Galois cohomology of real semisimple groups via Kac labelings

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- Let $\mathbf{G}$ be a real algebraic group.
- Say, $\mathbf{G} \subseteq \mathrm{GL}(n, \mathbb{C})$ is defined by polynomials with *real* coefficients.
- We have $\mathbf{G}(\mathbb{R})$, $\mathbf{G}(\mathbb{C})$, $\sigma\colon \mathbf{G}(\mathbb{C}) \to \mathbf{G}(\mathbb{C}),\ g \mapsto \bar{g}$. Then σ is anti-holomorphic and $\sigma^2 = \mathrm{id}$. We have also $G := \mathbf{G} \times_{\mathbb{R}} \mathbb{C}$.
- Conversely, from a connected reductive *complex* algebraic group G and an anti-holomorphic involution

$$\sigma\colon G(\mathbb{C}) \to G(\mathbb{C}),$$

  by Galois descent we obtain a *real* algebraic group $\mathbf{G}$.
- We say that σ is a *real structure* on G. We write

$$\mathbf{G} = (G, \sigma).$$

- Let $\mathbf{G} = (G, \sigma)$. Write

$$Z^1(\mathbb{R}, \mathbf{G}) = \{a \in G \mid a \cdot \sigma(a) = 1\}.$$

- The group $G(\mathbb{C})$ acts on $Z^1(\mathbb{R}, \mathbf{G})$ by

$$g\colon\ a \mapsto g \cdot a \cdot \sigma(g^{-1}),$$

By definition, $H^1(\mathbb{R}, \mathbf{G})$ is the set of orbits of this action. It has a *neutral element*, the class $[1] \in H^1(\mathbb{R}, \mathbf{G})$.

- The Galois cohomology answers important questions. In particular, if $\mathbf{H} \subset \mathbf{G}$ is an $\mathbb{R}$-subgroup, and $\mathbf{Y} = \mathbf{G}/\mathbf{H}$, then the set of orbits of $\mathbf{G}(\mathbb{R})$ in $\mathbf{Y}(\mathbb{R})$ is in a canonical bijection with

$$\ker[H^1(\mathbb{R}, \mathbf{H}) \to H^1(\mathbb{R}, \mathbf{G})].$$

- We write

$$\mathbf{G}(\mathbb{R})_2 = \{g \in \mathbf{G}(\mathbb{R}) \mid g^2 = 1\}.$$

- If $a \in \mathbf{G}(\mathbb{R})_2$, then

$$a \cdot \sigma(a) = a^2 = 1.$$

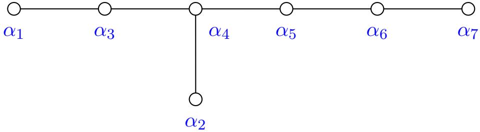
  Thus $\mathbf{G}(\mathbb{R})_2 \subset Z^1(\mathbb{R}, \mathbf{G})$.

- From now on $\mathbf{G} = (G, \sigma)$ is a (connected) *semisimple* $\mathbb{R}$-group. I describe a simple combinatorial algorithm for $H^1(\mathbb{R}, \mathbf{G})$ in terms of the *affine Dynkin diagram* of $\mathbf{G}$.

- The algorithm is due to Victor Kac (1968) for adjoint groups. The idea in the simply connected case is due to Ernest B. Vinberg (an e-mail message to the speaker dated 2008).

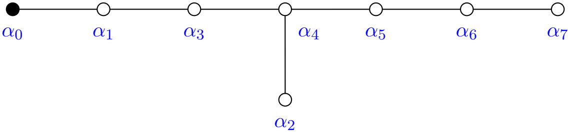
I will explain the algorithm in the example of E_7. I start from the *adjoint* case (due to Kac). So we have:

$G = G^{\mathrm{ad}}$ of type E_7,
$T \subset G$ a maximal torus,
B a Borel subgroup
$R = R(G, T)$ the root system,
$S = S(G, T, B) = \{\alpha_1, \dots, \alpha_7\}$ the system of simple roots.



This is *not* what we need.

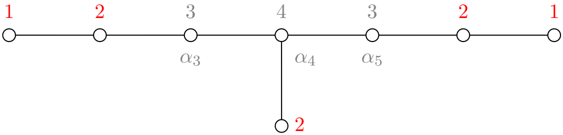
- We need the *affine* or *extended* Dynkin diagram



- Now we have 8 roots in a 7-dimensional space (namely, α_0 is the *lowest root*, $\alpha_0 = -\alpha_h$, where α_h is the highest root in R with respect to S). There is a linear relation

$$m_0\alpha_0 + m_1\alpha_1 + \cdots + m_7\alpha_7 = 0,$$

normalized such that $m_0 = 1$. Here all m_i are positive integers, see below:

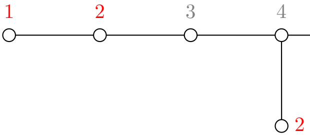


Definition

A *Kac labeling of* $\widetilde{D}$ is a family (q_i) of numerical labels $q_i \in \mathbb{Z}_{\geq 0}$, $i = 0, 1, \ldots, 7$, such that

$$\sum_i m_i q_i = 2.$$

Clearly, $q_i \leq 2$ for all i. Moreover, $q_3 = q_4 = q_5 = 0$.



The Kac labelings of $\widetilde{D}$ are:

$$\begin{matrix} 2000000 \\ 0 \end{matrix} \qquad \begin{matrix} 0000002 \\ 0 \end{matrix}$$

$$\begin{matrix} 1000001 \\ 0 \end{matrix}$$

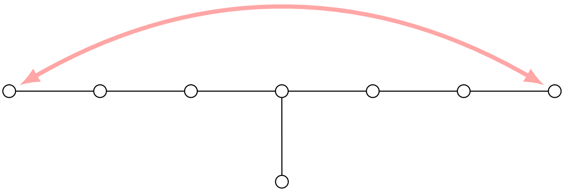
$$\begin{matrix} 0100000 \\ 0 \end{matrix} \qquad \begin{matrix} 0000010 \\ 0 \end{matrix}$$

$$\begin{matrix} 0000000 \\ 1 \end{matrix}$$

We denote by $\mathcal{K}(\widetilde{D})$ the set of all Kac labelings of $\widetilde{D}$.

We consider the *adjoint* group $G = G^{\mathrm{ad}}$ with fundamental group $F = \pi_1(G^{\mathrm{ad}})$ of order 2: $F = \{1, \rho\}$.

The element ρ acts on $\widetilde{D}$ by the only nontrivial automorphism.



So F acts on $\mathcal{K}(\widetilde{D})$. Let $\mathcal{K}(\widetilde{D})/F$ denote the set of orbits.

Theorem (Kac 1969)

Let $\mathbf{G}^{\mathrm{ad}} = (G^{\mathrm{ad}}, \sigma_c)$ *denote the compact adjoint group of type* E_7. *Then*

$$H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}}) \cong \mathcal{K}(\widetilde{D})/F.$$

$$\begin{array}{ccc} \begin{smallmatrix}2000000\\0\end{smallmatrix} & & \begin{smallmatrix}0000002\\0\end{smallmatrix} \\ & \begin{smallmatrix}1000001\\0\end{smallmatrix} & \\ \begin{smallmatrix}0100000\\0\end{smallmatrix} & & \begin{smallmatrix}0000010\\0\end{smallmatrix} \\ & \begin{smallmatrix}0000000\\1\end{smallmatrix} & \end{array}$$

Four orbits

Let T be a $\mathbb{C}$-torus,

$\mathsf{X}^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ the character group,

$\mathsf{X}_*(T) = \mathrm{Hom}(\mathbb{C}^\times, T)$ the cocharacter group,

$$\langle\,,\rangle : \ \mathsf{X}^*(T) \times \mathsf{X}_*(T) \to \mathbb{Z}.$$

Namely, for $\chi \in \mathsf{X}^*(T), \quad \nu \in \mathsf{X}_*(T)$ we consider

$$\chi \circ \nu \colon \ \mathbb{C}^\times \xrightarrow{\ \nu\ } T \xrightarrow{\ \chi\ } \mathbb{C}^\times.$$

We have

$$\chi \circ \nu = (z \mapsto z^n)$$

for some $n \in \mathbb{Z}$, and we set

$$\langle \chi, \nu \rangle = n.$$

- $\mathbf{G}^{\mathrm{ad}} = (G^{\mathrm{ad}}, \sigma_c)$ compact, $\mathbf{T}^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}$ a maximal torus
- $Q = \mathsf{X}^*(T^{\mathrm{ad}})$, $P^\vee = \mathsf{X}_*(T^{\mathrm{ad}})$, $\langle\, , \rangle\colon Q \times P^\vee \to \mathbb{Z}$.
- $q = (q_i) \in \mathcal{K}(\widetilde{D})$ a Kac labeling. Define $\widehat{q} \in P^\vee$ by

$$\langle \alpha_i, \widehat{q} \rangle = q_i \quad \text{for the simple roots } \alpha_i\,, \quad i = 1, \dots 7$$

Then $\widehat{q} : \mathbb{C}^\times \to T^{\mathrm{ad}}$. Take

$$a_q = \widehat{q}(-1) \in T^{\mathrm{ad}}(\mathbb{C})_2 = \mathbf{T}^{\mathrm{ad}}(\mathbb{R})_2 \subset Z^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}})$$

$$\mathbb{C}^\times \xrightarrow{\widehat{q}} T^{\mathrm{ad}}, \quad -1 \mapsto a_q := \widehat{q}(-1) \in \mathbf{T}^{\mathrm{ad}}(\mathbb{R})_2\,.$$

$$\mathcal{K}(\widetilde{D})/F \xrightarrow{\sim} H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}}), \quad q \mapsto [a_q].$$

- To $q \in \mathcal{K}(\widetilde{D})$ we associate a *twisted form*

$$\mathbf{G}_q^{\mathrm{ad}} = (G^{\mathrm{ad}}, \sigma_q), \quad \sigma_q = a_q \circ \sigma_c .$$

$\overset{2000000}{0}$	$\overset{0000002}{0}$	E_7	Compact
$\overset{1000001}{0}$		$\mathsf{E}_6 \cdot T_1$	$\mathsf{E}_{7(-25)}$
$\overset{0100000}{0}$	$\overset{0000010}{0}$	$\mathsf{A}_1 \cdot \mathsf{D}_6$	$\mathsf{E}_{7(-5)}$
$\overset{0000000}{1}$		A_7	$\mathsf{E}_{7(7)}$

- $\mathcal{K}(\widetilde{D})/F \xrightarrow{\sim} H^1(\mathbb{R}, \mathbf{G}_q^{\mathrm{ad}}) \colon \; p \mapsto [a_{p,q}]$

$$\mathbb{C}^\times \xrightarrow{\widehat{p}-\widehat{q}} T^{\mathrm{ad}}, \quad -1 \longmapsto a_{p,q} := (\widehat{p} - \widehat{q})(-1) \in \mathbf{T}^{\mathrm{ad}}(\mathbb{R})_2$$

- For $p = q$ we have $a_{p,q} = 1$.

Now we consider the *simply connected* E_7.

- $\mathbf{G}_q^{\mathrm{sc}} = (G^{\mathrm{sc}}, \sigma_q)$ the *universal cover* of $\mathbf{G}_q^{\mathrm{ad}}$,
- T^{sc} the preimage of T^{ad} in G^{sc},
- $P = \mathsf{X}^*(T^{\mathrm{sc}})$, the character group of T^{sc}.

- Then $[P : Q] = 2$, $P = \langle Q, \lambda \rangle$,

$$\lambda = \sum_{\alpha \in S} c_\alpha \alpha = \frac{1}{2}(\alpha_2 + \alpha_5 + \alpha_7).$$

- Set

$$\mathcal{K}(\widetilde{D}, q) = \left\{ p \in \mathcal{K}(\widetilde{D}) \;\middle|\; \sum_{\alpha \in S} c_\alpha p_\alpha \equiv \sum_{\alpha \in S} c_\alpha q_\alpha \pmod{\mathbb{Z}} \right\}.$$

- Write
$$Q^\vee = \mathsf{X}_*(T^{\mathrm{sc}}) := \mathrm{Hom}(\mathbb{C}^\times, T^{\mathrm{sc}}).$$

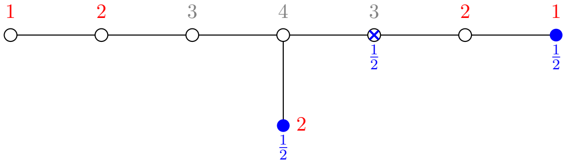
- Then $[P^\vee : Q^\vee] = 2$. For $p \in \mathcal{K}(\widetilde{D})$,

$$p \in \mathcal{K}(\widetilde{D}, q) \iff \widehat{p} - \widehat{q} \in Q^\vee \subset P^\vee, \quad \widehat{p} - \widehat{q} \colon \mathbb{C}^\times \to T^{\mathrm{sc}}.$$

- Hence $(\widehat{p} - \widehat{q})(-1) \in T^{\mathrm{sc}}(\mathbb{C})_2 = T^{\mathrm{sc}}(\mathbb{R})_2 \subset Z^1(\mathbb{R}, \mathbf{G}_q^{\mathrm{sc}})$

Theorem (Vinberg, Borovoi, Timashev)

$$\mathcal{K}(\widetilde{D}, q) \xrightarrow{\sim} H^1(\mathbb{R}, \mathbf{G}_q^{\mathrm{sc}})$$
$$p \mapsto [(\widehat{p} - \widehat{q})(-1)] \in H^1(\mathbb{R}, \mathbf{G}_q^{\mathrm{sc}}).$$



$$\sum c_\alpha q_\alpha \equiv \tfrac{1}{2}(q_2 + q_5 + q_7) = \tfrac{1}{2}(q_2 + q_7)$$

(because $q_5 = 0$). Write

$$s(q) = q_2 + q_7 \text{ (the sum over the blue vertices).}$$

Then

$$p \in \mathcal{K}(\widetilde{D}, q) \iff s(p) \equiv s(q) \pmod 2.$$

- We say that a Kac labeling q is *even* if $s(q)$ is even.
- Even and odd Kac labelings:

$$\mathcal{K}^{\text{even}}: \quad \begin{smallmatrix}2000000\\0\end{smallmatrix} \qquad \begin{smallmatrix}0000002\\0\end{smallmatrix} \qquad \begin{smallmatrix}0100000\\0\end{smallmatrix} \qquad \begin{smallmatrix}0000010\\0\end{smallmatrix}$$

$$\mathcal{K}^{\text{odd}}: \quad \begin{smallmatrix}1000001\\0\end{smallmatrix} \qquad \begin{smallmatrix}0000000\\1\end{smallmatrix}$$

- If q is even, then $H^1(\mathbb{R}, \mathbf{G}_q^{\text{sc}}) \simeq \mathcal{K}^{\text{even}}$, hence $\#H^1(\mathbb{R}, \mathbf{G}_q^{\text{sc}}) = 4$.
- If q is odd, then $H^1(\mathbb{R}, \mathbf{G}_q^{\text{sc}}) \simeq \mathcal{K}^{\text{odd}}$, hence $\#H^1(\mathbb{R}, \mathbf{G}_q^{\text{sc}}) = 2$.

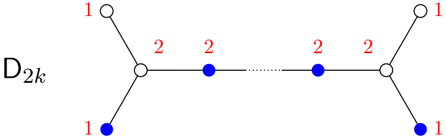
- Skip Garibaldi and Nikita Semenov, 2010, $H^1(\mathbb{R}, \mathsf{E}^{\text{sc}}_{7(-5)})$, $\#H^1 = 4$.
- Brian Conrad, 2016, $H^1(\mathbb{R}, \mathsf{E}^{\text{sc}}_{7(7)})$, $\#H^1 = 2$.
- Adams and Taïbi 2018, B. and Evenor 2016 - all four cases.

HALF-SPIN GROUP

- We consider the *half-spin group*, the compact group $\mathbf{G} = (G, \sigma_c)$ of type ${}^1\mathrm{D}_\ell$ with even $\ell = 2k \geq 4$ with the cocharacter lattice

$$X^\vee = \langle Q^\vee, \omega_{\ell-1}^\vee \rangle.$$

- It is neither simply connected nor adjoint. It is not isomorphic to $\mathrm{SO}_{2\ell}$ unless $\ell = 4$.



The coefficients m_α are in red.

- As above,

$$\mathcal{K}(\widetilde{D}) = \left\{ q = (q_\alpha) \mid \sum_{\alpha \in \widetilde{D}} m_\alpha q_\alpha = 2 \right\}.$$

- The character lattice $X = \mathsf{X}^*(T)$ is generated by Q and the weight

$$\lambda := (\alpha_1 + \alpha_3 + \cdots + \alpha_{\ell-3} + \alpha_\ell)/2.$$

  In the diagram, the vertices corresponding to these simple roots $\alpha_1, \alpha_3, \dots, \alpha_\ell$ are painted in blue.

- We write

$$s(q) = \sum_{\alpha \text{ blue}} q_\alpha .$$

  Then

$$\mathcal{K}(\widetilde{D}, q) = \{ p \in \mathcal{K}(\widetilde{D}) \mid s(p) \equiv s(q) \pmod 2 \}.$$

- The fundamental group

$$F = \pi_1(G) = \mathsf{X}_*(T)/Q^\vee = \{1, \rho\}$$

is of order 2. The nontrivial element ρ acts by the reflection with respect to the *vertical* axis of symmetry of $\widetilde{D}$.

- To each Kac labeling $q \in \mathcal{K}(\widetilde{D})$ we associate a real form $\mathbf{G}_q = (G, \sigma_q)$ of $\mathbf{G}$ with

$$\sigma_q = a_q \circ \sigma_c, \quad a_q = \widehat{q}(-1) \in \mathbf{T}^{\mathrm{ad}}(\mathbb{R})_2,$$

and we have a canonical bijection

$$(*) \quad \mathcal{K}(\widetilde{D}, q)/F \xrightarrow{\sim} H^1(\mathbb{R}, \mathbf{G}_q), \quad p \mapsto [(\widehat{p} - \widehat{q})(-1)].$$

- In this case we have both a *congruence* coming from $\mathsf{X}^*(T)/Q$ in the definition of $\mathcal{K}(\widetilde{D}, q)$, and an *action of the fundamental group* $F = \mathsf{X}_*(T)/Q^\vee$.

Example: $\ell = 6$. Even orbits: five:

$$\begin{array}{ccccc} 1 & & & & 1 \\ & 0 & 0 & 0 & \\ 0 & & & & 0 \end{array}$$

$$\begin{array}{ccccc} 0 & & & & 0 \\ & 0 & 0 & 0 & \\ 1 & & & & 1 \end{array}$$

$$\begin{array}{ccccc} 2 & & & & 0 \\ & 0 & 0 & 0 & \\ 0 & & & & 0 \end{array} \qquad \begin{array}{ccccc} 0 & & & & 2 \\ & 0 & 0 & 0 & \\ 0 & & & & 0 \end{array}$$

$$\begin{array}{ccccc} 0 & & & & 0 \\ & 0 & 0 & 0 & \\ 2 & & & & 0 \end{array} \qquad \begin{array}{ccccc} 0 & & & & 0 \\ & 0 & 0 & 0 & \\ 0 & & & & 2 \end{array}$$

$$\begin{array}{ccccc} 0 & & & & 0 \\ & 1 & 0 & 0 & \\ 0 & & & & 0 \end{array} \qquad \begin{array}{ccccc} 0 & & & & 0 \\ & 0 & 0 & 1 & \\ 0 & & & & 0 \end{array}$$

Example: $\ell = 6$ (cont.) Odd orbits: three:

$$\begin{matrix}1&&&&0\\&0&0&0&\\1&&&&0\end{matrix}\qquad\qquad\begin{matrix}0&&&&1\\&0&0&0&\\0&&&&1\end{matrix}$$

$$\begin{matrix}1&&&&0\\&0&0&0&\\0&&&&1\end{matrix}\qquad\qquad\begin{matrix}0&&&&1\\&0&0&0&\\1&&&&0\end{matrix}$$

$$\begin{matrix}0&&&&0\\&0&1&0&\\0&&&&0\end{matrix}$$

- If q is *even*, then $\#H^1(\mathbb{R}, \mathbf{G}_q) = 5$, and if q is *odd*, then $\#H^1(\mathbb{R}, \mathbf{G}_q) = 3$.
- Formula $(*)$ gives explicit cocycles: $\quad p \mapsto [(\widehat{p} - \widehat{q})(-1)]$.

- For a half-spin group we have both a congruence and a finite group action, because our half-spin group is neither adjoint nor simply connected.
- If we wish to compute $H^1(\mathbb{R}, \mathbf{G}_q^{\mathrm{ad}})$ for the corresponding *adjoint* group $\mathbf{G}_q^{\mathrm{ad}}$, then we have no congruences, but an action of the fundamental group

$$\pi_1(G^{\mathrm{ad}}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

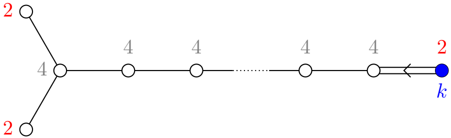
of order 4. We obtain 6 orbits (for $\ell = 6$). They correspond to the real forms

$$\mathrm{PGO}(12),\ \mathrm{PGO}(10,2),\ \mathrm{PGO}(8,4),\ \mathrm{PGO}(6,6),\ \mathrm{PGO}^*(12),\ \mathrm{PGO}^*(12).$$

- If we wish to compute $H^1(\mathbb{R}, \mathbf{G}^{\mathrm{sc}})$ for the corresponding *simply connected* group $\mathbf{G}_q^{\mathrm{sc}}$, then $\pi_1(G^{\mathrm{sc}}) = \{1\}$, so we have no group action, but we have *two* congruences modulo 2.

- In the case when our group is an *outer* form of a compact group, we should consider *twisted affine* Dynkin diagrams as in the note of Kac and in the books by Onishchik and Vinberg. For example:

${}^2\mathsf{A}_{2k-1}$



- Here, when G is *adjoint*, the fundamental group of G acts on $\widetilde{D}$ by the reflection with respect to the horizontal symmetry axis (the only nontrivial automorphism of $\widetilde{D}$).
- When G is *simply connected*, $p \in \mathcal{K}(\widetilde{D}, q)$ if and only if $p_k = q_k$, where k is the blue vertex.
- The case of a *semisimple* group is similar.

Idea of proof (the idea is due to Onishchik and Vinberg).

Assume that $\mathbf{G} = \mathbf{G}$ is a *compact* connected (reductive) $\mathbb{R}$-group, $\mathbf{T} \subset \mathbf{G}$ is a maximal torus.

Theorem (Borel and Serre 1964)

The inclusion $\mathbf{T}(\mathbb{R})_2 \hookrightarrow Z^1(\mathbb{R}, \mathbf{G})$ *induces a bijection* $\mathbf{T}(\mathbb{R})_2/W \xrightarrow{\sim} H^1(\mathbb{R}, \mathbf{G})$, *where* W *is the Weyl group.*

- Let $\mathbf{T}$ be a compact $\mathbb{R}$-torus. The cocharacter group $X_* = \mathsf{X}_*(T)$ naturally embeds into $\mathfrak{t} = \mathrm{Lie}\, T$. We have a surjective homomorphism

$$e\colon \mathfrak{t}^{\mathbb{R}} \to \mathbf{T}(\mathbb{R}), \quad x \mapsto \exp 2\pi x$$

with kernel $\boldsymbol{i}X_*$, where $\mathfrak{t}^{\mathbb{R}} = \mathrm{Lie}\, \mathbf{T}$. We identify $\mathbf{T}(\mathbb{R}) = \mathfrak{t}^{\mathbb{R}}/\boldsymbol{i}X_*$.

- We assume that $\mathbf{G} = \mathbf{G}^{\mathrm{ad}}$ is simple and *adjoint*. Instead of computing $\mathbf{T}^{\mathrm{ad}}(\mathbb{R})_2/W$, we compute

$$\begin{aligned}\mathbf{T}^{\mathrm{ad}}(\mathbb{R})/W &= (\mathfrak{t}^{\mathbb{R}}/\boldsymbol{i}P^\vee)/W \\ &= \mathfrak{t}^{\mathbb{R}}/(\boldsymbol{i}P^\vee \rtimes W) \\ &= (\mathfrak{t}^{\mathbb{R}}/\widetilde{W})/(\boldsymbol{i}P^\vee/\boldsymbol{i}Q^\vee),\end{aligned}$$

  where $\mathfrak{t}^{\mathbb{R}} = \mathrm{Lie}\,\mathbf{T}^{\mathrm{ad}}$, $P^\vee = \mathsf{X}_*(T^{\mathrm{ad}})$, $Q^\vee = \mathsf{X}_*(T^{\mathrm{sc}})$, $\boldsymbol{i}^2 = -1$, $\boldsymbol{i}Q^\vee \subseteq \boldsymbol{i}P^\vee \subset \mathfrak{t}^{\mathbb{R}}$, $\widetilde{W} = \boldsymbol{i}Q^\vee \rtimes W$.

- The *affine Weyl group* $\widetilde{W} := \boldsymbol{i}Q^\vee \rtimes W \subseteq \boldsymbol{i}P^\vee \rtimes W$ is a reflection group acting on $\mathfrak{t}^{\mathbb{R}}$, and it has a fundamental domain Δ described in Bourbaki's book *Lie-456*. It is a simplex, which in suitable coordinates is given by the equation and inequalities

$$(*) \quad m_0q_0 + m_1q_1 + \cdots + m_\ell q_\ell = 2, \quad q_i \in \mathbb{R},\ q_i \geq 0.$$

- We obtain $\mathfrak{t}^{\mathbb{R}}/\widetilde{W} = \Delta$ and

$$\begin{aligned}\mathbf{T}^{\mathrm{ad}}(\mathbb{R})/W &= \mathfrak{t}^{\mathbb{R}}/(\boldsymbol{i}P^{\vee} \rtimes W)\\ &= (\mathfrak{t}^{\mathbb{R}}/(\boldsymbol{i}Q^{\vee} \rtimes W))/(\boldsymbol{i}P^{\vee}/\boldsymbol{i}Q^{\vee})\\ &= (\mathfrak{t}^{\mathbb{R}}/\widetilde{W})/(\boldsymbol{i}P^{\vee}/\boldsymbol{i}Q^{\vee})\\ &= \Delta/F,\end{aligned}$$

where the fundamental group $F = \boldsymbol{i}P^{\vee}/\boldsymbol{i}Q^{\vee} = \pi_1(G^{\mathrm{ad}})$ acts on Δ by automorphisms of $\widetilde{D}$.

- We obtain $\mathbf{T}^{\mathrm{ad}}(\mathbb{R})_2/W$ instead of $\mathbf{T}(\mathbb{R})/W$ if we take *integer* values $q_i \in \mathbb{Z}_{\geq 0}$ such that $(*)$ holds, that is, if we take $q \in \mathcal{K}(\widetilde{D})$. Thus

$$H^1(\mathbb{R}, \mathbf{G}^{\mathrm{ad}}) \cong \mathbf{T}^{\mathrm{ad}}(\mathbb{R})_2/W \cong \mathcal{K}(\widetilde{D})/F.$$

If we take the *simply connected* group $\mathbf{G}^{\rm sc}$ instead of the *adjoint* group $\mathbf{G}^{\rm ad}$, then

$$\mathbf{T}^{\rm sc}(\mathbb{R})/W = (\mathfrak{t}^{\mathbb{R}}/\boldsymbol{i}Q^\vee)/W = \mathfrak{t}^{\mathbb{R}}/(\boldsymbol{i}Q^\vee \rtimes W) = \mathfrak{t}^{\mathbb{R}}/\widetilde{W} = \Delta,$$

no need to take quotient by $\pi_1(G^{\rm sc}) = \{1\}$. However, to guarantee that our point $p = (p_i)$ comes from $\mathbf{T}^{\rm sc}(\mathbb{R})_2$, we need to take $p_i \in \mathbb{Z}_{\geq 0}$ and to add *congruences* on (p_i). We obtain

$$H^1(\mathbb{R}, \mathbf{G}^{\rm sc}) \cong \mathbf{T}^{\rm sc}(\mathbb{R})_2/W \cong \mathcal{K}(\widetilde{D}, q).$$

Our results

- For a (connected) semisimple $\mathbb{R}$-group $\mathbf{G}$ we give $H^1(\mathbb{R}, \mathbf{G})$ with *explicit cocycles*.

- For a *normal* homomorphism of semisimple $\mathbb{R}$-groups $\varphi\colon \mathbf{G} \to \mathbf{G}'$ we compute

$$\varphi_*\colon H^1(\mathbb{R}, \mathbf{G}) \to H^1(\mathbb{R}, \mathbf{G}').$$

A homomorphism $\varphi\colon \mathbf{G} \to \mathbf{G}'$ is called *normal* if $\varphi(\mathbf{G})$ is normal in $\mathbf{G}'$. For example, any surjective homomorphism is normal.

- For a short exact sequence

$$1 \to \mathbf{A} \to \mathbf{G} \to \mathbf{G}' \to 1$$

with central finite $\mathbb{R}$-subgroup $\mathbf{A}$ of a semisimple $\mathbb{R}$-group $\mathbf{G}$, we compute a 7-term exact sequence

$$\mathbf{A}(\mathbb{R}) \to \pi_0\mathbf{G}(\mathbb{R}) \to \pi_0\mathbf{G}'(\mathbb{R}) \xrightarrow{\delta} H^1(\mathbb{R},\mathbf{A})$$
$$\to H^1(\mathbb{R},\mathbf{G}) \to H^1(\mathbb{R},\mathbf{G}') \xrightarrow{\Delta} H^2(\mathbb{R},\mathbf{A}).$$

- For $a \in Z^1(\mathbb{R},\mathbf{G})$ we compute the *twisting bijection*

$$\tau_a\colon H^1(\mathbb{R}, {}_a\mathbf{G}) \to H^1(\mathbb{R},\mathbf{G})$$

sending $[1] \in H^1(\mathbb{R}, {}_a\mathbf{G})$ to $[a] \in H^1(\mathbb{R},\mathbf{G})$.

Thank you!

Merci beaucoup!