BRAUER EQUIVALENCE IN A HOMOGENEOUS SPACE WITH CONNECTED STABILIZER

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Abstract. Let G be a simply connected semisimple algebraic group over a field k of characteristic 0, H a connected k-subgroup of G, $X = G/H$. When k is a local field or a number field, we compute the set of Brauer equivalence classes in $X(k)$.

0. Introduction

In this note we investigate the Brauer equivalence in a homogeneous space $X = G/H$, where G is a simply connected semisimple algebraic group over a local field or a number field, and H is a connected subgroup of G.

In more detail, let k be a field of characteristic 0, and let $\bar{k}$ be a fixed algebraic closure of k. For a smooth algebraic variety Y over k, set $\overline{Y} = Y_{\bar{k}} = Y \times_k \bar{k}$. Let $\operatorname{Br} Y$ denote the cohomological Brauer group of Y, $\operatorname{Br} Y = H^2_{\text{ét}}(Y, \mathbb{G}_m)$. Set $\operatorname{Br}_1 Y = \ker[\operatorname{Br} Y \to \operatorname{Br} \overline{Y}]$. There is a canonical pairing

$$Y(k) \times \operatorname{Br}_1 Y \to \operatorname{Br} k, \qquad (y, b) \mapsto b(y) \tag{0.1}$$

called the Manin pairing. We define the Brauer equivalence on $Y(k)$ as follows: $y_1 \sim y_2$ if $(y_1, b) = (y_2, b)$ for all $b \in \operatorname{Br}_1 Y$. We denote the set of classes of Brauer equivalence in $Y(k)$ by $Y(k)/\operatorname{Br}$. Note that we define the Brauer equivalence in terms of $\operatorname{Br}_1 Y$, not in terms of $\operatorname{Br}_1 Y^c$ or $\operatorname{Br} Y^c$, where Y^c is a smooth compactification of Y.

The notion of B-equivalence for a subgroup B of the Brauer group $\operatorname{Br} Y$ was introduced by Manin [Ma1], [Ma2]. Colliot-Thélène and Sansuc [CT/Sa1] investigated the Brauer equivalence in algebraic tori (they defined the Brauer equivalence in terms of the Brauer group of a smooth

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compactification). The Brauer equivalence in reductive groups was studied in [Th].

Let G be a simply connected semisimple algebraic group over k. Let H be a connected subgroup of G. We denote by H^{tor} the biggest toric quotient group of H. We are interested in the Brauer equivalence in the set $X(k)$ where $X = G/H$.

We compute $X(k)/\mathrm{Br}$ when k is a local field. Namely, we prove that there is a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \mathrm{im}\,[\ker[H^1(k,H) \to H^1(k,G)] \to H^1(k,H^{\mathrm{tor}})]$$

(Theorem 2.1). Moreover, when k is a nonarchimedean local field, we prove that there is a bijection $X(k)/\mathrm{Br} \xrightarrow{\sim} H^1(k,H^{\mathrm{tor}})$ (Theorem 2.2).

We also compute $X(k)/\mathrm{Br}$ when k is a number field. We prove that there is a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \mathrm{im}\left[\ker[H^1(k,H) \to H^1(k,G)] \to \bigoplus_v H^1(k_v,H^{\mathrm{tor}})\right]$$

(Theorem 3.1), where v runs over the set of places of k. Moreover when k is a totally imaginary number field, we prove that there is a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} H^1(k,H^{\mathrm{tor}})/Ш^1(k,H^{\mathrm{tor}})$$

(Theorem 3.4), where $Ш^1$ denotes the Shafarevich–Tate kernel.

In Example 3.9 we compute $X(k)/\mathrm{Br}$ when X is a symmetric space of a simply connected almost simple group over a totally imaginary number field k.

Remark 0.1. It would be interesting to compute the set of Brauer equivalence classes in $X(k)$, where $X = G/H$, with respect to the Brauer equivalence defined by the group $\mathrm{Br}\, X^c$, where X^c is a smooth compactification of X. Unfortunately the group $\mathrm{Br}\, X^c$ is not known, there is only a conjecture of Colliot-Thélène and the second author [CT/K]. Note that if k is a number field and $X = G/H$ is a symmetric space of a simply connected semisimple k-group G, then it follows from the conjecture of [CT/K] that $\mathrm{Br}\, X^c = \mathrm{Br}\, k$, and therefore there is only one equivalence class in $X(k)$.

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1. Generalities over an arbitrary field

1.1. We introduce some notation. For a smooth algebraic variety Y over a field k of characteristic 0 let $U(Y) = k[Y]^\times/k^\times$. Let $\operatorname{Pic} Y$ denote the Picard group of Y. Let $\operatorname{Br} Y$ and $\operatorname{Br}_1 Y$ be as in the Introduction. Set $\operatorname{Br}_a Y = \operatorname{coker}[\operatorname{Br} k \to \operatorname{Br}_1 Y]$. Assume that Y has a k-rational point y, and define

$$\operatorname{Br}_y Y = \ker[\operatorname{Br}_1 Y \xrightarrow{y^*} \operatorname{Br} k]$$

where y^* is the specialization map.

We prove that $\operatorname{Br}_y Y \simeq \operatorname{Br}_a Y$. Consider the composed map

$$\operatorname{Br} k \to \operatorname{Br}_1 Y \xrightarrow{y^*} \operatorname{Br} k\,,$$

it is the identity of $\operatorname{Br} k$. It follows that the exact sequence

$$0 \to \operatorname{Br}_y Y \to \operatorname{Br}_1 Y \xrightarrow{y^*} \operatorname{Br} k \to 0$$

splits, and we obtain an isomorphism $\operatorname{Br}_y Y \oplus \operatorname{Br} k \simeq \operatorname{Br}_1 Y$. Thus we obtain an isomorphism $\operatorname{Br}_y Y \to \operatorname{Br}_a Y$ and a splitting $\operatorname{Br}_a Y \to \operatorname{Br}_1 Y$ of the exact sequence

$$0 \to \operatorname{Br} k \to \operatorname{Br}_1 Y \to \operatorname{Br}_a Y \to 0.$$

1.2. We wish to investigate the Brauer equivalence in homogeneous spaces. Let G be a simply connected semisimple algebraic group over a field k of characteristic 0. Let $H \subset G$ be a connected k-subgroup. Set $X = G/H$; then X is a left homogeneous space of G with connected stabilizer. The variety X has a distinguished k-rational point x_0, the image in $X(k)$ of the unit element $e \in G(k)$.

We recall the definition of the connecting map $\delta\colon X(k) \to H^1(k, H)$, cf. [Se], I-5.4. Let $\pi\colon G \to G/H = X$ denote the canonical morphism. The group H acts on the right on G by $g * h = gh$ where $g \in G$, $h \in H$. Let $x \in X(k)$; then $\pi^{-1}(x)$ is a right torsor under H. By definition $\delta(x)$ is the class of the torsor $\pi^{-1}(x)$ in $H^1(k, H)$. Note that the map δ induces a canonical bijection

$$G(k)\backslash X(k) \xrightarrow{\sim} \ker[H^1(k, H) \to H^1(k, G)]$$

(cf. [Se], I-5.4, Cor. 1 of Prop. 36), where $G(k)\backslash X(k)$ is the quotient of $X(k)$ by the left action of $G(k)$.

We construct a map $X(k) \to H^1(k, H^{\mathrm{tor}})$ taking x_0 to 1. Composing the map $\delta\colon X(k) \to H^1(k, H)$ with the canonical map $H^1(k, H) \to H^1(k, H^{\mathrm{tor}})$ induced by the homomorphism $H \to H^{\mathrm{tor}}$, we obtain a map

$$\alpha\colon X(k) \to H^1(k, H^{\mathrm{tor}})\,. \tag{1.1}$$

Clearly this map is constant on the orbits of $G(k)$ in $X(k)$.

Let $\mathbf{X}(H)$ denote the group of k-characters of H, i.e. $\mathbf{X}(H) = \mathrm{Hom}_k(H, \mathbb{G}_m)$. We have $\mathbf{X}(H) = \mathbf{X}(H^{\mathrm{tor}})$.

Proposition 1.3. *There is a canonical isomorphism* $\mathbf{X}(H) \xrightarrow{\sim} \mathrm{Pic}\, X$.

Proof. By [Sa], 6.10 there is an exact sequence

$$U(G) \to \mathbf{X}(H) \to \mathrm{Pic}\, X \to \mathrm{Pic}\, G.$$

By Rosenlicht's theorem [Ro] $U(G) = \mathbf{X}(G)$, and clearly $\mathbf{X}(G) = 1$ because G is semisimple, so $U(G) = 1$. By [Sa], 6.9(iv) we have $\mathrm{Pic}\, G = 1$. Thus we obtain an isomorphism $\mathbf{X}(H) \xrightarrow{\sim} \mathrm{Pic}\, X$. □

1.3.1. *Remark.* In the case when k is algebraically closed, Proposition 1.3 was proved in [Po], Cor. of Thm. 4.

1.4. We have seen in the proof of Proposition 1.3 that $U(\overline{G}) = 1$. It follows that $U(\overline{X}) = 1$.

Since $X(k) \neq \varnothing$ and $U(\overline{X}) = 1$, we have by [Sa], 6.3(iii)

$$\mathrm{Br}_a\, X = H^1(k, \mathrm{Pic}\, \overline{X}).$$

We have $\mathrm{Br}_{x_0}\, X \simeq \mathrm{Br}_a\, X$. By Proposition 1.3, $\mathrm{Pic}\, \overline{X} = \mathbf{X}(\overline{H})$. We obtain

$$\mathrm{Br}_{x_0}\, X = H^1(k, \mathbf{X}(\overline{H})) = H^1(k, \mathbf{X}(\overline{H}^{\mathrm{tor}})). \tag{1.2}$$

There is a canonical cup product pairing

$$H^1(k, H^{\mathrm{tor}}) \times H^1(k, \mathbf{X}(\overline{H}^{\mathrm{tor}})) \to \mathrm{Br}\, k. \tag{1.3}$$

The pairing (1.3) together with the map (1.1) $X(k) \to H^1(k, H^{\mathrm{tor}})$ and the isomorphism (1.2) defines a pairing

$$X(k) \times \mathrm{Br}_{x_0}\, X \to \mathrm{Br}\, k. \tag{1.4}$$

Theorem 1.5. *The pairing (1.4) up to sign coincides with the restriction of the Manin pairing (0.1) to* $X(k) \times Br_{x_0} X \subset X(k) \times \mathrm{Br}_1\, X$.

Proof. We use the description of the Manin pairing with the help of torsors given in [CT/Sa2], §2.

We regard the canonical map $G \to X = G/H$ as a right (non-abelian) X-torsor under H. Set $S = H^{\mathrm{tor}}$ and denote by H^{ssu} the kernel of the natural homomorphism $\psi\colon H \to S$. This homomorphism induces push-forward maps in cohomology: $H^1(k, H) \to H^1(k, S)$ and $H^1(X, H) \to H^1(X, S)$ sending non-abelian torsors under H to abelian torsors under S (explicitly, a torsor Z under H goes to the torsor Z/H^{ssu} under S). Let $Y = G/H^{\mathrm{ssu}}$ be the torsor under S obtained from X by push-forward. Note that by Proposition 1.3 we have an isomorphism $\mathbf{X}(\overline{S})\xrightarrow{\sim}\operatorname{Pic}\overline{X}$.

Let $\theta_Y\colon X(k)\to H^1(k,S)$ be the canonical evaluation map associated to Y, i.e. θ_Y takes $x\in X(k)$ to the isomorphism class of the fibre of Y at x. Notice that θ_Y coincides with the map α defined by (1.1). Indeed, α is the composition $X(k)\to H^1(k,H)\to H^1(k,H^{\mathrm{tor}})$ where the first arrow is the connecting map δ defined in 1.2 and the second one is the push-forward map induced by ψ. Recall that $\delta(x)$ coincides with the isomorphism class of the fibre of $G\to X$ at x. Since push-forward commutes with specialization, $\alpha(x)$ coincides with the isomorphism class of the fibre of Y at x, and thus $\alpha=\theta_Y$.

To finish the proof, it only remains to recall the isomorphism (1.2) and to apply the diagram

$$\begin{array}{ccccc} X(k) & \times & \operatorname{Br}_1 X & \longrightarrow & \operatorname{Br} k \\ {\scriptstyle\theta_Y}\downarrow & & \uparrow & & \| \\ H^1(k,S) & \times & H^1(k,\mathbf{X}(\overline{S})) & \longrightarrow & \operatorname{Br} k \end{array}$$

Here the top row is the Manin pairing and the bottom row is the cup-product. The diagram is commutative up to sign (cf. [CT/Sa2], Prop. 2.7.10), which proves the theorem. ☐

2. Brauer equivalence over a local field

Theorem 2.1. *Let G, H, X be as in 1.2. Assume that k is a local field of characteristic 0 (archimedean or not). Then the map (1.1) $\alpha\colon X(k)\to H^1(k,H^{\mathrm{tor}})$ induces a bijection*

$$X(k)/\mathrm{Br}\xrightarrow{\sim}\operatorname{im}\left[\ker[H^1(k,H)\to H^1(k,G)]\to H^1(k,H^{\mathrm{tor}})\right].$$

Proof. It follows from Theorem 1.5 that two points $x_1,x_2\in X(k)$ are Brauer-equivalent if and only if $(\alpha(x_1),\eta)=(\alpha(x_2),\eta)$ for every $\eta\in H^1(k,\mathbf{X}(\overline{H}^{\mathrm{tor}}))$. Since k is a local field, the cup product pairing (1.3) is perfect (Tate–Nakayama duality, cf. [Mi], Cor. I-2.4), and it follows that x_1 and x_2 are Brauer-equivalent if and only if $\alpha(x_1)=\alpha(x_2)$. Thus the set of classes of Brauer equivalence is in a bijective correspondence with $\operatorname{im}\alpha$. We see that we must only describe the image of $X(k)$ in $H^1(k,H^{\mathrm{tor}})$. But the image of $X(k)$ in $H^1(k,H)$ is the same as the image of $G(k)\backslash X(k)$ and it equals $\ker[H^1(k,H)\to H^1(k,G)]$. Hence the image of $X(k)$ in $H^1(k,H^{\mathrm{tor}})$ is

$$\operatorname{im}\left[\ker[H^1(k,H)\to H^1(k,G)]\to H^1(k,H^{\mathrm{tor}})\right],$$

and the assertion of the theorem follows. ☐

Theorem 2.2. *Let G, H, X be as in 1.2, and assume that k is a nonarchimedean local field of characteristic* 0*. Then the map (1.1) α induces a bijection*

$$X(k)/\mathrm{Br} \xrightarrow{\sim} H^1(k, H^{\mathrm{tor}}).$$

Proof. Since G is a simply connected group, by Kneser's theorem (see [Pl/Ra], 6.1, Thm. 4) $H^1(k, G) = 1$. We see now from Theorem 2.1 that $X(k)/\mathrm{Br}$ is in a bijective correspondence with $\mathrm{im}\,[H^1(k, H) \to H^1(k, H^{\mathrm{tor}})]$. Let H^{ssu} denote $\ker[H \to H^{\mathrm{tor}}]$, it is an extension of a semisimple group by a unipotent group. Since k is local nonarchimedean and $(H^{\mathrm{ssu}})^{\mathrm{tor}} = 1$, the map $H^1(k, H) \to H^1(k, H^{\mathrm{tor}})$ is surjective, cf. [Bo], Cor. 6.4. This proves the theorem. □

3. Brauer equivalence over a number field

Theorem 3.1. *Let k be a number field, and let G, H, X be as in 1.2. Then the map*

$$X(k) \to G(k)\backslash X(k) \xrightarrow{\sim} \ker[H^1(k, H) \to H^1(k, G)] \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})$$

induces a bijection

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \mathrm{im}\left[\ker[H^1(k, H) \to H^1(k, G)] \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})\right]$$

where v runs over the set of places of k.

To prove Theorem 3.1 we need a lemma.

Lemma 3.2. (cf. [Ma/Ts], 4.5). *Let Y be a variety over a number field k. Then the map $Y(k)/\mathrm{Br} \to \prod_v Y(k_v)/\mathrm{Br}$ is injective, where v runs over the set of places of k and $Y(k_v)/\mathrm{Br}$ denotes the set of Brauer equivalence classes in $Y(k_v)$.*

Proof. Let $y_1, y_2 \in Y(k)$, and assume that y_1 and y_2 are Brauer-equivalent in $Y(k_v)$ for all places v of k. This means that for every $b_v \in \mathrm{Br}_1 Y_{k_v}$, $(y_1, b_v) = (y_2, b_v)$. Let now $b \in \mathrm{Br}_1 Y$. We wish to compare (y_1, b) and (y_2, b). Consider $\mathrm{loc}_v(y_i, b) \in \mathrm{Br}\, k_v$, $i = 1, 2$, where loc means localization. We have $\mathrm{loc}_v(y_i, b) = (y_i, \mathrm{loc}_v b)$, where $\mathrm{loc}_v b \in \mathrm{Br}_1 Y_{k_v}$. By assumption we have $(y_1, \mathrm{loc}_v b) = (y_2, \mathrm{loc}_v b)$. We see that $\mathrm{loc}_v(y_1, b) = \mathrm{loc}_v(y_2, b)$ for all v. It follows that $(y_1, b) = (y_2, b)$, because the map $\mathrm{loc}\colon \mathrm{Br}\, k \to \prod_v \mathrm{Br}\, k_v$ is injective. Thus y_1 and y_2 are Brauer-equivalent in $Y(k)$. □

3.3. *Proof of Theorem 3.1.* Note that $\mathrm{Br}_1 G = \mathrm{Br}\, k$ (cf. [Sa], 6.9(iv)), hence every orbit of $G(k)$ in $X(k)$ is contained in one class of Brauer equivalence. It follows that the map $X(k) \to X(k)/\mathrm{Br}$ factors through $G(k)\backslash X(k)$:

$$X(k) \to G(k)\backslash X(k) \to X(k)/\mathrm{Br}$$

and these maps are surjective.

Consider the commutative diagram

$$\begin{array}{ccccc} X(k)/\mathrm{Br} & \longrightarrow & H^1(k,H) & \longrightarrow & H^1(k,H^{\mathrm{tor}}) \\ {\scriptstyle a}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle d} \\ \prod_v X(k_v)/\mathrm{Br} & \xrightarrow{\ b\ } & \prod_v H^1(k_v,H) & \xrightarrow{\ c\ } & \prod_v H^1(k_v,H^{\mathrm{tor}}) \end{array} \tag{3.1}$$

The image of the map d is contained in $\bigoplus_v H^1(k_v, H^{\mathrm{tor}})$ (cf. e.g. [Vo], 11.3, Cor. 1 of Prop. 1), and we obtain a map

$$X(k)/\mathrm{Br} \to H^1(k,H) \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}}).$$

Consider the maps

$$X(k) \xrightarrow{e} H^1(k,H) \xrightarrow{f} \bigoplus_v H^1(k_v, H^{\mathrm{tor}}). \tag{3.2}$$

Since in diagram (3.1) the map a is injective by Lemma 3.2, and the map $c \circ b$ is injective by Theorem 2.1, we see that in (3.2) the fibres of the map $f \circ e$ are exactly the Brauer equivalence classes in $X(k)$, and so

$$X(k)/\mathrm{Br} \xrightarrow{\sim} \mathrm{im}\,(f \circ e) = f(\mathrm{im}\, e),$$

whence Theorem 3.1. □

Theorem 3.4. *In Theorem 3.1 assume that k is a totally imaginary number field. Then the bijection of Theorem 3.1 induces a bijection*

$$X(k)/\mathrm{Br} \xrightarrow{\sim} H^1(k,H^{\mathrm{tor}})/\text{Ш}^1(k,H^{\mathrm{tor}}).$$

To prove Theorem 3.4 we need a proposition and two corollaries.

Proposition 3.5. *Let k be a totally imaginary number field, and $L = (\overline{F}, \kappa)$ a k-kernel (k-lien) (see [Sp], [Bo], [F/S/S] for a definition), where $\overline{F}$ is a connected linear $\bar{k}$-group such that $\overline{F}^{\mathrm{tor}} = 1$. Then every element of $H^2(k,L)$ is neutral.*

Proof. The proposition follows from [Bo], Thm. 6.8(iii) and Thm. 6.3(ii). Note that in the case when $\overline{F}$ is semisimple, the proposition was proved by Douai ([Do], Cor. 5.1), see also [Bo], Cor. 6.9. The proposition follows also from Douai's result and [Bo], Prop. 4.1. □

Corollary 3.6. *Let k be a totally imaginary number field and let*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be an exact sequence of linear k-groups. If G_1 is connected and $G_1^{\mathrm{tor}} = 1$, then the map $H^1(k, G_2) \to H^1(k, G_3)$ is surjective.

Proof. We argue as in [Bo], the proof of Cor. 6.4. Let $\xi \in H^1(k, G_3)$, and let $\psi \in Z^1(k, G_3)$ be a cocycle from the class ξ. According to Springer ([Sp], 1.20), one can associate to ψ a k-kernel $L_\psi = (G_{1\bar{k}}, \kappa_\psi)$ and a cohomology class $\Delta(\psi) \in H^2(k, L_\psi)$ which is the obstruction to lifting ξ to $H^1(k, G_2)$. Since $G_{1\bar{k}}^{\mathrm{tor}} = 1$, by Prop. 3.5 the class $\Delta(\psi)$ is neutral, and hence ξ comes from $H^1(k, G_2)$. □

Corollary 3.7. *Let F be a connected linear group over a totally imaginary number field k. Then the map $H^1(k, F) \to H^1(k, F^{\mathrm{tor}})$ is surjective.*

Proof. We have an exact sequence

$$1 \to F^{\mathrm{ssu}} \to F \to F^{\mathrm{tor}} \to 1$$

where $(F^{\mathrm{ssu}})^{\mathrm{tor}} = 1$. Now the corollary follows from Corollary 3.6. □

3.8. *Proof of Theorem 3.4.* Since G is simply connected and k is a totally imaginary number field, $H^1(k, G) = 1$ (Kneser–Harder–Chernousov, see [Pl/Ra] §6.1, Thm. 6). Thus $\ker[H^1(k, H) \to H^1(k, G)] = H^1(k, H)$. By Theorem 3.1 $X(k)/\mathrm{Br}$ is in a bijective correspondence with

$$\mathrm{im}\left[H^1(k, H) \to H^1(k, H^{\mathrm{tor}}) \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})\right].$$

By Corollary 3.7 the map $H^1(k, H) \to H^1(k, H^{\mathrm{tor}})$ is surjective. We see that $X(k)/\mathrm{Br}$ is in a bijective correspondence with

$$\mathrm{im}\left[H^1(k, H^{\mathrm{tor}}) \to \bigoplus_v H^1(k_v, H^{\mathrm{tor}})\right] = H^1(k, H^{\mathrm{tor}})/\text{Ш}^1(k, H^{\mathrm{tor}}).$$

□

Example 3.9. Let G be a simply connected absolutely almost simple group over a number field k, $H \subset G$ a connected k-subgroup, $X = G/H$. Assume that X is a symmetric space, i.e. H is the group of invariants of an involution of G. From the classification of involutions of simple Lie algebras (see e.g. [He], X-5, p. 514) it follows that $\dim H^{\mathrm{tor}} \leq 1$.

If $H^{\mathrm{tor}} = 1$ or H^{tor} is a one-dimensional split torus, then $H^1(k_v, H^{\mathrm{tor}}) = 1$ for all v, and by Theorem 3.1 $X(k)/\mathrm{Br}$ consists of one element.

If $H^{\rm tor}$ is a one-dimensional nonsplit torus, then $H^{\rm tor}$ splits over a quadratic extension K of k. Assume in addition that k is totally imaginary. Then by Theorem 3.4 $X(k)/{\rm Br} = H^1(k, H^{\rm tor})/Ш^1(k, H^{\rm tor})$. Since K/k is cyclic, we have $Ш^1(k, H^{\rm tor}) = 1$ ([Vo], 11.6, Cor. 3), and we see that

$$X(k)/{\rm Br} = H^1(k, H^{\rm tor}) = k^\times / N_{K/k}K^\times$$

where $N_{K/k}$ denotes the norm map.

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