THE HODGE GROUP AND ENDOMORPHISM ALGEBRA OF AN ABELIAN VARIETY

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Abstract. This is an English translation of the author's 1981 note in Russian, published in a Yaroslavl collection. We prove that if an Abelian variety over $\mathbb{C}$ has no nontrivial endomorphisms, then its Hodge group is $\mathbb{Q}$-simple.

In this note we prove that if an Abelian variety A over $\mathbb{C}$ has no nontrivial endomorphisms, then its Hodge group $\mathrm{Hg}\, A$ is a $\mathbb{Q}$-simple algebraic group. Actually a slightly more general result is obtained. The note was inspired by Tankeev's paper [1]. The author is very grateful to Yu.G. Zarhin for useful discussions.

Let A be an Abelian variety over $\mathbb{C}$. Set $V = H_1(A, \mathbb{Q})$. Denote by T^1 the compact one-dimensional torus over $\mathbb{R}$: $T^1 = \{z \in \mathbb{C} : |z| = 1\}$. Denote by $\varphi\colon T^1 \to \mathrm{GL}(V_\mathbb{R})$ the homomorphism defining the complex structure in $V_\mathbb{R} = H_1(A, \mathbb{R})$. By definition, the Hodge group $\mathrm{Hg}\, A$ is the smallest algebraic subgroup $H \subset \mathrm{GL}(V)$ defined over $\mathbb{Q}$ such that $H_\mathbb{R} \supset \mathrm{im}\, \varphi$. Denote by $\mathrm{End}\, A$ the ring of endomorphisms of A, and set $\mathrm{End}^\circ A = \mathrm{End}\, A \otimes_\mathbb{Z} \mathbb{Q}$.

Theorem. *Let A be a polarized Abelian variety. Let F denote the center of* $\mathrm{End}^\circ A$*, and let F_0 denote the subalgebra of fixed points in F of the Rosati involution induced by the polarization. Set $G = \mathrm{Hg}\, A$ and denote by r the number of factors in the decomposition of the commutator subgroup $G' = (G, G)$ of G into an almost direct product of $\mathbb{Q}$-simple groups. Then* $r \le \dim_\mathbb{Q} F_0$.

Corollary 1. *If $F = \mathbb{Q}$ (in particular, if $\mathrm{End}^\circ A = \mathbb{Q}$), then $\mathrm{Hg}\, A$ is a $\mathbb{Q}$-simple group.*

Before proving the Theorem and deducing Corollary 1, we describe the necessary properties of the Hodge group.

Proposition (see [2])**.** *Let A be an Abelian variety over $\mathbb{C}$. Then*
(a) *The Hodge group $G = \mathrm{Hg}\, A$ is a connected reductive group.*
(b) *The centralizer K of $\mathrm{im}\, \varphi$ in $G_\mathbb{R}$ is a maximal compact subgroup of $G_\mathbb{R}$.*
(c) *$G'_\mathbb{R}$ is a group of Hermitian type (i.e., its symmetric space admits a structure of a Hermitian symmetric space).*

M.V. Borovoi, The Hodge group and the algebra of endomorphisms of an Abelian variety (Russian). In: "Problems in Group Theory and Homological Algebra", pp. 124–126, Yaroslav. Gos. Univ., Yaroslavl, 1981.

(d) *The algebra* $\mathrm{End}^\circ A$ *is the centralizer of* $\mathrm{Hg}\, A$ *in* $\mathrm{End}\, V$.
(e) *For any polarization* P *of* A, *the Hodge group* $\mathrm{Hg}\, A$ *respects the corresponding nondegenerate skew-symmetric form* ψ_P *on the space* V.

Deduction of Corollary 1 from the Theorem. Denote by C the center of G. From assertion (d) of the Proposition, it follows that $C \subset F^*$. Hence, under the hypotheses of the corollary we have $C \subset \mathbb{Q}^*$. From assertion (b) of the Proposition it follows that $C_\mathbb{R}$ is a compact group, hence, C is a finite group and $G = G'$. By virtue of the Theorem, $r = 1$ and G is a $\mathbb{Q}$-simple group.

Lemma 1. *Let* H *be a normal subgroup of* G *defined over* $\mathbb{Q}$ *such that the* $\mathbb{R}$*-group* $H_\mathbb{R}$ *is compact. Then* $H \subset C$ *(where* C *denotes the center of* G*).*

Proof. By assertion (b) of the Proposition we have $H_\mathbb{R} \subset K$, where K is the centralizer of $\mathrm{im}\,\varphi$ in $G_\mathbb{R}$. Therefore, $\mathrm{im}\,\varphi$ is contained in the centralizer (defined over $\mathbb{Q}$) $\mathcal{Z}(H)$ of H in G. Then it follows from the definition of the Hodge group that $G \subset \mathcal{Z}(H)$. Hence $H \subset C$. $\square$

Lemma 2. *Consider the natural representation* ρ *of the group* $G'_\mathbb{R}$ *in the vector space* $V_\mathbb{R}$. *Denote by* r_ρ *the number of pairwise nonequivalent summands in the decomposition of* ρ *into a direct sum of* $\mathbb{R}$*-irreducible representations. Then* $r_\rho = \dim_\mathbb{Q} F_0$.

Proof. We set $\mathfrak{A} = \mathrm{End}^\circ A \otimes_\mathbb{Q} \mathbb{R}$ and write the decomposition

$$\mathfrak{A}_1 + \cdots + \mathfrak{A}_{r_\mathfrak{A}}.$$

of the semisimple $\mathbb{R}$-algebra $\mathfrak{A}$ into a sum of simple $\mathbb{R}$-algebras. By the assertion (d) of the Proposition we have $r_\rho = r_\mathfrak{A}$. Furthermore, it is known (see [3, Section 21]) that the Rosati involution acts on the center F_i of the algebra $\mathfrak{A}_i$ trivially if $F_i = \mathbb{R}$, and as the complex conjugation if $F_i = \mathbb{C}$. It follows that

$$F_0 \otimes_\mathbb{Q} \mathbb{R} = \mathbb{R} + \cdots + \mathbb{R}$$

($r_\mathfrak{A}$ summands) whence $\dim_\mathbb{Q} F_0 = r_\mathfrak{A}$. Thus $\dim_\mathbb{Q} F_0 = r_\mathfrak{A} = r_\rho$. $\square$

Proof of the theorem. Denote by r_{nc} the number of noncompact groups in the decomposition

$$G'_\mathbb{R} = G_1 \cdot G_2 \cdots G_N$$

of the group $G'_\mathbb{R}$ in an almost direct product of simple $\mathbb{R}$-groups. It is known from results of Satake [4, Theorem 2] that for each $\mathbb{R}$-irreducible representation ρ' in the decomposition of the representation ρ into a direct sum of $\mathbb{R}$-irreducibles, there exist not more that one noncompact group G_i ($1 \le i \le N$) such that the restriction $\rho'|_{G_i}$ is nontrivial. Therefore, $r_{\mathrm{nc}} \le r_\rho$. Taking in account Lemma 2, we obtain that $r_{\mathrm{nc}} \le \dim_\mathbb{Q} F_0$.

Further, since $G'_\mathbb{R}$ is of Hermitian type, we see that all the groups $G_1, \dots, G_N$ are of Hermitian type as well, and hence they are absolutely simple. Consider the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of the simple factors $G_1, \dots, G_N$. The orbits of the Galois group bijectively correspond to $\mathbb{Q}$-simple normal subgroups of G'. By Lemma 1 each orbit contains at least one noncompact group G_i. Thus the number of orbits, i.e., the number r of $\mathbb{Q}$-simple normal subgroups of G', does not exceed the number r_{nc} of noncompact groups among $G_1, \dots, G_N$. We obtain that $r \le r_{\mathrm{nc}} \le \dim_{\mathbb{Q}} F_0$, which completes the proof of the theorem. □

Corollary 2. *Assume that* $\operatorname{End}^\circ A = \mathbb{Q}$. *Write the decomposition*

$$\rho_{\mathbb{C}} = \rho_1 \otimes \cdots \otimes \rho_N$$

of the irreducible representation $\rho_{\mathbb{C}}$ *of the semisimple group* $G_{\mathbb{C}}$ *in the vector space* $V_{\mathbb{C}}$ *into a tensor product of irreducible representations* ρ_i *of the universal coverings* $\widetilde{G}_{i\mathbb{C}}$ $(i = 1, \dots, N)$ *of the simple factors* $G_{i\mathbb{C}}$ *of* $G_{\mathbb{C}}$. *Then each of the representations* ρ_i *respects a nondegenerate skew-symmetric bilinear form, and the number* N *is odd.*

Proof. Indeed, by Corollary 1 the Galois group permutes transitively the groups $G_{i\mathbb{C}}$ and the representations ρ_i. Now Corollary 2 follows from assertion (e) of the Proposition. □

References

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Submitted on November 17, 1980.