FAST DCT-BASED ALGORITHMS FOR SIGNAL CONVOLUTION AND TRANSLATION

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ABSTRACT

Fast DCT-based algorithms are presented for signal convolution and translation that are virtually free of boundary effects, characteristic for corresponding DFT-based fast algorithms. The properties of DCT relevant to the subject are summarized and compared to the corresponding properties of DFT.

Index Terms— DCT, convolution, translation

1. INTRODUCTION

DCT is a very important signal transform useful in many applications. Most known is the use of DCT for signal and image compression. It was also shown ([1], [2]) that DCT can be used for boundary effects free signal resampling and convolution. In this paper we derive properties of DCT relevant to this application and suggest new efficient algorithms for signal translation and convolution that provide boundary effects free substitution for commonly used corresponding FFT based algorithms.

2. FAST DCT-BASED ALGORITHMS FOR SIGNAL CONVOLUTION AND TRANSLATION

2.1. Definitions and known algorithms

Known DCT based algorithms for signal boundary effects free translation and convolution involve Discrete Cosine (DCT) and Discrete Cosine-Sine (DcST) Transforms defined as follows:

$$\alpha_r^{(DCT)} = \mathbf{DCT}\{a_k\} \triangleq \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} a_k \cos\left(\pi \frac{(k+1/2)\,r}{N}\right),$$

$$a_k = \mathbf{IDCT}\{\alpha_r^{(DCT)}\} \triangleq \frac{1}{\sqrt{2N}} \left\{\alpha_0^{(DCT)} + 2\sum_{r=1}^{N-1} \alpha_r^{(DCT)} \cos\left(\pi \frac{(k+1/2)\,r}{N}\right)\right\},$$

$$\alpha_r^{(DST)} = \mathbf{DcST}\left\{a_k\right\} \triangleq \sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} a_k \sin\left(\pi\frac{(k+1/2)\,r}{N}\right),$$

$$a_k = \mathbf{IDcST}\left\{\alpha_r^{(DST)}\right\} \triangleq \frac{1}{\sqrt{2N}}\left\{(-1)^k\alpha_N^{(DST)} + 2\sum_{r=1}^{N-1}\alpha_r^{(DST)}\sin\left(\pi\frac{(k+1/2)\,r}{N}\right)\right\}.$$

The cyclic convolution is defined in the following way ($k = 0, \ldots, N-1$):

$$\tilde{a}_k = a_k * h_k \triangleq \sum_{n=0}^{N-1} a_n h_{(k-n) \bmod N}.$$

The convolution property in the DCT domain is more complex than its DFT counterpart. It was investigated extensively [3], [4], [5] and the convolution algorithm was derived [1], [2]:

$$\tilde{a}_k = \sqrt{2N}\mathbf{IDCT}\left\{\alpha_r^{(DCT)}\eta_r^{re}\right\} + \sqrt{2N}\mathbf{IDcST}\left\{\alpha_r^{(DCT)}\eta_r^{im}\right\},$$

where $\alpha_r^{(DCT)}$ denotes the DCT spectrum of the original signal a_k and η_r denotes the DFT spectrum of the zero-padded interpolation kernel h_k:

$$\eta_r = \mathbf{DFT}\left\{h_k^{(z)}\right\},$$

where

$$h_k^{(z)} = \begin{cases} h_k & \text{if } k = 0, \ldots, N-1, \\ 0 & \text{if } k = N, \ldots, 2N-1. \end{cases}$$

Two important special cases of the convolution algorithm are the DCT-based translation algorithm [1], [2]:

$$\tilde{a}_{k-p} = \mathbf{IDCT}\left\{\alpha_r^{(DCT)}\cos\left(\frac{\pi pr}{N}\right)\right\} + \mathbf{IDcST}\left\{\alpha_r^{(DCT)}\sin\left(\frac{\pi pr}{N}\right)\right\},$$

where p is a fractional shift factor, and the DCT-based "zero-padding" scaling algorithm [6], [7], [8], [9]:

$$\tilde{a}_k^{(L)} = \sqrt{2N}\mathbf{IDCT}\left\{\mathbf{ZP}\left[\alpha_r^{(DCT)}\right]\right\},$$

where L is an integer scaling factor and $\mathbf{ZP}$ denotes the zero-padding operator (to length LN).

The interpolation kernels of the translation algorithm:

$$\tilde{a}_{k-p} = \sum_{k_0=0}^{N-1} a_{k_0}\left\{\mathrm{sincd}\left[2N-1; 2N; k+k_0-p+1\right] + \mathrm{sincd}\left[2N-1; 2N; k-k_0-p\right]\right\}$$

and of the scaling algorithm:

$$\tilde{a}_k^{(L)} = \sum_{k_0=0}^{N-1} a_{k_0}\left\{\mathrm{sincd}\left[2N-1; 2LN; k+Lk_0+(L+1)/2\right] + \mathrm{sincd}\left[2N-1; 2LN; k-Lk_0-(L-1)/2\right]\right\}$$

are both sum of two shifted discrete sinc functions sincd. These "double" kernels have smaller side lobes compared to the "single" sincd kernel involved in the DFT interpolation (Fig. 1), which proved to be perfect interpolation kernel for sampled data [10].

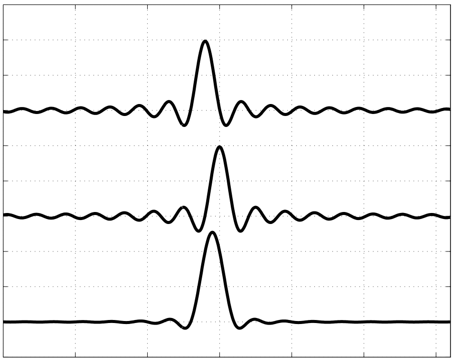

Fig. 1. Comparison of the sincd interpolation kernels used for the translation in the DCT domain. Top and central graphs: "single" kernels. Bottom graph: "double" kernel.

2.2. New algorithms

The DcST of the signal can be found as a flipped DCT of a signal with odd samples with inverted sign [11]:

$$[\mathbf{DcST}\{a_k\}]_r = \left[\mathbf{DCT}\left\{(-1)^k a_k\right\}\right]_{N-r}.$$

(The frequency index r runs from 1 to N for DcST and from 0 to $N-1$ for DCT.)

The method described above can be extended to computing of IDcST through IDCT. The IDcST is obtained from the IDCT of a flipped DCT of a signal with odd samples with inverted sign:

$$[\mathbf{IDcST}\{\alpha_r\}]_k = (-1)^k\,[\mathbf{IDCT}\{\alpha_{N-r}\}]_k. \tag{2.1}$$

On this base we can suggest a modified algorithm that involves two transforms only (DCT and IDCT):

$$\tilde{a}_k = \sqrt{2N}\mathbf{IDCT}\left\{\alpha_r^{(DCT)}\eta_r^{re} + (-1)^k\alpha_{N-r}^{(DCT)}\eta_{N-r}^{im}\right\},$$

where η_r is the DFT spectrum of the $2N$-periodic kernel $h_k^{(z)}$. The flow chart of the convolution DCT-based algorithm is shown in the Fig. 2.

The known translation algorithm [2] with complexity $\mathcal{O}(3N\log N)$ consists of three transforms (DCT, IDCT and IDcST). Using the Eq. 2.1, we derive a modified algorithm with complexity $\mathcal{O}(2N\log N)$ that consists of two transforms only (DCT and IDCT):

$$\tilde{a}_{k-p} = \mathbf{IDCT}\left\{\alpha_r^{(DCT)}\cos\frac{\pi pr}{N} + (-1)^k\alpha_{N-r}^{(DCT)}\sin\frac{\pi p(N-r)}{N}\right\}.$$

The IDCT is computed twice (for even and odd k) for $N/2$ output samples each time.

The flow chart of the DCT/IDCT-based translation algorithm is shown in the Fig. 3.

The shift and convolution properties of the DCT transform are summarized and compared to the corresponding properties of the DFT in the Table 1 along with the DFT-based and DCT-based fast signal convolution and translation algorithms.

3. CONCLUSIONS

Boundary effects free algorithms for fast DCT-based convolution and translation are presented. A list of relevant properties of DCT and the corresponding properties of DFT for reference and comparison is provided. The similarity of the structure of interpolation kernels for the translation and scaling is demonstrated.

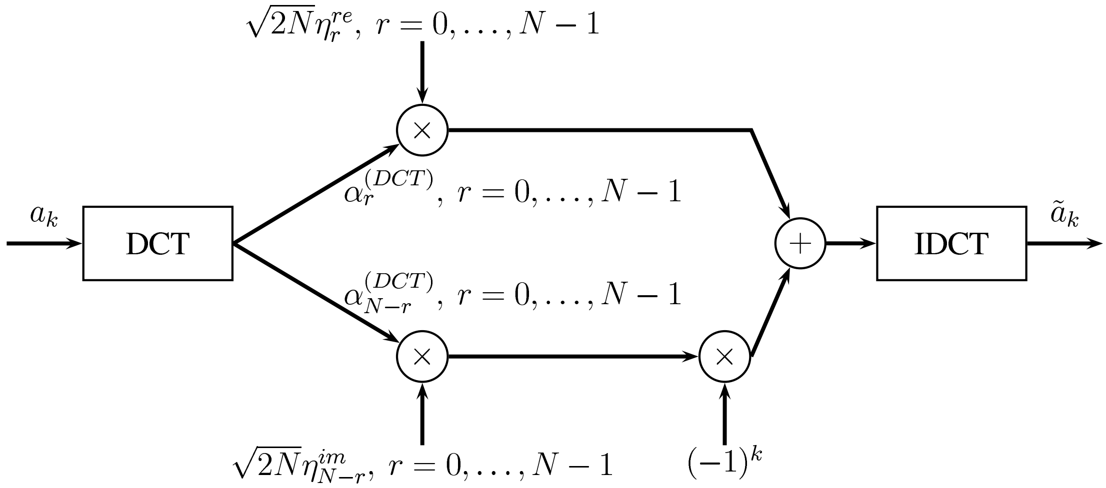

Fig. 2. Flow chart of the convolution through IDCT algorithm.

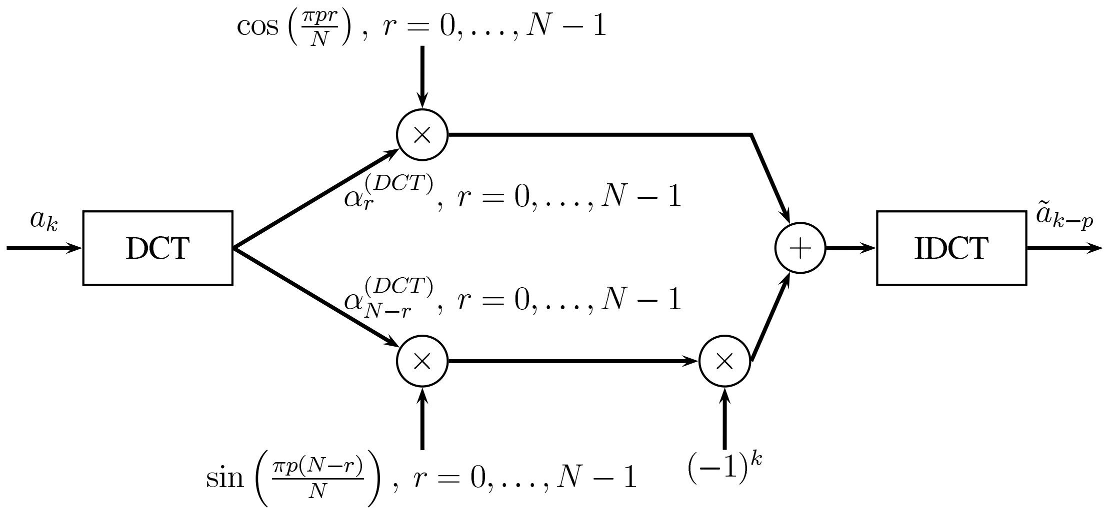

Fig. 3. Flow chart of the translation through IDCT algorithm.

Table 1. Summary of definitions, properties and algorithms in DFT and DCT domains.

	DFT	DCT
Definitions of transforms	$\alpha_r = \mathbf{DFT}_N\left\{a_k\right\} \triangleq$ $\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} a_k \exp\left(i2\pi\frac{kr}{N}\right)$ $a_k = \mathbf{IDFT}_N\left\{\alpha_r\right\} \triangleq$ $\frac{1}{\sqrt{N}}\sum_{r=0}^{N-1} \alpha_r \exp\left(-i2\pi\frac{kr}{N}\right)$	$\alpha_r^{(DCT)} = \mathbf{DCT}_N\left\{a_k\right\} \triangleq$ $\sqrt{\frac{2}{N}}\sum_{k=0}^{N-1} a_k \cos\left(\pi\frac{(k+1/2)r}{N}\right)$ $a_k = \mathbf{IDCT}_N\left\{\alpha_r^{(DCT)}\right\} \triangleq$ $\frac{1}{\sqrt{2N}}\left\{\alpha_0^{(DCT)} + 2\sum_{r=1}^{N-1}\alpha_r^{(DCT)}\cos\left(\pi\frac{(k+1/2)r}{N}\right)\right\}$
Shift theorem	$\mathbf{DFT}_N\left\{\tilde{a}_{k-p}\right\} =$ $\mathbf{DFT}_N\left\{a_k\right\}\exp\left[i\frac{2\pi pr}{N}\right]$	$\mathbf{DCT}_N\left\{\tilde{a}_{k-p}\right\} =$ $\mathbf{DCT}_N\left\{a_k\right\}\cos\left[\frac{\pi pr}{N}\right]$ $-\left[\mathbf{DCT}_N\left\{(-1)^k a_k\right\}\right]_{N-r}\sin\left[\frac{\pi pr}{N}\right]$
Translation algorithm	$\tilde{a}_{k-p} = \mathbf{IDFT}_N\left\{\alpha_r\exp\left(i\frac{2\pi pr}{N}\right)\right\}$	$\tilde{a}_{k-p} = \mathbf{IDCT}_N\left\{\alpha_r^{(DCT)}\cos\left(\frac{\pi pr}{N}\right)\right\}$ $+(-1)^k\left[\mathbf{IDCT}_N\left\{\alpha_{N-r}^{(DCT)}\sin\left(\frac{\pi p(N-r)}{N}\right)\right\}\right]_k$
Convolution theorem	$\mathbf{DFT}_N\left\{\tilde{a}_k = a_k * h_k\right\} =$ $\sqrt{N}\mathbf{DFT}_N\left\{a_k\right\}\mathbf{DFT}_N\left\{h_k\right\},$ h_k is convolution kernel, $*$ denotes convolution	$\mathbf{DCT}_N\left\{\tilde{a}_k = a_k * h_k\right\} =$ $\sqrt{2N}\left\{\mathbf{DCT}_N\left\{a_k\right\}\eta_r^{re}\right.$ $\left.-\left[\mathbf{DCT}_N\left\{(-1)^k a_k\right\}\right]_{N-r}\eta_r^{im}\right\},$ $\eta_r = \mathbf{DFT}_{2N}\left\{h_k^{(z)}\right\},$ $h_k^{(z)} = \{h_k, \text{if } k = 0,\ldots,N-1;\ 0, \text{if } k = N,\ldots,2N-1\}$
Convolution algorithm	$\tilde{a}_k = a_k * h_k =$ $\sqrt{N}\mathbf{IDFT}_N\left\{\mathbf{DFT}_N\left\{a_k\right\}\mathbf{DFT}_N\left\{h_k\right\}\right\}$	$\tilde{a}_k = a_k * h_k = \sqrt{2N}\left\{\mathbf{IDCT}_N\left\{\alpha_r^{(DCT)}\eta_r^{re}\right\}\right.$ $\left.+(-1)^k\left[\mathbf{IDCT}_N\left\{\alpha_{N-r}^{(DCT)}\eta_{N-r}^{im}\right\}\right]_k\right\}$

4. REFERENCES

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