Signal Reconstruction from FROG Measurements

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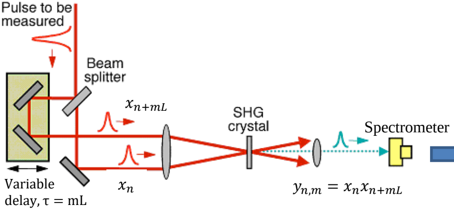
- But we cannot recover a 1D signal from its power spectrum
- The Frequency-Resolved Optical Gating techniques were introduced in 1991 by Daniel J. Kane and Rick Trebino
- Nowadays, FROG is a commonly–used method for full characterization of ultra-short optical pulses due to its simplicity and good experimental performance

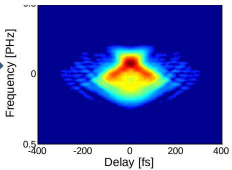
The FROG technique

The FROG trace is the Fourier magnitude of product of the signal with a translated version of itself, for several different translations.

$$|\hat{y}_{k,m}|^2 = \left| \sum_{n=0}^{N-1} x_n x_{n+mL} e^{-2\pi i k n/N} \right|^2, \quad m = 0, \ldots, \lceil N/L \rceil - 1$$

It is a **quartic** intensity map $\mathbb{C}^N \mapsto \mathbb{R}^{N \times N/L}$.





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- **Goal**: Conditions on the number of samples required to determine a signal uniquely, up to trivial ambiguities, from its FROG trace

STFT phase retrieval and XFROG

- In some cases, one can use a known reference pulse g:

$$|\hat{y}_{k,m}|^2 = \left| \sum_{n=0}^{N-1} x_n g_{n+mL} e^{-2\pi i k n / N} \right|^2, \quad m = 0, \ldots, \lceil N/L \rceil - 1$$

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- Data–driven initialization based on approximating xx^* with theoretical analysis

FROG symmetries

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Claim

The following signals have the same FROG trace as $x \in \mathbb{C}^N$:

1. *the rotated signal $xe^{i\psi}$ for some $\psi \in \mathbb{R}$;*
2. *the translated signal $x_n^\ell = x_{n-\ell}$ for some $\ell \in \mathbb{Z}$;*
3. *the reflected signal $\tilde{x}_n = \overline{x_{-n}}$.*

FROG symmetries for bandlimited signals

- The translation symmetry implies that signal with Fourier transfrom $\hat{x}_k e^{-2\pi i \ell k/N}$ for some $\ell \in \mathbb{Z}$ has the same FROG trace as x

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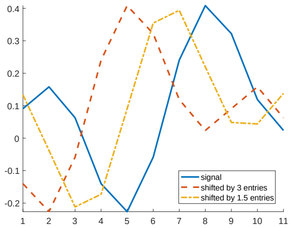
- Example for signal with Fourier transform $[1, i, -i, 0, 0, 0, 0, 0, 0, i, -i]$:



Uniqueness

Theorem

Let $x \in \mathbb{C}^N$ be a B–bandlimited signal for some $B \leq N/2$.

If $L \leq N/4$, then almost all signals are determined uniquely from their FROG trace, modulo the trivial ambiguities, from $3B$ measurements.

If in addition we have access to the signal's power spectrum and $L \leq N/3$, then $2B$ measurements are sufficient.

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- Random phase retrieval requires $4N-4$ measurements (to recover all signals)

Few words on the proof

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- One can formulate the FROG measurements as

$$\hat{y}_{k,m} = \frac{1}{N} \sum_{\ell=0}^{N-1} \hat{x}_\ell \hat{x}_{k-\ell} \omega^{\ell m}, \quad \omega = e^{2\pi i L/N},$$

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- Because of the bandlimited assumption, one can form a pyramid structure of $\hat{x}_\ell \hat{x}_{k-\ell}$

$$\begin{gathered}
\widehat{x}_0^2, 0, \dots, 0 \\
\widehat{x}_0\widehat{x}_1, \widehat{x}_1\widehat{x}_0, 0, \dots, 0 \\
\widehat{x}_0\widehat{x}_2, \widehat{x}_1^2, \widehat{x}_2\widehat{x}_0, \dots, 0 \dots \\
\vdots \\
\widehat{x}_{N/2-1}\widehat{x}_0, \widehat{x}_{N/2-2}\widehat{x}_1, \dots, \widehat{x}_{N/2-1}\widehat{x}_0, 0, \dots, 0 \\
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- Because of the two continuous symmetries, we can fix $\widehat{x}_0$ and $\widehat{x}_1$ to be real and $\Im\widehat{x}_2 \geq 0$
- The rest of the coefficients are determined recursively. Given the first $(k-1)$ Fourier coefficients, we get a quadratic system for the kth coefficient

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- When $L = 1$, the two signals are determined uniquely, up to symmetries [B., Sidorenko and Eldar, 2017].
- How many measurements do we need to determine a generic pair of signals from their blind FROG trace?

Additional open questions

- Many FROG non-linearities to consider. For example, a setup for characterization of Attosecond pulses, called FROG for Complete Reconstruction of Attosecond Bursts (FROG CRAB), is modeled as:

$$|\hat{y}_{k,m}|^2 = \left| \sum_{n=0}^{N-1} u_n e^{iv_{n+mL}} e^{-2\pi ikn/N} \right|^2, \quad m = 0, \dots, \lceil N/L \rceil - 1$$

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- Analysis of FROG algorithms. Currently, the most popular algorithm is the Principal Component Generalized Projections which alternates between the known intensities and the form of the non-linear interaction.

References

1. Trebino, R.. Frequency-resolved optical gating: the measurement of ultrashort laser pulses. Springer Science & Business Media, 2012.
2. Bendory, T., Eldar. Y. and Boumal N. "Non-convex phase retrieval from STFT measurements". To appear in IEEE Transactions on Information Theory, 2017.
3. Bendory, T., Sidorenko P. and Eldar Y. "On the uniqueness of FROG methods." IEEE Signal Processing Letters, vol. 24, issue 5, pp. 722-726, 2017.
4. Bendory, T., Edidin D., and Eldar Y. "On Signal Reconstruction from FROG Measurements." arXiv preprint arXiv:1706.08494, 2017.

New benchmark for crystallography problems

1

Benchmark problems for phase retrieval

Veit Elser and Ti-Yen Lan

***Abstract*—The hardest instances of phase retrieval arise in crystallography, where the signal is periodic and comprised of atomic distributions arranged uniformly in the unit cell of the crystal. We have constructed a graded set of benchmark problems for evaluating algorithms that perform this type of phase retrieval. A simple iterative algorithm was used to establish baseline runtimes that empirically grow exponentially in the sparsity of the signal autocorrelation. We also review the algorithms used by the leading software packages for crystallographic phase retrieval.**

***Index Terms*—phase retrieval, periodic signals, reconstruction algorithms**

of the problem. Not having to deal with complicating factors incidental to phase retrieval, *e.g.* space groups, may also be a contributing factor. In any case, this state of affairs is easily addressed by making available instances of phase retrieval[1] that (*i*) are seen as challenging by crystallographers and (*ii*) are presented with an eye toward accessibility for non-crystallographers. Below we describe the construction of a set of benchmark problems with these characteristics. In addition to providing a basis for comparing different algorithms, the graded difficulty of the instances will provide evidence of the complexity behavior of individual algorithms. We did not

Thanks for your attention!