group of) invariants.

We define 0-hypercohomology. We write Maps(Γ, F) for the set of continuous maps $\varphi\colon \Gamma \to F$ and set

$$\begin{aligned} C^0 &= \text{Maps}(\Gamma, F) \times G \text{ (we regard } C^0 \text{ as a set)} \\ Z^0 &= \{(\varphi, g) \in C^0 \mid \varphi(\sigma\tau) = \varphi(\sigma) \cdot {}^\sigma\varphi(\tau),\ {}^\sigma g = \alpha(\varphi(\sigma)^{-1}) \cdot g,\ \sigma, \tau \in \Gamma\} \end{aligned}$$

The group F acts on the set of 0-cocycles Z^0 on the right by

$$(\varphi, g) * f = (\varphi', g'), \quad \varphi'(\sigma) = f^{-1} \cdot \varphi(\sigma) \cdot {}^\sigma f, \quad g' = \alpha(f)^{-1} \cdot g,$$

(where $f \in F$), and we set

$$\mathbf{H}^0(F \to G) = Z^0/F$$

The set $\mathbf{H}^0(F \to G)$ has a neutral element, namely the class of $(1, 1)$. We write $\text{Cl}(\varphi, g)$ for the hypercohomology class of a cycycle (φ, g).

1.3 *Morphisms of complexes.* A morphism of (short) complexes $(F_1 \to G_1) \to (F_2 \to G_2)$ is a commutative diagram

$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \downarrow & & \downarrow \\ G_1 & \longrightarrow & G_2 \end{array}$$

of Γ-groups. Such a morphism induces a canonical homomorphism

$$\mathbf{H}^{-1}(F_1 \to G_1) \to \mathbf{H}^{-1}(F_2 \to G_2)$$

and a canonical map

$$\mathbf{H}^0(F_1 \to G_1) \to \mathbf{H}^0(F_2 \to G_2)$$

1.4 Examples

(1) $\mathbf{H}^0(1 \to G) = H^0(G) = G^\Gamma$.

(2) $\mathbf{H}^0(F \to 1) = H^1(F)$. To $\text{Cl}(\varphi, 1) \in \mathbf{H}^0(F \to 1)$ we associate $\text{Cl}(\varphi) \in H^1(F)$.

(3) If $\alpha\colon F \to G$ is injective, then the morphism of complexes $(F \to G) \to (1 \to G/\alpha(F))$ induces a canonical bijection $\mathbf{H}^0(F \to G) \xrightarrow{\sim} H^0(\text{coker}\,\alpha)$.

(4) If $\alpha\colon F \to G$ is surjective, then the embedding $(\ker \alpha \to 1) \hookrightarrow (F \to G)$ of complexes induces a canonical bijection $H^1(\ker \alpha) \xrightarrow{\sim} \mathbf{H}^0(F \to G)$.

In the rest of this section we define the hypercohomology exact sequence associated to a short exact sequence of complexes of Γ-groups.

1.5 *Exact sequences.* A short exact sequence of complexes of Γ-groups is a sequence

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

such that the rows in the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1
\end{array}$$

are exact. We regard F_1 and G_1 as subgroups of F_2 and G_2, respectively. For such an exact sequence we define the connecting map

$$\delta_{-1}\colon \mathbf{H}^{-1}(F_3 \to G_3) \to \mathbf{H}^0(F_1 \to G_1)$$

as follows.

Let $f_3 \in \mathbf{H}^{-1}(F_3 \to G_3) = (\ker \alpha_3)^\Gamma$. Choose $f \in F_2$ such that $f (\operatorname{mod} F_1) = f_3$. We define a 0-cochain $(\varphi_1, g_1) \in C^0(F_2 \to G_2)$ by

$$\varphi_1(\sigma) = f \cdot {}^\sigma f^{-1}, \; g_1 = \alpha_2(f)$$

It is easy to show that $(\varphi_1, g_1) \in Z^0(F_1 \to G_1)$.

We set $\delta_{-1}(f_3) = \mathrm{Cl}(\varphi_1, g_1) \in \mathbf{H}^0(F_1 \to G_1)$. We leave to the reader to check that the map δ_{-1} is defined correctly, i.e. $\delta_{-1}(f_3)$ does not depend on the choice of the representative $f \in F_2$ of f_3.

1.6 Proposition. *Let*

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

be an exact sequence of complexes of Γ-groups. Then the hypercohomology sequence

$$\begin{aligned}
(1.6.1) \qquad & 1 \longrightarrow \mathbf{H}^{-1}(F_1 \to G_1) \xrightarrow{i_*} \mathbf{H}^{-1}(F_2 \to G_2) \xrightarrow{j_*} \mathbf{H}^{-1}(F_3 \to G_3) \\
& \xrightarrow{\delta_{-1}} \mathbf{H}^0(F_1 \to G_1) \xrightarrow{i_*} \mathbf{H}^0(F_2 \to G_2) \xrightarrow{j_*} \mathbf{H}^0(F_3 \to G_3)
\end{aligned}$$

is exact.

Note that exactness makes sense because $\mathbf{H}^{-1}(F_k \to G_k)$ is a group and $\mathbf{H}^0(F_k \to G_k)$ is a pointed set $(k = 1, 2, 3)$.

Proof. We prove the exactness at $\mathbf{H}^{-1}(F_3 \to G_3)$. It follows immediately from the definition of δ_{-1} that $\delta_{-1} \circ j_* = 1$. Conversely, suppose that $f_3 \in (\ker \alpha_3)^\Gamma$ and $\delta_{-1}(f_3) = 1$. Let f be a representative of f_3 in F_2. Then there exists $f_1 \in F_1$ such that

$$\begin{aligned}
&\alpha_1(f_1)^{-1} \cdot \alpha_2(f) = 1 \\
&f_1^{-1} \cdot f \cdot {}^\sigma f^{-1} \cdot {}^\sigma f_1 = 1,
\end{aligned}$$

hence

$$\alpha_2(f_1^{-1}f) = 1,\ {}^{\sigma}(f_1^{-1}f) = f_1^{-1}f$$

Set $f' = f_1^{-1}f$. Then $f'(\operatorname{mod} F_1) = f_3$, $\alpha_2(f') = 1$, and ${}^{\sigma}f' = f'$ for any $\sigma \in \Gamma$. Thus

$$f' \in (\ker \alpha_2)^{\Gamma} = \mathbf{H}^{-1}(F_2 \to G_2)$$

and $f_3 = j_*(f')$. We have proved that $f_3 \in \operatorname{im} j_*$.

We leave the proof of the exactness at the other terms to the reader.

2 Crossed modules and $\mathbf{H}^1$

To define $\mathbf{H}^1(F \to G)$ we need an additional structure on $F \to G$, namely the structure of *crossed module.*

2.1 Definition. A crossed module is a short complex (homomorphism) $\alpha: F \to G$, endowed with a left action of G on F (denoted by $(g, f) \mapsto {}^{g}f$) satisfying

(2.1.1) $$ff'f^{-1} = {}^{\alpha(f)}f'$$

(2.1.2) $$\alpha({}^{g}f) = g \cdot \alpha(f) \cdot g^{-1}$$

for any $f, f' \in F$, $g \in G$.

We say that a group Γ acts on a crossed module $\alpha: F \to G$, if Γ acts on F and G such that

$$\alpha({}^{\sigma}f) = {}^{\sigma}(\alpha(f)),\ {}^{\sigma}({}^{g}f) = {}^{{}^{\sigma}g}({}^{\sigma}f) \quad \text{for any} \quad f \in \Gamma,\ g \in G,\ \sigma \in \Gamma.$$

2.2 Examples *of crossed modules.*

(1) $\alpha: F \to G$ where F is any (abelian) G-module, α is trivial.

(2) $\alpha: F \hookrightarrow G$ where F is a normal subgroup of G, $\alpha: F \hookrightarrow G$ is the inclusion, ${}^{g}f = gfg^{-1}$.

(3) $\alpha: F \to G$ where $F \to G$ is any surjective homomorphism with central kernel. An element $g \in G$ acts on F by ${}^{g}f = \tilde{g}f\tilde{g}^{-1}$ where $\tilde{g}$ is any lifting of g to F.

(4) $F \to \operatorname{Aut} F$ for any group F, $f \mapsto \operatorname{int}(f)$.

(5) Let X be a "nice" topological space, $Y \subset X$ a subspace and $x_0 \in Y$ a point. Then $\pi_1(Y, x_0)$ acts on $\pi_2(X, Y, x_0)$, and the complex $\pi_2(X, Y, x_0) \xrightarrow{\partial} \pi_1(Y, x_0)$ (where ∂ is the boundary homomorphism) is a crossed module.

(6) Deligne's crossed module $\rho: G^{\mathrm{sc}} \to G$ of algebraic groups, described in the Introduction.

2.3 *Remark.* J. H. C. Whitehead [W1], [W2], who introduced the notion of a crossed module, considered the crossed module 2.2(5). Dedecker showed in [Ded1], [Ded2] that a crossed module $F \to G$ suits to define hypercohomology $\mathbf{H}^1(X, F \to G)$ where X is a group, a topological space and so on. For a survey on crossed modules see [BHu].

2.4. Lemma (cf. [BHu]). *Let $F \xrightarrow{\alpha} G$ be a crossed module. Then*

(i) *the group* $\ker \alpha$ *is central in F;*

(ii) $\ker \alpha$ *is G-invariant;*

(iii) $\operatorname{im}\alpha$ *is normal in G.*

Proof. (i) follows from (2.1.1); (ii) and (iii) follow from (2.1.2).

2.5 Corollary. *The action of G on F induces an action of* $\operatorname{coker}\alpha$ *on the abelian group* $\ker \alpha$*.*

2.6 *The group structure on H^0.* Let $F \to G$ be a crossed group with a Γ-action. We show that $C^0 = C^0(F \to G)$, $Z^0(F \to G)$ and $\mathbf{H}^0(F \to G)$ have natural group structures.

The group G acts on $\mathrm{Maps}(\Gamma, F)$ by $({}^g\varphi)(\sigma) = {}^g(\varphi(\sigma))$ ($\varphi \in \mathrm{Maps}(\Gamma, F)$, $\sigma \in \Gamma$). We define a group structure on C^0 by

$$(\varphi_1, g_1) \cdot (\varphi_2, g_2) = ({}^{g_1}\varphi_2 \cdot \varphi_1,\ g_1 g_2).$$

One can check that Z^0 is a subgroup of C^0 with respect to this group structure.

Consider the map $\nu\colon F \to Z^0$ defined by the formula $\nu(f) = (\varphi, \alpha(f))$ where $\varphi(\sigma) = f \cdot {}^\sigma f^{-1}$. One can easily check that ν is a group homomorphism and its image is normal in Z^0. Moreover the right action of F on Z^0 defined by

$$((\varphi, g), f) \longmapsto \nu(f^{-1}) \cdot (\varphi, g)$$

coincides with the action $*$ of 1.2. Thus $\mathbf{H}^0(F \to G) = Z^0/\operatorname{im}\nu$, and therefore $\mathbf{H}^0(F \to G)$ has a canonical group structure. This group structure depends functorially on the crossed module $F \to G$.

2.7 *Hypercohomology in degree* 1. Let $F \to G$ be a crossed module with a Γ-action. Following Dedecker [Ded3] we define the first hypercohomology as follows.

Let Z^1 denote the set of pairs $(h, \psi) \in \mathrm{Maps}(\Gamma \times \Gamma, F) \times \mathrm{Maps}(\Gamma, G)$ such that for any $\sigma, \tau, \upsilon \in \Gamma$

$$\alpha(h(\sigma,\tau))^{-1} \cdot \psi(\sigma\tau) = \cdot\psi(\sigma) \cdot {}^\sigma\psi(\tau)$$
$$h(\sigma, \tau\upsilon) \cdot {}^{\psi(\sigma)\sigma}h(\tau, \upsilon) = h(\sigma\tau, \upsilon) \cdot h(\sigma, \tau).$$

We define a right action $Z^1 \times C^0 \to Z^1$ of the group of 0-cochains C^0 on the set of 1-cocycles Z^1. For $(a, g) \in C^0$ we set

$$(h, \psi) * (a, g) = (h', \psi')$$

where

$$\psi'(\sigma) = g^{-1} \cdot \alpha(a(\sigma)) \cdot \psi(\sigma) \cdot {}^{\sigma}g$$
$$h'(\sigma,\tau) = {}^{g^{-1}}\left[a(\sigma\tau) \cdot h(\sigma,\tau) \cdot {}^{\psi(\sigma)\sigma}a(\tau)^{-1} \cdot a(\sigma)^{-1}\right]$$

One can easily check that this is a group action.

Now we set

$$\mathbf{H}^1(F \to G) = Z^1/C^0.$$

The set $\mathbf{H}^1(F \to G)$ has *a neutral element*, namely the class of the trivial cocycle $(1,1) \in Z^1$. We write $\mathrm{Cl}(h,\psi)$ for the hypercohomology class of 1-cocycle (h,ψ).

2.8 *Morphisms of crossed modules.* A morphism $\varepsilon\colon (F_1 \to G_1) \to (F_2 \to G_2)$ of crossed modules is a pair of homomorphisms $(\varepsilon_0\colon G_1 \to G_2,\ \varepsilon_{-1}\colon F_1 \to F_2)$ such that the diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{\varepsilon_{-1}} & F_2 \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ G_1 & \xrightarrow[\varepsilon_0]{} & G_2 \end{array}$$

commutes and ${}^{\varepsilon_0(g)}\varepsilon_{-1}(f) = \varepsilon_{-1}({}^g f)$ for any $g \in G_1,\ f \in F_1$.

A morphism ε of crossed modules with Γ-action defines homomorphisms

$$\varepsilon_*\colon \mathbf{H}^i(F_1 \to G_1) \to \mathbf{H}^i(F_2 \to G_2) \quad (i = -1, 0)$$

and a map $\varepsilon_*\colon \mathbf{H}^1(F_1 \to G_1) \to \mathbf{H}^1(F_2 \to G_2)$ that takes the neutral element to the neutral element. Thus $\mathbf{H}^{-1}$, $\mathbf{H}^0$ and $\mathbf{H}^1$ are functors.

2.9 Examples.

(1) $\mathbf{H}^1(1 \to G) = H^1(G)$.

(2) $\mathbf{H}^1(F \to 1) = H^2(F)$ (note that in this case F is abelian and therefore $H^2(F)$ makes sense). To $\mathrm{Cl}(h,1) \in \mathbf{H}^1(F \to 1)$ we associate $\mathrm{Cl}(h) \in H^2(F)$.

(3) If $F \xrightarrow{\alpha} G$ is a crossed module and α is injective, then the morphism of complexes $(F \to G) \to (1 \to G/\alpha(F))$ induces a canonical bijection $\mathbf{H}^1(F \to G) \xrightarrow{\sim} H^1(\mathrm{coker}\,\alpha)$.

(4) If α is surjective, then the embedding $(\ker\,\alpha \to 1) \hookrightarrow (F \to G)$ of crossed modules induces a bijection $H^2(\ker\,\alpha) \xrightarrow{\sim} \mathbf{H}^1(F \to G)$. One can check that the map $H^i(G) \to \mathbf{H}^i(F \to G) = H^{i+1}(\ker\alpha)$ $(i = 0,1)$ coinsides with the connecting map $\delta_i\colon H^i(G) \to H^{i+1}(\ker\,\alpha)$ associated to the short exact sequence $1 \to \ker\,\alpha \to F \to G \to 1$ (see [Se], Ch. I, §5, for the definition of δ_i).

In the rest of this section we prolong the hypercohomology exact sequence (1.6.1).

2.10 Let

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

be an exact sequence of complexes of groups. We identify $(F_1 \to G_1)$ with its image in $(F_2 \to G_2)$. Assume that $(F_1 \to G_1)$ and $(F_2 \to G_2)$ are endowed with structures of crossed modules such that i is a morphism of crossed modules. We assume also that

(2.10.1) *F_1 is G_2-invariant in F_2.*

Then G_2 acts on $F_3 \simeq F_2/F_1$. We do not assume that $(F_3 \to G_3)$ is a crossed module.

We define a left action of the group $\mathbf{H}^0(F_2 \to G_2)$ on the set $\mathbf{H}^0(F_3 \to G_3)$ by

$$\mathrm{Cl}(\varphi_2, g_2) \cdot \mathrm{Cl}(\varphi_3, g_3) = \mathrm{Cl}({}^{g_2}\varphi_3 \cdot j(\varphi_2), j(g_2) \cdot g_3)$$

One can check that this is a correctly defined group action.

2.11 *The connecting map.* Let a short exact sequence

$$1 \to (F_1 \to G_1) \to (F_2 \to G_2) \to (F_3 \to G_3) \to 1$$

be as in 2.10. We define the connecting map

$$\delta_0 \colon \mathbf{H}^0(F_3 \to G_3) \to \mathbf{H}^1(F_1 \to G_1)$$

as follows.

Let $\xi_3 \in \mathbf{H}^0(F_3 \to G_3)$, $\xi_3 = \mathrm{Cl}(\varphi_3, g_3)$, $(\varphi_3, g_3) \in Z^0(F_3 \to G_3)$. We lift (φ_3, g_3) to some (φ, g), $\varphi \in \mathrm{Maps}(\Gamma, F_2)$, $g \in G_2$. We set

$$\psi_1(\sigma) = g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot {}^{\sigma}g$$
$$h_1(\sigma, \tau) = {}^{g^{-1}}[\varphi(\sigma\tau) \cdot {}^{\sigma}\varphi(\tau)^{-1} \cdot \varphi(\sigma)^{-1}].$$

Then $\psi_1(\sigma) \in G_1$ and $h_1(\sigma, \tau) \in F_1$ for any $\sigma, \tau \in \Gamma$ (we use (2.10.1)).

We show that $(h_1, \psi_1) \in Z^1(F_1 \to G_1)$. We have

$$\begin{aligned}
\psi_1(\sigma) \cdot {}^{\sigma}\psi_1(\tau) &= g^{-1} \cdot \alpha_2(\varphi(\sigma)) \cdot {}^{\sigma}g \cdot {}^{\sigma}g^{-1} \cdot \alpha_2({}^{\sigma}\varphi(\tau)) \cdot {}^{\sigma\tau}g \\
&= g^{-1} \cdot \alpha_2(\varphi(\sigma) \cdot {}^{\sigma}\varphi(\tau) \cdot \varphi(\sigma\tau)^{-1}) \cdot g \cdot g^{-1} \cdot \alpha_2(\varphi(\sigma\tau)) \cdot {}^{\sigma\tau}g \\
&= \alpha_2(h_1(\sigma, \tau))^{-1} \cdot \psi_1(\sigma\tau);
\end{aligned}$$

$$\begin{aligned}
&h_1(\sigma, \tau\upsilon) \cdot {}^{\psi_1(\sigma)\sigma}h_1(\tau, \upsilon) \\
&\quad = {}^{g^{-1}}[\varphi(\sigma\tau\upsilon) \cdot {}^{\sigma}\varphi(\tau\upsilon)^{-1} \cdot \varphi(\sigma)^{-1}] \\
&\qquad \cdot {}^{g^{-1}}[\varphi(\sigma) \cdot {}^{\sigma}\varphi(\tau\upsilon) \cdot {}^{\sigma\tau}\varphi(\upsilon)^{-1} \cdot {}^{\sigma}\varphi(\tau)^{-1} \cdot \varphi(\sigma)^{-1}] \\
&\quad = {}^{g^{-1}}[\varphi(\sigma\tau\upsilon) \cdot {}^{\sigma\tau}\varphi(\upsilon)^{-1} \cdot {}^{\sigma}\varphi(\tau)^{-1} \cdot \varphi(\sigma)^{-1}] \\
&\quad = {}^{g^{-1}}[\varphi(\sigma\tau\upsilon) \cdot {}^{\sigma\tau}\varphi(\upsilon)^{-1} \cdot \varphi(\sigma\tau)^{-1}] \cdot {}^{g^{-1}}[\varphi(\sigma\tau) \cdot {}^{\sigma}\varphi(\tau)^{-1} \cdot \varphi(\sigma)^{-1}] \\
&\quad = h_1(\sigma\tau, \upsilon) \cdot h_1(\sigma, \tau).
\end{aligned}$$

Hence $(h_1, \psi_1) \in Z^1(F_1 \to G_1)$.

We set $\delta_0(\xi_3) = \mathrm{Cl}(h_1, \psi_1) \in \mathbf{H}^1(F_1 \to G_1)$. We leave to the reader to check that $\delta_0(\xi_3)$ is defined correctly.

2.12 Proposition. *Let*

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1 \tag{2.12.1}$$

be a short exact sequence of complexes of Γ*-groups where* i *is an embedding of crossed modules with* Γ*-action. We identify* $(F_1 \to G_1)$ *with its image in* $(F_2 \to G_2)$ *and assume that the subgroup* $F_1 \subset F_2$ *is* G_2*-invariant. Then*

(i) *the sequence*

$$\begin{aligned} 1 \to \mathbf{H}^{-1}(F_1 \to G_1) &\xrightarrow{i_*} \mathbf{H}^{-1}(F_2 \to G_2) \xrightarrow{j_*} \mathbf{H}^{-1}(F_3 \to G_3) \\ \xrightarrow{\delta_{-1}} \mathbf{H}^0(F_1 \to G_1) &\xrightarrow{i_*} \mathbf{H}^0(F_2 \to G_2) \xrightarrow{j_*} \mathbf{H}^0(F_3 \to G_3) \\ \xrightarrow{\delta_0} \mathbf{H}^1(F_1 \to G_1) &\xrightarrow{i_*} \mathbf{H}^1(F_2 \to G_2) \end{aligned} \tag{2.12.2}$$

is exact.

(ii) δ_0 *defines a bijection*

$$\mathbf{H}^0(F_2 \to G_2)\backslash \mathbf{H}^0(F_3 \to G_3) \xrightarrow{\sim} \ker[\mathbf{H}^1(F_1 \to G_1) \to \mathbf{H}^1(F_2 \to G_2)]. \tag{2.12.3}$$

Proof. We leave the proof of (ii) to the reader. To prove (i) we must prove the exactness at the terms $\mathbf{H}^0(F_3 \to G_3)$ and $\mathbf{H}^1(F_1 \to G_1)$. We leave the proof for $\mathbf{H}^0(F_3 \to G_3)$ to the reader.

We prove the exactness at $\mathbf{H}^1(F_1 \to G_1)$. It is clear that $i_* \circ \delta_0 = 1$. Indeed, the cocycle $(h_1, \psi_1) \in Z^1(F_1 \to G_1)$ constructed in 2.11 is cohomologous to $(1,1)$ in $Z^1(F_2 \to G_2)$.

Conversely, let $\eta_1 \in \mathbf{H}^1(F_1 \to G_1)$, $\eta_1 = \mathrm{Cl}(h_1, \psi_1)$. Assume that $i_*(\eta_1) = 1$. Then

$$\begin{aligned} \psi_1(\sigma) &= g^{-1} \cdot \alpha_2(a(\sigma)) \cdot {}^\sigma g \\ h_1(\sigma, \tau) &= {}^{g^{-1}}[a(\sigma\tau) \cdot {}^\sigma a(\tau)^{-1} \cdot a(\sigma)^{-1}] \end{aligned}$$

for some $a\colon \Gamma \to F_2$, $g \in G_2$. Set $g_3 = g (\operatorname{mod} G_1) \in G_3$, $\varphi_3(\sigma) = a(\sigma) (\operatorname{mod} F_1) \in F_3$. Using (2.10.1) one can easily check that $(\varphi_3, g_3) \in Z^0(F_3 \to G_3)$. Set $\xi_3 = \mathrm{Cl}(\varphi_3, g_3)$; then $\eta_1 = \delta_0(\xi_3)$. Thus $\eta_1 \in \operatorname{im} \delta_0$, which was to be proved.

2.12.4 *Remark.* The hypercohomology exact sequence (2.12.2) depends on the short exact sequence (2.12.1) functorially.

2.13 Corollary. *Let* $F \xrightarrow{\alpha} G$ *be a crossed module with* Γ*-action.*

(i) ([Br1], (4.2.2)). *There is an exact sequence*

$$\begin{aligned} 1 \longrightarrow \mathbf{H}^{-1}(F \to G) \xrightarrow{\lambda_{-1}} H^0(F) &\xrightarrow{\alpha_*} H^0(G) \xrightarrow{\varkappa_0} \mathbf{H}^0(F \to G) \\ \xrightarrow{\lambda_0} H^1(F) &\xrightarrow{\alpha_*} H^1(G) \xrightarrow{\varkappa_1} \mathbf{H}^1(F \to G). \end{aligned} \tag{2.13.1}$$

(ii) *The map* $\alpha_*\colon H^1(F) \to H^1(G)$ *defines a bijection*

$$\mathbf{H}^0(F \to G)\backslash H^1(F) \xrightarrow{\sim} \ker[\varkappa_1\colon H^1(G) \to \mathbf{H}^1(F \to G)] \tag{2.13.2}$$

Here $\alpha_*\colon H^i(F) \to H^i(G)$ are the canonical maps induced by α. The maps λ_{-1}, $\varkappa_0$, λ_0 and $\varkappa_1$ can be described as follows:

$$\lambda_1\colon \mathbf{H}^{-1}(F \to G) = (\ker\,\alpha)^\Gamma \hookrightarrow F^\Gamma = H^0(F),\ f \longmapsto f$$
$$\varkappa_0\colon H^0(G) = G^\Gamma \to \mathbf{H}^0(F \to G),\ g \longmapsto \mathrm{Cl}(1,g)$$
$$\lambda_0\colon \mathrm{Cl}(\varphi, g) \longmapsto \mathrm{Cl}(\varphi)$$
$$\varkappa_1\colon \mathrm{Cl}(\psi) \longmapsto \mathrm{Cl}(1,\psi)$$

2.13.3 *Remark.* The exact sequence (2.13.1), but without the last term, was earlier constructed by Deligne ([Del], (2.4.3.1)).

Proof. Consider the short exact sequence

$$1 \to (1 \to G) \to (F \to G) \to (F \to 1) \to 1$$

of complexes of Γ-groups, where $(1 \to G) \to (F \to G)$ is a morphism of crossed modules. The exact sequence (2.12.2) takes in our case the form (2.13.1), and the bijection (2.12.3) takes the form (2.13.2).

2.14 *Twisting.* To describe the fibers of the map $\varkappa_1\colon H^1(G) \to \mathbf{H}^1(F \to G)$ we need twisting.

The group G acts on the crossed module $(F \to G)$. An element $g_* \in G$ acts by

$$f \mapsto {}^{g_*}f,\ g \mapsto g_* g g_*^{-1}\ \ (f \in F,\, g \in G)$$

Let $\psi \in Z^1(G)$. We can define the twisted crossed module ${}_\psi(F \to G) = ({}_\psi F \to {}_\psi G)$, where the twisted groups ${}_\psi F$ and ${}_\psi G$ are the same F and G as abstract groups, but Γ acts differently, namely,

$${}^{\sigma *}f = {}^{\psi(\sigma)\sigma}f,\ {}^{\sigma *}g = \psi(\sigma)\cdot {}^{\sigma}g \cdot \psi(\sigma)^{-1}\ (\sigma \in \Gamma,\ f \in F,\ g \in G).$$

We define a map

$$t_\psi\colon \mathbf{H}^1({}_\psi(F \to G)) \to \mathbf{H}^1(F \to G)$$

taking $1 \in \mathbf{H}^1({}_\psi(F \to G))$ to $\mathrm{Cl}(1,\psi) \in \mathbf{H}^1(F \to G)$. Let $(h', \psi') \in Z^1({}_\psi(F \to G))$. By definition this means that

$$\psi'(\sigma)\cdot\psi(\sigma)\cdot {}^{\sigma}\psi'(\tau)\cdot\psi(\sigma)^{-1} = \alpha(h'(\sigma,\tau))^{-1}\cdot\psi'(\sigma\tau)$$
$$h'(\sigma,\tau\upsilon)\cdot {}^{\psi'(\sigma)\psi(\sigma)\sigma}h'(\tau,\upsilon) = h'(\sigma\tau,\upsilon)\cdot h'(\sigma,\tau).$$

We set

$$t_\psi(\mathrm{Cl}(h', \psi')) = \mathrm{Cl}(h', \psi'\psi)$$

One can easily check that the map t_ψ is defined correctly.

We can define a map $t_\psi\colon H^1({}_\psi G) \to H^1(G)$ in a similar way. The diagram

$$\begin{array}{ccc} H^1({}_\psi G) & \xrightarrow{t_\psi} & H^1(G) \\ {\scriptstyle {}_\psi\varkappa_1}\downarrow & & \downarrow{\scriptstyle \varkappa_1} \\ \mathbf{H}({}_\psi(F \to G)) & \xrightarrow{t_\psi} & \mathbf{H}^1(F \to G) \end{array} \tag{2.14.1}$$

commutes.

2.15 Proposition. *Let* $(F \xrightarrow{\alpha} G)$ *be a crossed module with* Γ*-action. Consider the exact sequence* (2.13.1)*. Let* $\eta \in H^1(G)$, $\eta = \mathrm{Cl}(\psi)$, $\psi \in Z^1(G)$*. Then the fiber of* $\varkappa_1$ *over* $\varkappa_1(\eta)$ *is in canonical bijection with the quotient set*

$$\mathbf{H}^0({}_\psi(F \to G))\backslash H^1({}_\psi F).$$

Proof. The map $t_\psi\colon \mathbf{H}^1({}_\psi(F \to G)) \to \mathbf{H}^1(F \to G)$ takes 1 to $\mathrm{Cl}(1, \psi) = \varkappa_1(\eta)$. Since the diagram (2.14.1) is commutative, the map $t_\psi\colon H^1({}_\psi G) \to H^1(G)$ takes the kernel of ${}_\psi\varkappa_1$ to the fiber of $\varkappa_1$ over $\varkappa_1(\eta)$. By Corollary 2.13 (ii) the kernel of ${}_\psi\varkappa_1$ is in canonical bijection with $\mathbf{H}^0({}_\psi(F \to G))\backslash H^1({}_\psi F)$. This proves the proposition.

2.16 Proposition ([Br1], (5.1.3)). *Let*

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

be an exact sequence of crossed modules with Γ*-action. Then the sequence*

$$\begin{aligned} &1 \longrightarrow \mathbf{H}^{-1}(F_1 \to G_1) \xrightarrow{i_*} \mathbf{H}^{-1}(F_2 \to G_2) \xrightarrow{j_*} \mathbf{H}^{-1}(F_3 \to G_3) \\ &\xrightarrow{\delta_{-1}} \mathbf{H}^0(F_1 \to G_1) \xrightarrow{i_*} \mathbf{H}^0(F_2 \to G_2) \xrightarrow{j_*} \mathbf{H}^0(F_3 \to G_3) \\ &\xrightarrow{\delta_0} \mathbf{H}^1(F_1 \to G_1) \xrightarrow{i_*} \mathbf{H}^1(F_2 \to G_2) \xrightarrow{j_*} \mathbf{H}^1(F_3 \to G_3) \end{aligned} \tag{2.16.1}$$

is defined and exact.

Proof. Since j is a morphism of crossed modules, the subgroup $F_1 \subset F_2$ is G_2-invariant, and therefore the map δ_0 is defined. We must prove only the exactness at $\mathbf{H}^1(F_2 \to G_2)$; we leave it to the reader.

2.17 *The case of a normal abelian submodule.* We want to prolong the exact sequence (2.16.1). We assume that the crossed submodule $(F_1 \to G_1) \subset (F_2 \to G_2)$ is *abelian*, i.e. F_1 and G_1 are abelian groups and G_1 acts on F_1 trivially. We assume also that

(2.17.1) $\alpha_2(F_2)$ *commutes with* G_1 *in* G_2*;*

(2.17.2) F_1 *is central in* F_2.

It follows from (2.17.1) and (2.17.2) that the group G_3 acts on the complex $(F_1 \to G_1)$ through $\operatorname{coker}\alpha_3$. A cocycle $(h_3, \psi_3) \in Z^1(F_3 \to G_3)$ defines a cocycle $\bar\psi_3 \in Z^1(\operatorname{coker}\alpha_3)$, namely $\bar\psi_3(\sigma) = \psi_3(\sigma)(\operatorname{mod}\alpha_3(F_3))$. Since $\operatorname{coker}\alpha_3$ acts on the complex $(F_1 \to G_1)$, we can define the twisted complex ${}_{\bar\psi_3}(F_1 \to G_1)$. We write ${}_{\psi_3}(F_1 \to G_1)$ for ${}_{\bar\psi_3}(F_1 \to G_1)$.

We define a hypercohomology class $\Delta_1(h_3, \psi_3) \in \mathbf{H}^2({}_{\psi_3}(F_1 \to G_1))$ as follows. We lift ψ_3 to some continuous map $\psi\colon \Gamma \to G_2$ and lift h_3 to some continuous map $h\colon \Gamma \times \Gamma \to F_2$. Then we set

$$d_1(\sigma,\tau) = \psi(\sigma)\cdot {}^{\sigma}\psi(\tau)\cdot \psi(\sigma\tau)^{-1}\cdot \alpha_2(h(\sigma,\tau))$$
$$\mathfrak{a}_1(\sigma,\tau,\upsilon) = {}^{\psi(\sigma)\sigma}h(\tau,\upsilon)^{-1}\cdot h(\sigma,\tau\upsilon)^{-1}\cdot h(\sigma\tau,\upsilon)\cdot h(\sigma,\tau)$$

It is clear that $d_1(\sigma,\tau) \in G_1$, $\mathfrak{a}_1(\sigma,\tau,\upsilon) \in F_1$. We must show now that $(\mathfrak{a}_1, d_1) \in Z^2({}_{\psi_3}(F_1 \to G_1))$, i.e.

$${}^{\psi(\sigma)\sigma}d_1(\tau,\upsilon)^{-1}d_1(\sigma,\tau)d_1(\sigma\tau,\upsilon)d_1(\sigma,\tau\upsilon)^{-1} = \alpha_1(\mathfrak{a}_1(\sigma,\tau,\upsilon)),$$
$${}^{\psi(\sigma)\sigma}\mathfrak{a}_1(\tau,\upsilon,\rho)\cdot \mathfrak{a}_1(\sigma\tau,\upsilon,\rho)^{-1}\cdot \mathfrak{a}_1(\sigma,\tau\upsilon,\rho)\cdot \mathfrak{a}_1(\sigma,\tau,\upsilon\rho)^{-1}\mathfrak{a}_1(\sigma,\tau,\upsilon) = 1$$

We skip this tedious (though not easy) calculation.

We set $\Delta_1(h_3, \psi_3) = \operatorname{Cl}(\mathfrak{a}_1, d_1) \in \mathbf{H}^2({}_{\psi_3}(F_1 \to G_1))$. We must check that the cohomology class $\delta_1(h_3, \psi_3)$ is defined correctly, i.e. it does not depend on the choice of the lifting (h, ψ) of (h_3, ψ_3). We leave the check to the reader.

2.18 Proposition. *Let*

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

be an exact sequence of crossed modules with Γ*-action such that the crossed module* $(F_1 \to G_1)$ *is abelian and* (2.17.1) *and* (2.17.2) *hold. Let* $(h_3, \psi_3) \in Z^1(F_3 \to G_3)$*. Then* $\operatorname{Cl}(h_3, \psi_3) \in \operatorname{im} j_*$ *if and only if* $\Delta_1(h_3, \psi_3) = 1$.

Proof. Left to the reader.

2.19 *The fibers of* j_*. Let the exact sequence

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

be as in Proposition 2.18. We want to describe the fibers of the map $j_*\colon \mathbf{H}^1(F_2 \to G_2) \to \mathbf{H}^1(F_3 \to G_3)$.

Let

$$(h,\psi) \in Z^1(F_2 \to G_2),\ \eta_2 = \mathrm{Cl}(h,\psi) \in \mathbf{H}^1(F_2 \to G_2),$$
$$(h_3,\psi_3) = j(h,\psi),\ \eta_3 = j_*(\eta_2) = \mathrm{Cl}(h_3,\psi_3) \in \mathbf{H}^1(F_3 \to G_3).$$

We define a map $t_{(h,\psi)}\colon \mathbf{H}^1({}_{\psi_3}(F_1 \to G_1)) \to \mathbf{H}^1(F_2 \to G_2)$ which takes 1 to η_2. We set

$$t_{(h,\psi)}(\mathrm{Cl}(h_1,\psi_1)) = \mathrm{Cl}(hh_1,\psi_1\psi)$$

One can check that $(hh_1,\psi_1\psi) \in Z^1(F_2 \to G_2)$ and that the map $t_{(h,\psi)}$ is defined correctly.

2.19.1 Lemma. *The fiber of the map $j_*\colon \mathbf{H}^1(F_2 \to G_2) \to \mathbf{H}^1(F_3 \to G_3)$ over $\eta_3 = j_*(\eta_2)$ is the image of the map $t_{(h,\psi)}$.*

Proof. Easy.

2.20 Example. Let $F \xrightarrow{\alpha} G$ be a crossed module with Γ-action. Consider the canonical exact sequence of crossed modules

(2.20.1) $$1 \to (\ker\alpha \to 1) \xrightarrow{i} (F \to G) \xrightarrow{j} (F/\ker\alpha \hookrightarrow G) \to 1$$

The complex $(\ker\alpha \to 1)$ is abelian, and

$$\mathbf{H}^i(\ker\alpha \to 1) = H^{i+1}(\ker\alpha) \quad (i \geq -1),$$
$$\mathbf{H}^i(F/\ker\alpha \hookrightarrow G) = H^i(\operatorname{coker}\alpha) \qquad \text{for } i = 0, 1.$$

Note that conditions (2.17.1) and (2.17.2) are satisfied. By Propositions 2.16 and 2.18, to the short exact sequence (2.20.1) we can associate the hypercohomology exact sequence

(2.20.2)
$$1 \longrightarrow H^1(\ker\alpha) \xrightarrow{i_*} \mathbf{H}^0(F \to G) \xrightarrow{j_*} (\operatorname{coker}\alpha)^\Gamma \xrightarrow{\delta_0} H^2(\ker\alpha)$$
$$\longrightarrow \mathbf{H}^1(F \to G) \xrightarrow{j_*} H^1(\operatorname{coker}\alpha) \dashrightarrow H^3({}_{\psi_3}(\ker\alpha))$$

The arrow $\dashrightarrow$ in (2.20.2) is not a map, it just indicates that if $\eta_3 \in H^1(\operatorname{coker}\alpha)$, $\eta_3 = \mathrm{Cl}(\psi_3)$, where $\psi_3 \in Z^1(\operatorname{coker}\alpha)$, then η_3 comes from $\mathbf{H}^1(F \to G)$ if and only if $\Delta_1(\psi_3) = 1$. The fiber $j_*^{-1}(\eta_3)$ is described in Lemma 2.19.1

2.21 *The case of a central submodule.* Let

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

be a short exact sequence. We identify the crossed module $(F_1 \to G_1)$ with its image in $(F_2 \to G_2)$.

We say that $(F_1 \to G_1)$ is *central* in $(F_2 \to G_2)$, if G_1 is central in G_2, F_1 is central in F_2, and G_2 acts trivially on F_1. Assume that $(F_1 \to G_1)$ is central in $(F_2 \to G_2)$. Then we can define the connecting map $\delta_1 \colon \mathbf{H}^1(F_3 \to G_3) \to \mathbf{H}^2(F_1 \to G_1)$.

Let $\eta_3 \in \mathbf{H}^1(F_3 \to G_3)$, $\eta_3 = \mathrm{Cl}(h_3, \psi_3)$. Then $\Delta_1(h_3, \psi_3) \in \mathbf{H}^2(F_1 \to G_1)$ (we write $\mathbf{H}^2(F_1 \to G_1)$ instead of $\mathbf{H}^2({}_{\psi_3}(F_1 \to G_1))$ because $(F_1 \to G_1)$ is central in $(F_2 \to G_2)$). One can check that $\Delta_1(h_3, \psi_3)$ does not depend on the choice of the cocycle (h_3, ψ_3) representing η_3. We set

$$\delta_1(\eta_3) = \Delta_1(h_3, \psi_3).$$

Propositions 2.16 and 2.18 imply

2.22 Proposition. *Let*

$$1 \to (F_1 \to G_1) \xrightarrow{i} (F_2 \to G_2) \xrightarrow{j} (F_3 \to G_3) \to 1$$

be a short exact sequence of crossed modules with Γ*-action, where the crossed submodule* $(F_1 \to G_1)$ *is central in* $(F_2 \to G_2)$*. Then the sequence*

$$\mathbf{H}^1(F_1 \to G_1) \xrightarrow{i_*} \mathbf{H}^1(F_2 \to G_2) \xrightarrow{j_*} \mathbf{H}^1(F_3 \to G_3) \xrightarrow{\delta_1} \mathbf{H}^2(F_1 \to G_1) \qquad (2.22.1)$$

is exact.

3 Quasi-isomorphisms

Let $(F_1 \xrightarrow{\alpha_1} G_1) \to (F_2 \xrightarrow{\alpha_2} G_2)$ be a morphism of crossed modules. Such a morphism induces group homomorphisms $\ker \alpha_1 \longrightarrow \ker \alpha_2$, $\operatorname{coker} \alpha_1 \longrightarrow \operatorname{coker} \alpha_2$.

3.1 Definition. A morphism $(F_1 \xrightarrow{\alpha_1} G_1) \to (F_2 \xrightarrow{\alpha_2} G_2)$ is called *a quasi-isomorphism* if the induced homomorphisms $\ker \alpha_1 \to \ker \alpha_2$ and $\operatorname{coker} \alpha_1 \to \operatorname{coker} \alpha_2$ are isomorphisms.

3.2 Examples.

(1) Let $(F \xrightarrow{\alpha} G)$ be a crossed module. If α is injective, then $(F \to G) \to (1 \to \operatorname{coker} \alpha)$ is a quasi-isomorphism. If α is surjective, then $(\ker \alpha \to 1) \to (F \to G)$ is a quasi-isomorphism.

(2) The morphism of crossed modules of algebraic groups $(Z^{(\mathrm{sc})} \to Z) \hookrightarrow (G^{\mathrm{sc}} \to G)$, described in the introduction, is a quasi-isomorphism.

3.3 Theorem. *Let* $\varepsilon \colon (F_1 \to G_1) \to (F_2 \to G_2)$ *be a quasi-isomorphism of crossed modules with* Γ*-action. Then* ε *induces bijections*

$$\varepsilon_* \colon \mathbf{H}^i(F_1 \to G_1) \longrightarrow \mathbf{H}^i(F_2 \to G_2)$$

for $i = -1, 0, 1$.

Proof. For $i = -1$ the assertion is obvious.

Let $i = 0$. ¿From 2.20 we obtain a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H^1(\ker\,\alpha_1) & \longrightarrow & \mathbf{H}^0(F_1 \to G_1) & \longrightarrow & (\operatorname{coker}\alpha_1)^\Gamma & \longrightarrow & H^2(\ker\,\alpha_1) \\
 & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\
1 & \longrightarrow & H^1(\ker\,\alpha_2) & \longrightarrow & \mathbf{H}^0(F_2 \to G_2) & \longrightarrow & (\operatorname{coker}\alpha_2)^\Gamma & \longrightarrow & H^2(\ker\,\alpha_2)
\end{array}$$

with exact rows. Three vertical arrows in this diagram are isomorphisms because ε is a quasi-isomorphism. Then by the five-lemma the map $\varepsilon_*\colon \mathbf{H}^0(F_1 \to G_1) \to \mathbf{H}^0(F_2 \to G_2)$ is also an isomorphism, which was to be proved.

Let $i = 1$. ¿From 2.20 we obtain a commutative diagram

$$\begin{array}{ccccccccc}
(\operatorname{coker}\alpha_1)^\Gamma & \to & H^2(\ker\,\alpha_1) & \to & \mathbf{H}^1(F_1 \to G_1) & \to & H^1(\operatorname{coker}\alpha_1) & \dashrightarrow & H^3({}_{\psi_3^{(1)}}(\ker\,\alpha_1)) \\
\downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\
(\operatorname{coker}\alpha_2)^\Gamma & \to & H^2(\ker\,\alpha_2) & \to & \mathbf{H}^1(F_2 \to G_2) & \to & H^1(\operatorname{coker}\alpha_2) & \dashrightarrow & H^3({}_{\psi_3^{(2)}}(\ker\,\alpha_2))
\end{array}$$

with exact rows. Four vertical arrows in this diagram are bijections because ε is a quasi-isomorphism. We prove the assertion by diagram chasing. To prove the surjectivity of the map $\varepsilon_*\colon \mathbf{H}^1(F_1 \to G_1) \to \mathbf{H}^1(F_2 \to G_2)$ we use Lemma 2.19.1.

4 Abelianization maps

Let K be a field of characteristic 0, and $\bar{K}$ a an algebraic closure of K. We set $\Gamma = \mathrm{Gal}(\bar{K}/K)$.

The notions of a crossed module of algebraic groups and a quasi-isomorphism of crossed modules of algebraic groups are defined in the obvious way. If $F \to G$ is a crossed module of algebraic groups, then $F(\bar{K}) \to G(\bar{K})$ is a (discrete) crossed module with a Γ-action. We define the Galois hypercohomology of $F \to G$ by

$$\mathbf{H}^i(K, F \to G) = \mathbf{H}^i(\Gamma, F(\bar{K}) \to G(\bar{K})) \qquad (i = -1, 0, 1).$$

We often abbreviate $\mathbf{H}^i(K, F \to G)$ to $\mathbf{H}^i(F \to G)$.

If $(F_1 \to G_1) \to (F_2 \to G_2)$ is a quasi-isomorphism of crossed modules of K-groups, then

$$(F_1(\bar{K}) \to G_1(\bar{K})) \to (F_2(\bar{K}) \to G_2(\bar{K}))$$

is a quasi-isomorphism of crossed modules with Γ-action, and by Theorem 3.3 we have a bijection $\mathbf{H}^i(F_1 \to G_1) \xrightarrow{\sim} \mathbf{H}^i(F_2 \to G_2)$.

4.1 Let G be a connected reductive K-group. Let G^{ss} denote its derived group (which is semisimple), and let $G^{\mathrm{sc}} \to G^{\mathrm{ss}}$ be the universal covering of G^{ss}. Consider the composition

$$\rho: G^{\mathrm{sc}} \to G^{\mathrm{ss}} \to G.$$

Then G acts on G^{sc}, and $G^{\mathrm{sc}} \xrightarrow{\rho} G$ is a crossed module of K-groups. Let Z denote the center of G, and $Z^{(\mathrm{sc})}$ the center of G^{sc}.

Let $T \subset G$ be a maximal torus defined over K. We set $T^{(\mathrm{sc})} = \rho^{-1}(T)$. We define the abelian Galois cohomology $H^i_{\mathrm{ab}}(K, G)$ (which we usually abbreviate to $H^i_{\mathrm{ab}}(G)$) by

$$H^i_{\mathrm{ab}}(K, G) := \mathbf{H}^i(K, T^{(\mathrm{sc})} \to T) = \mathbf{H}^i(K, Z^{(\mathrm{sc})} \to Z) \qquad (i \geq -1),$$

where we identify the abelian groups $\mathbf{H}^i(K, T^{(\mathrm{sc})} \to T)$ and $\mathbf{H}^i(K, Z^{(\mathrm{sc})} \to Z)$ using the quasi-isomorphism $(Z^{(\mathrm{sc})} \to Z) \to (T^{(\mathrm{sc})} \to T)$ of abelian complexes. Note that $\mathbf{H}^i_{\mathrm{ab}}(K, \cdot)$ is a functor from the category of connected reductive K-group to the category of abelian groups. We are interested here in H^0_{ab} and H^1_{ab}.

4.1.1 Lemma. *For $i = 0, 1$ there is a canonical and functorial in G bijection $\mathbf{H}^i(G^{\mathrm{sc}} \to G) \xrightarrow{\sim} H^i_{\mathrm{ab}}(G)$, which is a group isomorphism when $i = 0$.*

Proof. The assertion follows from Theorem 3.3, applied to the quasi-isomorphisms

$$(Z^{(\mathrm{sc})} \to Z) \to (T^{(\mathrm{sc})} \to T) \to (G^{\mathrm{sc}} \to G).$$

4.1.2 *Remark* (essentially due to L. Breen). There is another, more intrinsic explanation of the fact that $\mathbf{H}^1(G^{\mathrm{sc}} \to G)$ has a canonical structure of abelian group. Deligne ([De], 2.0.2) noted that the commutator morphism

$$(g_1, g_2) \mapsto g_1 g_2 g_1^{-1} g_2^{-1} : \qquad G \times G \to G$$

can be uniquely lifted to a morphism

$$(g_1, g_2) \mapsto \{g_1, g_2\} : \quad G \times G \to G^{\mathrm{sc}},$$

and we have $\{g_1, g_2\} = \{g_2, g_1\}$. The crossed module $G^{\mathrm{sc}} \to G$ together with the map $\{\, ,\, \}$ is a stable crossed module in the terminology of Conduché ([Co], 3.1). To the crossed module $G^{\mathrm{sc}} \to G$ one associates a (fibered) Picard category $\mathcal{C}(G^{\mathrm{sc}} \to G)$ (cf. [Br2], the remark after Def. 1.1.6). The map $\{\, ,\, \}$ defines a commutativity constraint in $\mathcal{C}(G^{\mathrm{sc}} \to G)$, and thus turns it into a commutative Picard category (cf. [Br3]). A commutaive Picard category is a categoric analogue of an abelian group, so the set of isomorphism classes of torsors under such a category has a canonical structure of abelian group. Since $\mathbf{H}^1(G^{\mathrm{sc}} \to G)$ is the set of isomorphism classes of torsors under $\mathcal{C}(G^{\mathrm{sc}} \to G)$ ([Br1], 6.2), we see that $\mathbf{H}^1(G^{\mathrm{sc}} \to G)$ has a canonical structure of abelian group.

4.2 For $i = 0, 1$ we define the abelianization map ab^i as the composition

$$\mathrm{ab}^i: H^i(G) \xrightarrow{\varkappa_i} \mathbf{H}(G^{\mathrm{sc}} \to G) \xrightarrow{\sim} \mathbf{H}(Z^{(\mathrm{sc})} \to Z) = H^i_{\mathrm{ab}}(G),$$

where the map $\varkappa_i$ is induced by the imbedding $(1 \to \mathrm{G}) \to (G^{\mathrm{sc}} \to G)$ of crossed modules. By Corollary 2.13(i) we have an exact sequence

$$(4.2.1) \qquad G^{\mathrm{sc}}(K) \xrightarrow{\rho_*} G(K) \xrightarrow{\mathrm{ab}^0} H^0_{\mathrm{ab}}(K, G) \longrightarrow H^1(K, G^{\mathrm{sc}}) \xrightarrow{\rho_*} H^1(K, G) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(K, G)$$

In [Bo3] (see also [Bo1]) we prove

4.2.2 Proposition. *If K is a local field of characteristic 0 (archimedean or not) or a number field, then the map* ab^1 *is surjective.*

¿From Proposition 4.2.2 we deduce here

4.2.3 Corollary. *If K is a non-archimedean local field of characteristic 0, then the map* ab^1 *is bijective.*

Proof. Consider the exact sequence (4.2.1). By Proposition 2.15 any fiber of the map ab^1 comes from $H^1(K, {}_\psi G^{\mathrm{sc}})$ where $\psi \in Z^1(K,G)$. Since ${}_\psi G^{\mathrm{sc}}$ is simply connected, by Kneser's theorem ([Kn]) we have $H^1(K, {}_\psi G^{\mathrm{sc}}) = 1$, so the map ab^1 is injective. By Proposition 4.2.2 the map ab^1 is surjective. We conclude that the abelianization map ab^1 is bijective, which was to be proved.

We see that when K is a non-archimedean local field, the set $H^1(K,G)$ has a canonical and functorial structure of abelian group. (This result is due to Kottwitz [Ko1], [Ko2] in a slightly less functorial form.)

4.3 We can now describe the abelianization maps

$$\mathrm{ab}^i\colon H^i(K,G) \to H^i_{\mathrm{ab}}(K,G) = \mathbf{H}^i(K, Z^{(\mathrm{sc})} \to Z) \qquad (i = 0,1)$$

explicitly in terms of cocycles.

4.3.1 Proposition. *Let $g \in H^0(k,G) = G(K)$. Write $g = \rho(g')\cdot z$ where $g \in G^{\mathrm{sc}}(\bar K)$, $z \in Z(\bar K)$. Then $\mathrm{ab}^0(g) = \mathrm{Cl}(\varphi, z)$ where the map $\varphi\colon \Gamma \to Z^{(\mathrm{sc})}(\bar K)$ is defined by $\varphi(\sigma) = (g')^{-1} \cdot {}^\sigma g'$.*

Proof. We have $(1,g) * g' = (\varphi, z)$ with the notation of 1.2. Thus the 0-cocycles $(1,g)$ and (φ,z) are cohomological in $\mathbf{H}^0(K, G^{\mathrm{sc}} \to G)$. This proves the assertion.

4.3.2 Proposition. *Let $\xi \in H^1(K,G)$ be a cohomology class, $\xi = \mathrm{Cl}(\psi)$, $\psi \in Z^1(K,G)$. Write $\psi(\sigma) = \rho(\psi'(\sigma)) \cdot z(\sigma)$ for $\sigma \in \Gamma$, where $\psi'\colon \Gamma \to G^{\mathrm{sc}}(\bar K)$ and $z\colon \Gamma \to Z(\bar K)$ are continuous maps. Then $\mathrm{ab}^1(\xi) = \mathrm{Cl}(h,z)$, where the map $h\colon \Gamma\times\Gamma \to Z^{(\mathrm{sc})}(\bar K)$ is given by*

$$h(\sigma,\tau) = \psi'(\sigma)\cdot {}^\sigma\psi'(\tau)\cdot \psi'(\sigma\tau)^{-1}.$$

Proof. With the notation of 2.7 we have $(1,\psi) * ((\psi')^{-1}, 1) = (h,z)$. Thus the 1-cocycles $(1,\psi)$ and (h,z) are cohomological in $\mathbf{H}^1(K, G^{\mathrm{sc}} \to G)$. This proves the assertion.

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