# Convergence of an Inexact Majorization-Minimization Method for Solving a Class of Composite Optimization Problems

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**Abstract** We suggest a majorization-minimization method for solving nonconvex minimization problems. The method is based on minimizing at each iterate a properly constructed *consistent majorizer* of the objective function. We describe a variety of classes of functions for which such a construction is possible. We introduce an inexact variant of the method, in which only approximate minimization of the consistent majorizer is performed at each iteration. Both the exact and the inexact algorithms are shown to be descent methods whose accumulation points have a property which is stronger than standard stationarity. We give examples of cases in which the exact method can be applied. Finally, we show that the inexact method can be applied to a specific problem, called *sparse source localization*, by utilizing a fast optimization method on a smooth convex dual of its subproblems.

## 1 Introduction

In this chapter we consider the general optimization problem

$$\min\{F(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}, \tag{1}$$

where $F : \mathbb{R}^n \to (-\infty, \infty]$ is a proper, closed extended real-valued function satisfying that its domain $\mathrm{dom}(F) := \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) < \infty\}$ is a convex subset of $\mathbb{R}^n$. In addition, we assume that $F$ is *directionally differentiable*, that is, for any $\mathbf{x}, \mathbf{y} \in \mathrm{dom}(F)$, the directional derivative of $F$ at $\mathbf{x}$ in the direction $\mathbf{y} - \mathbf{x}$,

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$$F'(\mathbf{x};\mathbf{y}-\mathbf{x}) := \lim_{t\to 0^+} \frac{F(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))-F(\mathbf{x})}{t}$$

exists (finite or infinite). For the sake of simplicity of exposition, all the spaces are Euclidean $\mathbb{R}^n$ spaces with the endowed dot product, but all the results hold trivially for general Euclidean spaces. In this context, the gradient of a differentiable function $F:\mathbb{R}^n \to \mathbb{R}$, denoted by $\nabla F$, is the vector of all partial derivatives $\nabla F(\mathbf{x}) := \left(\frac{\partial F}{\partial x_1}(\mathbf{x}),\ldots,\frac{\partial F}{\partial x_n}(\mathbf{x})\right)^T$.

The optimization method that we suggest for solving (1) is based on the general *majorization-minimization (MM)* scheme. At each iteration, a *consistent majorizer* is computed around the current iterate, and the next iterate is an exact or an approximate minimizer of that majorizer. A consistent majorizer is an upper bound on $F$ that coincides with it up to first-order terms around a given point in its domain. Consistent majorizers and methods based on the MM-scheme have been extensively studied in the literature, see for example the book [13] as well as the review paper [12] and references therein for a variety of constructions of consistent majorizers.

A special focus in the literature is on the case where $F$ is given by a composition $F = \varphi \circ \mathbf{f} + g$, where $\mathbf{f}$ is a mapping comprising $m$ real-valued differentiable functions with Lipschitzian gradients, $\varphi$ is a support function of a nonnegative compact and convex subset of $\mathbb{R}^m$ and $g$ is a proper closed and convex function. An applicable method for various composite models is the *proximal Gauss-Newton* method (PGNM), also known as *prox-linear* method. Its general step is

$$x^{k+1} = \operatorname*{argmin}_{\mathbf{y}} \left\{ g(\mathbf{y}) + \varphi\left(\mathbf{f}(\mathbf{x}^k) + \mathbf{J}_{\mathbf{f}}(\mathbf{x}^k)(\mathbf{y}-\mathbf{x}^k)\right) + \frac{1}{2t}\|\mathbf{y}-\mathbf{x}^k\|^2 \right\},$$

for some parameter $t>0$, which depends on the smoothness parameters of $\nabla f_i$ and the global Lipschitz constant of $\varphi$, whose finiteness is guaranteed as $\varphi$ is a support function of a bounded set. The matrix $\mathbf{J}_{\mathbf{f}}(\mathbf{x})$ is the Jacobian of $\mathbf{f}$ at $\mathbf{x}$. For a convergence analysis of the method see for example [17]. The prox-linear method was further investigated and extended in the more recent works [9, 10, 11, 15]. We note that a special instance of the prox-linear method is the *proximal gradient* method aimed at solving the additive model $F = f + g$ where $f$ is differentiable and $g$ is proper closed and convex; see the references [5, 6, 18] for convergence analysis as well as extensions.

The exact version of the MM scheme that we consider can be seen as a generalization of the prox-linear method to a broader class of models. Our main goal is to establish convergence results that will hold for both the exact and inexact MM algorithms.

The chapter is organized as follows. In Section 2 we define explicitly the concept of a consistent majorizer of a function. We describe a variety of classes of functions for which consistent majorizers can be constructed. In Section 3 we introduce the concept of *strongly stationary* points of (1) with respect to a given consistent majorizer of $F$, and show that in the case of a nonconvex consistent majorizer, it might lead to a stronger condition than the usual stationarity/“no descent directions” property. The potential advantage of the new optimality condition is demonstrated on an example of minimizing a concave quadratic function over a box. In Section 4 we describe the MM method and its inexact variant, in which only approximate minimizers of the consistent majorizers are computed, and analyze their convergence properties. We also provide an implementable example demonstrating some practical advantages of the MM method over the gradient projection method. Finally, in Section 5 we study a class of problems consisting of minimizing the composition of a nondegenerate support function with a mapping comprising functions for which strongly convex majorizers are constructable. For this class, the inexact MM method is shown to be fully implementable, and its application on a specific problem which we call *the sparse source localization* is provided.

**Notations.** Vectors are written in lower case boldface letters, matrices in upper case boldface, scalars and sets in italic. We denote $\mathbf{e} = (1,1,\ldots,1)^T \in \mathbb{R}^n$, and for a given vector $\mathbf{d} \in \mathbb{R}^n$ the matrix $\mathrm{diag}(\mathbf{d})$ is the diagonal matrix whose $i$th diagonal entry is $d_i$ for $i = 1,\ldots,n$. For two symmetric matrices $\mathbf{A},\mathbf{B}$ we write $\mathbf{A} \succeq \mathbf{B}$ ($\mathbf{A} \succ \mathbf{B}$) if $\mathbf{A}-\mathbf{B}$ is positive semidefinite (positive definite). The notation $\lambda_{\max}(\mathbf{A})$ corresponds to the maximal eigenvalue of the matrix $\mathbf{A}$. The set $\Delta_n$ is the unit simplex in $\mathbb{R}^n$, namely, $\Delta_n := \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, \mathbf{x} \geq \mathbf{0}\}$. The norm notation $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$, i.e., $\|\mathbf{x}\| := \|\mathbf{x}\|_2 \equiv \sqrt{\langle \mathbf{x},\mathbf{x}\rangle}$. For a given closed convex set $B \subseteq \mathbb{R}^n$ the orthogonal projection on $B$ is defined by $P_B(\mathbf{x}) := \underset{\mathbf{y}\in B}{\mathrm{argmin}}\,\|\mathbf{y}-\mathbf{x}\|$.

# 2 Consistent Majorizers

## *2.1 Directionally Differentiable Functions*

We consider the minimization problem

$$\min\{F(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}, \tag{2}$$

where $F : \mathbb{R}^n \to (-\infty,\infty]$ is a proper, closed extended real-valued function which is directionally differentiable, a simple notion that is defined below.

**Definition 1 (directionally differentiable functions).** A function $F : \mathbb{R}^n \to (-\infty,\infty]$ is called **directionally differentiable** if it satisfies the following two properties:

- $\mathrm{dom}(F)$ is a convex set.
- For any $\mathbf{x},\mathbf{y} \in \mathrm{dom}(F)$, the directional derivative $F'(\mathbf{x};\mathbf{y}-\mathbf{x})$ exists (finite or infinite).

*Example 1 (additive composite model).* Suppose that $F = f + g$, where $f : \mathbb{R}^n \to \mathbb{R}$ is anywhere differentiable and $g : \mathbb{R}^n \to (-\infty,\infty]$ is convex. The function $F$ is indeed directionally differentiable since $\mathrm{dom}(F) = \mathrm{dom}(g)$ is convex and for any $\mathbf{x},\mathbf{y} \in \mathrm{dom}(F)$, by the convexity of $g$, $g'(\mathbf{x};\mathbf{y}-\mathbf{x})$ exists (finite or infinite) and thus also $F'(\mathbf{x};\mathbf{y}-\mathbf{x})$ exists and is given by

$$F'(\mathbf{x};\mathbf{y}-\mathbf{x}) = \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + g'(\mathbf{x};\mathbf{y}-\mathbf{x}).$$

*Example 2 (dc functions).* Let $F = f - g$ where $f : \mathbb{R}^n \to (-\infty,\infty]$ and $g : \mathbb{R}^n \to \mathbb{R}$ are convex. Then $\mathrm{dom}(F) = \mathrm{dom}(f)$ and by the convexity of $f$ and $g$ both possess directional derivatives at all feasible directions, and $g'(\mathbf{x};\mathbf{y}-\mathbf{x})$ is finite for all $\mathbf{y},\mathbf{x} \in \mathrm{dom}(F)$. In particular, for any $\mathbf{x},\mathbf{y} \in \mathrm{dom}(F)$:

$$F'(\mathbf{x};\mathbf{y}-\mathbf{x}) = f'(\mathbf{x};\mathbf{y}-\mathbf{x}) - g'(\mathbf{x};\mathbf{y}-\mathbf{x}).$$

## 2.2 Definition

A basic ingredient in the analysis in this paper is the concept of a *consistent majorizer*.

**Definition 2 (consistent majorizer).** Given a directionally differentiable function $F : \mathbb{R}^n \to (-\infty,\infty]$, a function $h : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty,\infty]$ is called a **consistent majorizer function of** $F$ if the following properties hold:

(A) $h(\mathbf{y},\mathbf{x}) \geq F(\mathbf{y})$ for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$.
(B) $h(\mathbf{y},\mathbf{y}) = F(\mathbf{y})$ for any $\mathbf{y} \in \mathrm{dom}(F)$.
(C) For any $\mathbf{x} \in \mathrm{dom}(F)$, the function $h_{\mathbf{x}}(\mathbf{y}) := h(\mathbf{y},\mathbf{x})$ is directionally differentiable and satisfies that $\mathrm{dom}(h_{\mathbf{x}}) = \mathrm{dom}(F)$ and

$$h'_{\mathbf{x}}(\mathbf{x};\mathbf{z}-\mathbf{x}) = F'(\mathbf{x};\mathbf{z}-\mathbf{x}) \text{ for any } \mathbf{z} \in \mathrm{dom}(F).$$

(D) For any $\mathbf{y} \in \mathrm{dom}(F)$ the function $\mathbf{x} \mapsto -h(\mathbf{y},\mathbf{x})$ is closed[1].

It is simple to show that the sum of two consistent majorizers is also a consistent majorizer.

**Theorem 1.** *Let $F_1$ and $F_2$ be two directionally differentiable functions where at least one of them, say $F_i$, satisfies $F_i'(\mathbf{x};\mathbf{y}-\mathbf{x}) \in \mathbb{R}$ for all $\mathbf{x},\mathbf{y} \in \mathrm{dom}(F_i)$. Suppose that $h_1,h_2$ are consistent majorizers of $F_1$ and $F_2$, respectively. Then $h_1+h_2$ is a consistent majorizer of $F_1+F_2$.*

*Proof.* Follows directly by the definition of consistent majorizers and the facts that (i) the sum of two closed functions is a closed function and (ii) the directional derivative is additive in the sense that $(h_1+h_2)'(\mathbf{x};\mathbf{d}) = h_1'(\mathbf{x};\mathbf{d}) + h_2'(\mathbf{x};\mathbf{d})$ for any $\mathbf{x},\mathbf{d} \in \mathbb{R}^n$ for which the relevant expressions are well-defined. ☐

## 2.3 Examples

Below are several examples of consistent majorizers in several important settings.

[1] which is the same as saying that the function $\mathbf{x} \mapsto h(\mathbf{y},\mathbf{x})$ is upper semicontinuous.

*Example 3 (dd).* If $F : \mathbb{R}^n \to (-\infty, \infty]$ is a directionally differentiable function, then obviously, the function $h(\mathbf{y}, \mathbf{x}) = F(\mathbf{y}) + \frac{\eta}{2}\|\mathbf{y} - \mathbf{x}\|^2$ is a consistent majorizer of $F$ for any $\eta \geq 0$.

*Example 4 (concave differentiable).* Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ which is concave and continuously differentiable. By the concavity of $f$, it follows that $f(\mathbf{y}) \leq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and therefore the function

$$h(\mathbf{y}, \mathbf{x}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$$

is a majorizer of $f$, meaning that property (A) holds. Property (B) holds since for any $\mathbf{y} \in \mathbb{R}^n$, $h(\mathbf{y}, \mathbf{y}) = f(\mathbf{y})$. The function $h_{\mathbf{x}}(\mathbf{y}) \equiv h(\mathbf{y}, \mathbf{x})$, as an affine function, is directionally differentiable and satisfies for any $\mathbf{z} \in \mathbb{R}^n$,

$$h'_{\mathbf{x}}(\mathbf{x}; \mathbf{z} - \mathbf{x}) = \langle \nabla f(\mathbf{x}), \mathbf{z} - \mathbf{x} \rangle = f'(\mathbf{x}; \mathbf{z} - \mathbf{x}),$$

establishing the validity of property (C). Since $f, \nabla f$ are continuous functions, it also holds that for a fixed $\mathbf{y}$, the function $\mathbf{x} \mapsto h(\mathbf{y}, \mathbf{x})$ is continuous over $\mathbb{R}^n$, and is in particular closed and thus property (D) holds.

*Example 5 (differentiable concave+dd).* Consider the function

$$F(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is concave and continuously differentiable and $g : \mathbb{R}^n \to (-\infty, \infty]$ is proper and directionally differentiable. By Examples 3 and 4, $h_1(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$ and $h_2(\mathbf{y}, \mathbf{x}) = g(\mathbf{y}) + \frac{\eta}{2}\|\mathbf{y} - \mathbf{x}\|^2$ are consistent majorizers of $f$ and $g$ respectively, and hence, by Theorem 1,

$$h(\mathbf{y}, \mathbf{x}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\eta}{2}\|\mathbf{y} - \mathbf{x}\|^2 + g(\mathbf{y})$$

is a consistent majorizer of $F$ for any $\eta \geq 0$.

*Example 6 ($C^{1,1}$).* Suppose that $f$ is $L$-smooth ($L > 0$) on $\mathbb{R}^n$, meaning that

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \text{ for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

The set of functions satisfying the above is denoted by $C_L^{1,1}$. By the descent lemma [8, Proposition A.24],

$$f(\mathbf{y}) \leq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{L}{2}\|\mathbf{x} - \mathbf{y}\|^2.$$

Thus, the function

$$h(\mathbf{y}, \mathbf{x}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{L}{2}\|\mathbf{x} - \mathbf{y}\|^2$$

is a majorizer of $f$, meaning that it satisfies property (A) in the definition of consistent majorizers. It is very simple to show that properties (B), (C) and (D) also hold and hence $h$ is a consistent majorizer of $f$.

*Example 7 ($C^{1,1}$+dd).* Consider the function

$$F(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is $L$-smooth and $g$ is a directionally differentiable function. By Examples 3 and 6 along with Theorem 1, it follows that

$$h(\mathbf{y},\mathbf{x}) = f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + g(\mathbf{y}) + \frac{L}{2}\|\mathbf{x}-\mathbf{y}\|^2$$

is a consistent majorizer of $F$.

The following table summarizes the above examples.

**Table 1**

| Model | Assumptions | Consistent majorizer $h(\mathbf{y},\mathbf{x})$ |
|---|---|---|
| $f+g$ | $f - C^1$, concave<br>$g -$ dd | $f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + g(\mathbf{y}) + \frac{\eta}{2}\|\mathbf{y}-\mathbf{x}\|^2$ $(\eta \geq 0)$ |
| $f+g$ | $f - C_L^{1,1} (L>0)$<br>$g -$ dd | $f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + g(\mathbf{y}) + \frac{L}{2}\|\mathbf{y}-\mathbf{x}\|^2$ |

*Example 8 (Polynomials).* Consider a polynomial function

$$F(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x}),$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ are *monomials*, that is,

$$f_i(\mathbf{x}) = a_i x_1^{p_{i,1}} x_2^{p_{i,2}} \cdots x_n^{p_{i,n}}, \quad i = 1,\dots,m,$$

where $a_1, a_2, \dots, a_m$ are real numbers, and $p_{i,j} \in \mathbb{N} \cup \{0\}$ for all $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$. A consistent majorizer of $F$ can be constructed as the sum of majorizers of the monomials $f_1,\dots,f_m$, invoking Theorem 1 (as for all $i$ $f_i$ is a differential real-valued function). We now show how one can define a consistent majorizer of a monomial. Given $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, by the Taylor formula we write the monomial as a polynomial in $\mathbf{y}$, developed around $\mathbf{x}$. Then we upper bound each non-pure[2] monomial of the obtained polynomial by a sum of pure monomials, through repeatedly applying the inequality

$$\alpha ab \leq \frac{1}{2}|\alpha|(a^2+b^2), \tag{3}$$

[2] A monomial is called *pure* if $\exists j \forall k \neq j\ p_{i,k} = 0$.

which holds for any real numbers $\alpha, a, b$.

We now demonstrate a construction of consistent majorizers on two numerical examples of third degree monomials.

$$\begin{aligned}
f_1(\mathbf{y}) &:= y_1^2 y_2 \\
&= x_1^2 x_2 + 2x_1x_2(y_1-x_1) + x_1^2(y_2-x_2) \\
&\quad + x_2(y_1-x_1)^2 + 2x_1(y_1-x_1)(y_2-x_2) + (y_1-x_1)^2(y_2-x_2) \\
&\leq x_1^2 x_2 + 2x_1x_2(y_1-x_1) + x_1^2(y_2-x_2) \\
&\quad + x_2(y_1-x_1)^2 + 2|x_1| \cdot \frac{1}{2}\left((y_1-x_1)^2 + (y_2-x_2)^2\right) + \frac{1}{2}\left((y_1-x_1)^4 + (y_2-x_2)^2\right) \\
&=: h_1(\mathbf{y},\mathbf{x}).
\end{aligned}$$

$$\begin{aligned}
f_2(\mathbf{y}) &:= y_1y_2y_3 \\
&= x_1x_2x_3 + x_2x_3(y_1-x_1) + x_1x_3(y_2-x_2) + x_1x_2(y_3-x_3) \\
&\quad + x_3(y_1-x_1)(y_2-x_2) + x_2(y_1-x_1)(y_3-x_3) + x_1(y_2-x_2)(y_3-x_3) \\
&\quad + (y_1-x_1)(y_2-x_2)(y_3-x_3) \\
&\leq x_1x_2x_3 + x_2x_3(y_1-x_1) + x_1x_3(y_2-x_2) + x_1x_2(y_3-x_3) \\
&\quad + |x_3| \cdot \frac{1}{2}\left((y_1-x_1)^2 + (y_2-x_2)^2\right) + |x_2| \cdot \frac{1}{2}\left((y_1-x_1)^2 + (y_3-x_3)^2\right) \\
&\quad + |x_1| \cdot \frac{1}{2}\left((y_2-x_2)^2 + (y_3-x_3)^2\right) \\
&\quad + \frac{1}{2}(y_1-x_1)^2 + \frac{1}{4}(y_2-x_2)^4 + \frac{1}{4}(y_3-x_3)^4 \\
&=: h_2(\mathbf{y},\mathbf{x}),
\end{aligned}$$

where the upper bound on $(y_1-x_1)(y_2-x_2)(y_3-x_3)$ is obtained by applying (3) twice

$$\begin{aligned}
(y_1-x_1)(y_2-x_2)(y_3-x_3) &\leq \frac{1}{2}(y_1-x_1)^2 + \frac{1}{2}\left((y_2-x_2)^2(y_3-x_3)^2\right) \\
&\leq \frac{1}{2}(y_1-x_1)^2 + \frac{1}{2} \cdot \left(\frac{1}{2}(y_2-x_2)^4 + \frac{1}{2}(y_3-x_3)^4\right) \\
&= \frac{1}{2}(y_1-x_1)^2 + \frac{1}{4}(y_2-x_2)^4 + \frac{1}{4}(y_3-x_3)^4.
\end{aligned}$$

By the construction, property (A) of consistent majorizers is satisfied. Since $\mathbf{y} \mapsto f_i(\mathbf{y})$ and $\mathbf{y} \mapsto h_i(\mathbf{y},\mathbf{x})$ are polynomials having the same constant and linear terms in their Taylor expansion around $\mathbf{x}$, properties (B) and (C) hold as well. Property (D) is also satisfied, as $\mathbf{x} \mapsto -h_i(\mathbf{y},\mathbf{x})$ is continuous in $\mathbf{x}$, and thus closed. Finally, $\mathbf{y} \mapsto h_i(\mathbf{y},\mathbf{x})$ are differentiable for all $i$, and in particular have finite directional derivatives at any point and in any direction. Thus, by repeatedly applying Theorem 1 we obtain that the function $h(\mathbf{y},\mathbf{x}) := \sum_{i=1}^m h_i(\mathbf{y},\mathbf{x})$ is a consistent majorizer of $F$.

An important property of the consistent majorizer of the form constructed above is its *separability*. It comprises $n$ pure monomials, each depending on one variable. This property facilitates its minimization over a box in $\mathbb{R}^n$.

*Example 9 (quadratic forms).* Let

$$F(\mathbf{x}) = \mathbf{x}^T \mathbf{Q}\mathbf{x}$$

for some $\mathbf{Q} \in \mathbb{S}^n$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ one has

$$F(\mathbf{y}) = \mathbf{y}^T \mathbf{Q}\mathbf{y} = \mathbf{x}^T \mathbf{Q}\mathbf{x} + 2(\mathbf{Q}\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + (\mathbf{y}-\mathbf{x})^T \mathbf{Q}(\mathbf{y}-\mathbf{x}).$$

Let $\Lambda$ be a **diagonal** matrix satisfying $\Lambda \succeq \mathbf{Q}$. Denote

$$h(\mathbf{y},\mathbf{x}) := \mathbf{x}^T \mathbf{Q}\mathbf{x} + 2(\mathbf{Q}\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + (\mathbf{y}-\mathbf{x})^T \Lambda(\mathbf{y}-\mathbf{x}). \tag{4}$$

Then, for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, the inequality $h(\mathbf{y},\mathbf{x}) \geq F(\mathbf{y})$ holds, $h(\mathbf{y},\mathbf{y}) = F(\mathbf{y})$, and $\nabla h_{\mathbf{x}}(\mathbf{x}) = \nabla F(\mathbf{x})$. The function $-h(\mathbf{y},\mathbf{x})$ is also continuous in $\mathbf{x}$, and hence closed. Thus, $h$ is a consistent majorizer of $F$. In addition, $h$ is **separable** in the components of $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$. Denote $\mathbf{e} := (1, 1, \ldots, 1)^T$. We mention two possible options (out of many) for choosing the diagonal matrix $\Lambda := \text{diag}(\bar{\lambda})$.

1. Defining $\bar{\lambda}$ as an optimal solution of the following SDP:

$$\text{(SDP)} \quad \min_{\lambda \in \mathbb{R}^n} \left\{ \mathbf{e}^T \lambda : \text{diag}(\lambda) \succeq \mathbf{Q} \right\}.$$

2. Setting $\bar{\lambda} := \lambda_{\max}(\mathbf{Q}) \cdot \mathbf{e}$.

## *2.4 Consistent Majorizers of Composite Functions*

Our objective in this section is to show how consistent majorizers of composite functions of the form

$$F(\mathbf{x}) = \varphi(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})), \tag{5}$$

can be computed under certain assumptions in case where consistent majorizers of the functions $f_1, f_2, \ldots, f_m$ are available. We will use the notation $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T$, so that

$$F(\mathbf{x}) = \varphi(\mathbf{f}(\mathbf{x})).$$

The construction of consistent majorizers of $F$ relies on Lemma 1 below that presents an expression for directional derivatives of functions of this form, but first, we explicitly write the required assumptions on $\varphi$ and $\mathbf{f}$.

**Assumption 1.** *(A)* $\varphi(\mathbf{x}) = \sigma_C(\mathbf{x}) := \max_{\mathbf{y}\in C}\langle \mathbf{x}, \mathbf{y}\rangle$, *where* $C \subseteq \mathbb{R}^m_+$ *is a nonnegative compact convex set.*

*(B) The functions* $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ *are closed and directionally differentiable with* $f_i'(\mathbf{x}; \mathbf{d}) \in \mathbb{R}$ *for all* $i \in \{1, \ldots, m\}$ *and* $\mathbf{x}, \mathbf{d} \in \mathbb{R}^n$.

An interesting example of a function satisfying property (A) of Assumption 1 is $\varphi(\mathbf{x}) = \max\{x_1, x_2, \ldots, x_n\}$, which corresponds to the choice $C = \Delta_n$, (with $m = n$). An interesting example of a composition $F = \varphi \circ \mathbf{f}$ where (A) and (B) are satisfied is $F(\mathbf{x}) = \|\mathbf{x}\|_1$ which corresponds to $C = (\Delta_2)^n$ (with $m = 2n$) and $\mathbf{f}(\mathbf{x}) = (x_1, -x_1, x_2, -x_2, \ldots, x_n, -x_n)^T$.

*Remark 1 (properties of* $\varphi$*).* Note that the fact that $\varphi$ is a support function of a compact set implies that it is real-valued convex, subadditive and positively homogenous. The fact that the underlying set is nonnegative implies that the function is in addition nonincreasing in the sense that $\mathbf{x} \leq \mathbf{y}$ implies that $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$.

In the following lemma we use the following notation: if the $m$ functions $s_1, s_2, \ldots, s_m$ have a directional derivative at $\mathbf{x}$ in the direction $\mathbf{d}$, then the corresponding directional derivative of the vector-valued function $\mathbf{s} = (s_1, s_2, \ldots, s_m)^T$ is denoted by $\mathbf{s}'(\mathbf{x}; \mathbf{d})$ and is the $m$-dimensional column vector given by

$$\mathbf{s}'(\mathbf{x}; \mathbf{d}) = (s_i'(\mathbf{x}; \mathbf{d}))_{i=1}^m.$$

**Lemma 1.** *Let*

$$S(\mathbf{x}) = \varphi(\mathbf{s}(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^n,$$

*where*

- $\varphi : \mathbb{R}^m \to \mathbb{R}$ *is a convex, subadditive and positively homogenous function.*
- $\mathbf{s} = (s_1, s_2, \ldots, s_m)^T$ *is a function from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$.

*Let* $\mathbf{x}, \mathbf{d} \in \mathbb{R}^n$ *and suppose that* $\mathbf{s}$ *is differentiable at* $\mathbf{x}$ *in the direction* $\mathbf{d}$ *with* $s_i'(\mathbf{x}; \mathbf{d}) \in \mathbb{R}$ *for all* $i$*. Then* $S$ *has a directional derivative at* $\mathbf{x}$ *in the direction* $\mathbf{d}$ *which is given by*

$$S'(\mathbf{x}; \mathbf{d}) = \varphi'(\mathbf{s}(\mathbf{x}); \mathbf{s}'(\mathbf{x}; \mathbf{d})). \tag{6}$$

*Proof.* Note that by the fact that the components of $\mathbf{s}$ have a directional derivative at $\mathbf{x}$ in the direction $\mathbf{d}$, it follows that there exists a function $\mathbf{o} : \mathbb{R}^+ \to \mathbb{R}^m$ satisfying $\lim_{t\to 0^+} \frac{\mathbf{o}(t)}{t} = \mathbf{0}$ for which

$$\mathbf{s}(\mathbf{x} + t\mathbf{d}) = \mathbf{s}(\mathbf{x}) + t\mathbf{s}'(\mathbf{x}; \mathbf{d}) + \mathbf{o}(t).$$

By the subadditivity and positive homogeneity of $\varphi$, it follows that

$$\begin{aligned}\frac{\varphi(\mathbf{s}(\mathbf{x}+t\mathbf{d})) - \varphi(\mathbf{s}(\mathbf{x}))}{t} &= \frac{\varphi(\mathbf{s}(\mathbf{x}) + t\mathbf{s}'(\mathbf{x}; \mathbf{d}) + \mathbf{o}(t)) - \varphi(\mathbf{s}(\mathbf{x}))}{t} \\ &\leq \frac{\varphi(\mathbf{s}(\mathbf{x}) + t\mathbf{s}'(\mathbf{x}; \mathbf{d})) - \varphi(\mathbf{s}(\mathbf{x}))}{t} + \varphi\left(\frac{\mathbf{o}(t)}{t}\right). \end{aligned} \tag{7}$$

Similarly,

$$\frac{\varphi(\mathbf{s}(\mathbf{x}+t\mathbf{d}))-\varphi(\mathbf{s}(\mathbf{x}))}{t} \geq \frac{\varphi(\mathbf{s}(\mathbf{x})+t\mathbf{s}'(\mathbf{x};\mathbf{d}))-\varphi(\mathbf{s}(\mathbf{x}))}{t} - \varphi\left(-\frac{\mathbf{o}(t)}{t}\right). \quad (8)$$

By the definition of the function $\mathbf{o}$, $\lim_{t\to 0^+}\frac{\mathbf{o}(t)}{t} = \mathbf{0}$, and thus, by the continuity of $\varphi$ (as it is a real-valued convex function), it follows that $\lim_{t\to 0^+}\varphi\left(\frac{\mathbf{o}(t)}{t}\right) = \lim_{t\to 0^+}\varphi\left(-\frac{\mathbf{o}(t)}{t}\right) = \varphi(\mathbf{0}) = 0$. It therefore follows by (7) and (8) that

$$\begin{aligned} S'(\mathbf{x};\mathbf{d}) &= \lim_{t\to 0^+}\frac{\varphi(\mathbf{s}(\mathbf{x}+t\mathbf{d}))-\varphi(\mathbf{s}(\mathbf{x}))}{t} = \lim_{t\to 0^+}\frac{\varphi(\mathbf{s}(\mathbf{x})+t\mathbf{s}'(\mathbf{x};\mathbf{d}))-\varphi(\mathbf{s}(\mathbf{x}))}{t} \\ &= \varphi'(\mathbf{s}(\mathbf{x});\mathbf{s}'(\mathbf{x};\mathbf{d})). \end{aligned}$$

□

Equipped with Lemma 1, we can now show how to construct a consistent majorizer of the function $F$ given in (1) out of consistent majorizers of $f_1, f_2, \ldots, f_m$.

**Theorem 2.** *Let*

$$F(\mathbf{x}) = \varphi(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})),$$

*where $\varphi$ and $\mathbf{f}$ satisfy the properties in Assumption 1. Assume that for any $i \in \{1,2,\ldots,m\}$ the function $h_i$ is a consistent majorizer of $f_i$. Then the function*

$$H(\mathbf{y},\mathbf{x}) = \varphi(h_1(\mathbf{y},\mathbf{x}), h_2(\mathbf{y},\mathbf{x}), \ldots, h_m(\mathbf{y},\mathbf{x}))$$

*is a consistent majorizer of $F$.*

*Proof.* We will show that the four properties in the definition of consistent majorizers hold:
(A). By the monotonicity of $\varphi$ (see Remark 1) and the fact that $h_i$ is a majorizer of $f_i$ for any $i$, it follows that for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$,

$$H(\mathbf{y},\mathbf{x}) = \varphi(h_1(\mathbf{y},\mathbf{x}), h_2(\mathbf{y},\mathbf{x}), \ldots, h_m(\mathbf{y},\mathbf{x})) \geq \varphi(f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_m(\mathbf{y})) = F(\mathbf{y}),$$

establishing property (A) for the pair $(F,H)$.
(B). Follows by the following simple computation:

$$H(\mathbf{y},\mathbf{y}) = \varphi(h_1(\mathbf{y},\mathbf{y}), h_2(\mathbf{y},\mathbf{y}), \ldots, h_m(\mathbf{y},\mathbf{y})) = \varphi(f_1(\mathbf{y}), f_2(\mathbf{y}), \ldots, f_m(\mathbf{y})) = F(\mathbf{y}).$$

(C). For a given $\mathbf{x} \in \mathrm{dom}(F)$, define the functions $h_{i,\mathbf{x}}(\mathbf{y}) := h_i(\mathbf{y},\mathbf{x}), i = 1,2,\ldots,m$ and the function

$$H_{\mathbf{x}}(\mathbf{y}) = H(\mathbf{y},\mathbf{x}) = \varphi(h_{1,\mathbf{x}}(\mathbf{y}), h_{2,\mathbf{x}}(\mathbf{y}), \ldots, h_{m,\mathbf{x}}(\mathbf{y})).$$

We need to prove that for any $\mathbf{x},\mathbf{z} \in \mathrm{dom}(F)$, $H'_{\mathbf{x}}(\mathbf{x};\mathbf{z}-\mathbf{x}) = F'(\mathbf{x};\mathbf{z}-\mathbf{x})$. Indeed, by Lemma 1 invoked with $\mathbf{s} = \mathbf{f}$, it follows that

$$F'(\mathbf{x};\mathbf{z}-\mathbf{x}) = \varphi'(\mathbf{f}(\mathbf{x});\mathbf{f}'(\mathbf{x};\mathbf{z}-\mathbf{x})). \quad (9)$$

Finally, for any $\mathbf{x},\mathbf{z} \in \mathrm{dom}(F)$, invoking Lemma 1 once more with $s_i(\mathbf{y}) := h_{i,\mathbf{x}}(\mathbf{y})$, and taking into account that $h_i$ is a consistent majorizer of $f_i$ for any $i$, we obtain

$$\begin{aligned}
H'_{\mathbf{x}}(\mathbf{x};\mathbf{z}-\mathbf{x}) &= \varphi'((h_{1,\mathbf{x}}(\mathbf{x}),\ldots,h_{m,\mathbf{x}}(\mathbf{x}))^T;(h'_{1,\mathbf{x}}(\mathbf{x};\mathbf{z}-\mathbf{x}),\ldots,h'_{m,\mathbf{x}}(\mathbf{x};\mathbf{z}-\mathbf{x}))^T) \\
&= \varphi'((f_1(\mathbf{x}),\ldots,f_m(\mathbf{x}))^T;(f'_1(\mathbf{x};\mathbf{z}-\mathbf{x}),\ldots,f'_m(\mathbf{x};\mathbf{z}-\mathbf{x}))^T) \\
&= \varphi'(\mathbf{f}(\mathbf{x});\mathbf{f}'(\mathbf{x};\mathbf{z}-\mathbf{x})) \\
&= F'(\mathbf{x};\mathbf{z}-\mathbf{x}).
\end{aligned}$$

(D). For a fixed $\mathbf{y} \in \mathbb{R}^n$ we need to show that $\mathbf{x} \mapsto -H(\mathbf{y},\mathbf{x})$ is closed. Specifically, let $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ be fixed; we need to establish the existence of $\delta > 0$ such that

$$H(\mathbf{y},\tilde{\mathbf{x}}) < H(\mathbf{y},\mathbf{x}) + \varepsilon$$

for all $\tilde{\mathbf{x}}$ such that $\|\tilde{\mathbf{x}}-\mathbf{x}\| < \delta$. That would show the equivalent assertion that $\mathbf{x} \mapsto H(\mathbf{y},\mathbf{x})$ is upper semicontinuous.

Indeed, by the continuity of $\varphi$, for any $\mathbf{z} \in \mathbb{R}^m$ there exists $\delta_{\mathbf{z}} > 0$ such that if $\|\tilde{\mathbf{z}}-\mathbf{z}\|_\infty < \delta_{\mathbf{z}}$, then $|\varphi(\tilde{\mathbf{z}})-\varphi(\mathbf{z})| < \varepsilon$. In particular, this holds for

$$\mathbf{z} := (h_1(\mathbf{y},\mathbf{x}),\ldots,h_m(\mathbf{y},\mathbf{x}))^T.$$

Since for any $i \in \{1,\ldots,m\}$ the function $\mathbf{x} \mapsto -h_i(\mathbf{y},\mathbf{x})$ is closed, there exists $\delta_i > 0$ such that if $\|\tilde{\mathbf{x}}-\mathbf{x}\| < \delta_i$, then

$$h_i(\mathbf{y},\tilde{\mathbf{x}}) < h_i(\mathbf{y},\mathbf{x}) + \delta_{\mathbf{z}}.$$

Define $\delta := \min\{\delta_1,\ldots,\delta_m\}$, and let $\tilde{\mathbf{x}}$ satisfy $\|\tilde{\mathbf{x}}-\mathbf{x}\| < \delta$. There exists two sets of indices

$$\begin{aligned}
I_{\tilde{\mathbf{x}}} &:= \{i \in \{1,\ldots,m\} : h_i(\mathbf{y},\tilde{\mathbf{x}}) \le h_i(\mathbf{y},\mathbf{x})\}, \\
J_{\tilde{\mathbf{x}}} &:= \{i \in \{1,\ldots,m\} : h_i(\mathbf{y},\mathbf{x}) < h_i(\mathbf{y},\tilde{\mathbf{x}}) < h_i(\mathbf{y},\mathbf{x}) + \delta_{\mathbf{z}}\}
\end{aligned}$$

satisfying $I_{\tilde{\mathbf{x}}} \cup J_{\tilde{\mathbf{x}}} = \{1,2,\ldots,m\}$ and $I_{\tilde{\mathbf{x}}} \cap J_{\tilde{\mathbf{x}}} = \emptyset$. Define a vector $\mathbf{u} \in \mathbb{R}^m$ as follows,

$$u_i = \begin{cases} h_i(\mathbf{y},\mathbf{x}), & i \in I_{\tilde{\mathbf{x}}}, \\ h_i(\mathbf{y},\tilde{\mathbf{x}}), & i \in J_{\tilde{\mathbf{x}}}. \end{cases}$$

By the monotonicity of $\varphi$ it follows that

$$H(\mathbf{y},\tilde{\mathbf{x}}) = \varphi(h_1(\mathbf{y},\tilde{\mathbf{x}}),\ldots,h_m(\mathbf{y},\tilde{\mathbf{x}})) \le \varphi(\mathbf{u}). \tag{10}$$

In addition, by the construction, $\|\mathbf{u}-\mathbf{z}\|_\infty < \delta_{\mathbf{z}}$. Thus,

$$\varphi(\mathbf{u}) < \varphi(\mathbf{z}) + \varepsilon = H(\mathbf{y},\mathbf{x}) + \varepsilon, \tag{11}$$

and the result follows by a summation of (10) and (11). □

*Example 10.* Suppose that

$$F(\mathbf{x}) = \max\{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})\} + g(\mathbf{x}),$$

where $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are $C^{1,1}$ functions and $g : \mathbb{R}^n \to (-\infty, \infty]$ is proper closed and convex. We assume specifically that $f_i \in C^{1,1}_{L_i}$ $(L_i > 0)$ for any $i$. Then by Example 6,

$$h_i(\mathbf{y}, \mathbf{x}) = f_i(\mathbf{x}) + \langle \nabla f_i(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{L_i}{2} \|\mathbf{x} - \mathbf{y}\|^2$$

is a consistent majorizer of $f_i$, and thus, by Theorem 2, which can be invoked since $f_i$ are directionally differentiable and $\varphi = \sigma_{\Delta_n}$, it follows that the function $(\mathbf{y}, \mathbf{x}) \mapsto \max_{i=1,2,\ldots,m}\{h_i(\mathbf{y}, \mathbf{x})\}$ is a consistent majorizer of $\mathbf{x} \mapsto \max_{i=1,2,\ldots,m} f_i(\mathbf{x})$. Consequently, by Theorem 1, it follows that

$$H(\mathbf{y}, \mathbf{x}) := \max_{i=1,2,\ldots,m}\{h_i(\mathbf{y}, \mathbf{x})\} + g(\mathbf{y})$$

is a consistent majorizer of $F$.

*Example 11.* Let

$$F(\mathbf{x}) = \sum_{i=1}^{m} |f_i(\mathbf{x})|,$$

where $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are differentiable convex functions. Note that $F$ can be rewritten as

$$F(\mathbf{x}) = \sum_{i=1}^{m} \max\{f_i(\mathbf{x}), -f_i(\mathbf{x})\},$$

meaning that $F = \varphi \circ \mathbf{t}$, where

$$\begin{aligned} \varphi(\mathbf{w}) &= \sum_{i=1}^{m} \max\{w_{2i-1}, w_{2i}\}, \\ t_{2i-1}(\mathbf{x}) &= f_i(\mathbf{x}), \quad i = 1, 2, \ldots, m, \\ t_{2i}(\mathbf{x}) &= -f_i(\mathbf{x}), \quad i = 1, 2, \ldots, m. \end{aligned}$$

Since $-f_i$ is concave, it follows that $(\mathbf{y}, \mathbf{x}) \mapsto -f_i(\mathbf{x}) - \langle \nabla f_i(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$ is a consistent majorizer of $-f_i$ (Example 4); in addition,$(\mathbf{y}, \mathbf{x}) \mapsto f_i(\mathbf{y})$ is a consistent majorizer of $\mathbf{x} \mapsto f_i(\mathbf{x})$ (Example 3 with $\eta = 0$). Thus, by Theorem 2, which can be invoked since the functions $t_1, t_2, \ldots, t_{2m}$ are directionally differentiable and $\varphi = \sigma_{(\Delta_2)^m}$, it follows that the function

$$H(\mathbf{y}, \mathbf{x}) = \sum_{i=1}^{m} \max\{f_i(\mathbf{y}), -f_i(\mathbf{x}) - \langle \nabla f_i(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle\}$$

is a consistent majorizer of $F$. It is interesting to note that this majorizer is a convex function w.r.t. $\mathbf{y}$.

## 3 Stationarity Measures and Conditions

Stationarity is a fundamental concept in optimization problems. For the optimization problem (2), perhaps the most natural stationarity condition of a given point is the following.

**Definition 3.** Let $F$ be a proper, closed, directionally differentiable function. A point $\mathbf{x}^* \in \mathrm{dom}(F)$ is called a **stationary** point of problem (2) if it satisfies

$$F'(\mathbf{x}^*;\mathbf{y}-\mathbf{x}^*) \geq 0 \quad \text{for all } \mathbf{y} \in \mathrm{dom}(F). \tag{12}$$

Stationarity is a well-known necessary optimality condition for problem (2), and it becomes also sufficient in the convex case, as stated in the following simple lemma. For the convenience of the reader we provide its proof in the appendix.

**Lemma 2.** *Let $F$ be a proper, closed, directionally differentiable function. If $\mathbf{x}^*$ is a local minimizer of (2), then it is a stationary point. If, in addition, $F$ is convex, then any stationary point $\mathbf{x}^*$ of (2) is a global minimizer.*

Most of the known first-order methods are designed such that their limit points would satisfy (12). Their analysis in many cases is based on some stationarity measure, which is a nonnegative function that vanishes exactly at stationary points. See e.g., [4, 5, 10, 16] and references therein for the wide usage of stationarity measures in analysis of first-order optimization algorithms.

In this section our main goal is to introduce stationarity measures that are based on consistent majorizers of $F$, the objective function of problem (2). At this point we introduce an additional property that will be assumed to be satisfied by consistent majorizers.

**Assumption 2.** *For any $\mathbf{x} \in \mathbb{R}^n$ the value $\min_{\mathbf{y}} h(\mathbf{y},\mathbf{x})$ is finite.*

Assumption 2 does not require that $\min_{\mathbf{y}} h(\mathbf{y},\mathbf{x})$ is attained; however, we always use the notation "min" rather than "inf". Now let $h : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty,\infty]$ be a consistent majorizer of $F$ such that Assumption 2 is satisfied. Define the function $S_{F,h} : \mathbb{R}^n \to (-\infty,\infty]$ by

$$S_{F,h}(\mathbf{x}) := F(\mathbf{x}) - \min_{\mathbf{y}} h(\mathbf{y},\mathbf{x}).$$

By Assumption 2, the function $S_{F,h}$ is well defined, and its domain coincides with $\mathrm{dom}(F)$. Though $S_{F,h}$ depends on $F$ and on the consistent majorizer $h$, from now on we simply denote

$$S \equiv S_{F,h},$$

omitting the subscripts $F$ and $h$ whenever they are clear from the context. The following lemma establishes the main properties of $S$.

**Lemma 3.** *Let $F$ be a proper, closed, directionally differentiable function, and $h$ be a consistent majorizer of $F$. Suppose that Assumption 2 holds. Then the function $S$ satisfies the following properties:*

*1.* $S(\mathbf{x}) \geq 0$ *for any* $\mathbf{x} \in \mathrm{dom}(F)$.
*2. Any* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{p} \in \operatorname*{argmin}_{\mathbf{y}} h(\mathbf{y},\mathbf{x})$ *satisfy*

$$F(\mathbf{x}) - F(\mathbf{p}) \geq S(\mathbf{x}).$$

*3.* $S$ *is lower semicontinuous, that is, if* $\mathbf{x}^k \to \tilde{\mathbf{x}}$ *as* $k \to \infty$, *then*

$$S(\tilde{\mathbf{x}}) \leq \liminf_{k\to\infty} S(\mathbf{x}^k).$$

*4.* $S(\mathbf{x}) = 0$ *if and only if* $\mathbf{x} \in \operatorname*{argmin}_{\mathbf{y}} h(\mathbf{y},\mathbf{x})$.
*5. If* $S(\mathbf{x}) = 0$, *then the inequality* $F'(\mathbf{x};\mathbf{y}-\mathbf{x}) \geq 0$ *holds for any* $\mathbf{y} \in \mathrm{dom}(F)$, *that is,* $\mathbf{x}$ *is a stationary point of (2).*
*If, in addition,* $\mathbf{y} \mapsto h(\mathbf{y},\mathbf{x})$ *is a convex function of* $\mathbf{y}$ *for any* $\mathbf{x} \in \mathrm{dom}(F)$, *then the converse is also true.*

*Proof.* 1. Let $\mathbf{x} \in \mathbb{R}^n$. Then by property (B) of consistent majorizers we get

$$\begin{aligned} S(\mathbf{x}) &= F(\mathbf{x}) - \min_{\mathbf{y}} h(\mathbf{y},\mathbf{x}) \\ &\geq F(\mathbf{x}) - h(\mathbf{x},\mathbf{x}) = F(\mathbf{x}) - F(\mathbf{x}) = 0. \end{aligned}$$

2. Let $\mathbf{x}$ and $\mathbf{p}$ be as in the assumption. Then by property (A) of consistent majorizers

$$F(\mathbf{x}) - F(\mathbf{p}) \geq F(\mathbf{x}) - h(\mathbf{p},\mathbf{x}) = F(\mathbf{x}) - \min_{\mathbf{y}} h(\mathbf{y},\mathbf{x}) = S(\mathbf{x}).$$

3. By property (D) of consistent majorizers, the function $-h(\mathbf{y},\cdot)$ is closed for any $\mathbf{y} \in \mathrm{dom}(F)$. Notice that

$$S(\mathbf{x}) = F(\mathbf{x}) - \min_{\mathbf{y}} h(\mathbf{y},\mathbf{x}) = F(\mathbf{x}) - \min_{\mathbf{y}\in\mathrm{dom}(F)} h(\mathbf{y},\mathbf{x}) = F(\mathbf{x}) + \max_{\mathbf{y}\in\mathrm{dom}(F)} \{-h(\mathbf{y},\mathbf{x})\}.$$

$S$ is closed (equivalently, lower semicontinuous) as the sum of the closed function and a pointwise maximum of closed functions.
4. $S(\mathbf{x}) = 0$ if and only if $F(\mathbf{x}) = \min_{\mathbf{y}} h(\mathbf{y},\mathbf{x})$ and by property (B) of consistent majorizers, the latter is valid if and only if $h(\mathbf{x},\mathbf{x}) = \min_{\mathbf{y}} h(\mathbf{y},\mathbf{x})$, which is equivalent to $\mathbf{x} \in \operatorname*{argmin}_{\mathbf{y}} h(\mathbf{y},\mathbf{x})$.
5. A necessary condition for $\mathbf{x}$ to be a global minimizer of $h_{\mathbf{x}}(\mathbf{y}) \equiv h(\mathbf{y},\mathbf{x})$ with respect to $\mathbf{y}$ (i.e., for $h_{\mathbf{x}}(\mathbf{x})$ to be the minimal value of $h_{\mathbf{x}}$) is (see Lemma 2)

$$(h_{\mathbf{x}})'(\mathbf{x};\mathbf{y}-\mathbf{x}) \geq 0 \quad \forall \mathbf{y} \in \mathrm{dom}(F). \tag{13}$$

By property (C) of consistent majorizers, the condition (13) is equivalent to

$$F'(\mathbf{x};\mathbf{y}-\mathbf{x}) \geq 0 \quad \forall \mathbf{y} \in \mathrm{dom}(F),$$

and the result follows.

If, in addition, the function $h_{\mathbf{x}}$ is convex in $\mathbf{y}$ for all $\mathbf{x} \in \mathrm{dom}(F)$, then the necessary condition (13) becomes also sufficient (see Lemma 2), namely, it also implies $\mathbf{x} \in \operatorname*{argmin}_{\mathbf{y}} h(\mathbf{y}, \mathbf{x})$.

□

As one can see by Lemma 3, if a point $\tilde{\mathbf{x}} \in \mathrm{dom}(F)$ satisfies $S(\tilde{\mathbf{x}}) = 0$, it is stationary, but in the nonconvex case there might exist some stationary points with $S(\tilde{\mathbf{x}}) > 0$. This observation leads us to formulate a necessary optimality condition, based on a property which is stronger than stationarity.

**Definition 4.** Let $F$ be a proper, closed, directionally differentiable function. We say that $\mathbf{x} \in \mathrm{dom}(F)$ is a **strongly stationary** point of problem (2) with respect to a consistent majorizer $h$ if $S(\mathbf{x}) = 0$.

The following lemma establishes a necessary optimality condition for the optimization problem (2).

**Lemma 4.** *Let $F$ be a proper, closed, directionally differentiable function and $h$ be a consistent majorizer of $F$. Suppose that Assumption 2 holds. Let $\mathbf{x}^* \in \mathrm{dom}(F)$ be a global optimal solution for problem (2). Then $\mathbf{x}^*$ is a strongly stationary point with respect to any consistent majorizer $h$.*

*Proof.* Assume otherwise, that is, $S(\mathbf{x}^*) > 0$. Then there exists $\mathbf{y} \in \mathrm{dom}(F)$ such that $h(\mathbf{y}, \mathbf{x}^*) < h(\mathbf{x}^*, \mathbf{x}^*)$. Since $\mathbf{x}^*$ is a global minimizer of $F$ over $\mathrm{dom}(F)$, for any $\mathbf{y} \in \mathrm{dom}(F)$ we have (utilizing properties (A) and (B) of consistent majorizers)

$$F(\mathbf{x}^*) \leq F(\mathbf{y}) \leq h(\mathbf{y}, \mathbf{x}^*) < h(\mathbf{x}^*, \mathbf{x}^*) = F(\mathbf{x}^*),$$

which yields a contradiction. □

By Lemma 4, any global minimizer of (2) is a strongly stationary point with respect to any consistent majorizer, and by Lemma 3 any such point is also a stationary point. These two observations might help in solving specific problems of the setting (2) if, for example, a certain algorithm can be shown to converge to a strongly stationary point rather than just to a stationary point, it might have better chances of converging to a global solution. The choice of the majorizer can affect the number of strongly stationary points.

*Example 12 (minimizing a concave quadratic form over a box).* Consider the optimization problem with the objective function defined in Example 9 for some $\mathbf{Q} \preceq \mathbf{0}$, and box constraints. That is, the minimization problem is given by

$$\text{(PQ)} \qquad \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ F(\mathbf{x}) := \mathbf{x}^T \mathbf{Q} \mathbf{x} : \quad -\mathbf{e} \leq \mathbf{x} \leq \mathbf{e} \right\}.$$

A concave function attains its minimal value over a compact convex set at least on one of its extreme points. Therefore, $F$ attains its minimal value over $[-1,1]^n$ at a vector in $\{-1,1\}^n$. A well known combinatorial optimization problem that can be reformulated in the form of (PQ) is the MAXCUT problem; see e.g. [7, Section 3.4.1] and references therein.

For problem (PQ), the stationary points are the vectors $\mathbf{x}^* \in [-1,1]^n$ satisfying

$$F'(\mathbf{x}^*;\mathbf{x}-\mathbf{x}^*) \geq 0 \quad \forall \mathbf{x} \in [-1,1]^n,$$

or equivalently,

$$\langle \mathbf{Q}\mathbf{x}^*, \mathbf{x}-\mathbf{x}^* \rangle \geq 0 \quad \forall \mathbf{x} \in [-1,1]^n.$$

We give two numerical examples with $n = 5,7$ by setting $\mathbf{Q} := \mathbf{Q}_j$ for $j = 1,2$, where

$$\mathbf{Q}_1 \equiv \begin{pmatrix} -24 & 2 & -8 & 0 & -5 \\ 2 & -26 & 0 & -6 & 1 \\ -8 & 0 & -22 & -7 & 0 \\ 0 & -6 & -7 & -18 & 5 \\ -5 & 1 & 0 & 5 & -34 \end{pmatrix} \quad \mathbf{Q}_2 \equiv \frac{1}{2}\begin{pmatrix} -24 & 2 & -8 & 0 & -5 & 0 & -6 \\ 2 & -26 & 0 & -6 & 1 & -1 & -3 \\ -8 & 0 & -22 & -7 & 0 & 4 & -1 \\ 0 & -6 & -7 & -18 & 5 & -1 & 1 \\ -5 & 1 & 0 & 5 & -34 & 0 & -3 \\ 0 & -1 & 4 & -1 & 0 & -28 & -7 \\ -6 & -3 & -1 & 1 & -3 & -7 & -32 \end{pmatrix}.$$

Since $\mathbf{Q}_1, \mathbf{Q}_2 \prec \mathbf{0}$, at least one global minimizer must be a vertex of $[-1,1]^n$. Therefore, we can reduce the discussion to the $2^n$ vertices $\{-1,1\}^n$.

For each vertex $\mathbf{x} \in \{-1,1\}^n$ we checked whether it is a stationary point. It is simple to show that stationarity in this case can be easily verified, utilizing the following explicit test. Denote by $\mathbf{q}_i$ the $i$th column of $\mathbf{Q}$. A vector $\mathbf{x}^* \in [-1,1]^n$ is a stationary point of (PQ) if and only if for each $i = 1,\ldots,n$ one of the following holds:

- $\mathbf{q}_i^T\mathbf{x}^* \leq 0$ and $x_i^* = 1$,
- $\mathbf{q}_i^T\mathbf{x}^* \geq 0$ and $x_i^* = -1$,
- $\mathbf{q}_i^T\mathbf{x}^* = 0$.

We also checked for each vertex whether it is a strongly stationary point with respect to the majorizers described in Example 9 from Sect. 2.3. Note that utilizing (4), a consistent majorizer of $F$ is given by

$$h(\mathbf{y},\mathbf{x}) := \sum_{i=1}^{n} h_i(y_i,\mathbf{x}) + \mathbf{x}^T\mathbf{Q}\mathbf{x},$$

where

$$h_i(y_i,\mathbf{x}) := 2(\mathbf{q}_i^T\mathbf{x})(y_i - x_i) + \bar{\lambda}_i(y_i - x_i)^2, \quad i = 1,\ldots,n,$$

with $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$ being the diagonal entries of a given diagonal matrix $\Lambda$ satisfying $\Lambda \succeq \mathbf{Q}$. Since $h$ is a separable sum of functions in the variables $y_i$, for a given $\mathbf{x} \in [-1,1]^n$ the test whether $S(\mathbf{x}) = 0$ amounts to computing $n$ numbers $y_1^*, y_2^*, \ldots, y_n^*$ satisfying the conditions

$$y_i^* \in \underset{y_i \in [-1,1]}{\operatorname{argmin}}\, h_i(y_i,\mathbf{x}), \quad i = 1,\ldots,n,$$

and testing whether $h(\mathbf{y}^*,\mathbf{x}) = h(\mathbf{x},\mathbf{x})$, where $\mathbf{y}^* := (y_1^*, y_2^*, \ldots, y_n^*)^T$.

Table 2 contains the number of stationary (W) and strongly stationary (S) points out of the $2^n$ vertices. The column (G) indicates how many vertices are global optimal solutions of (PQ). The results show that in these examples the standard stationarity condition almost does not rule out any of the vertices. Strong stationarity is a more restrictive condition, and its strictness depends on the chosen consistent majorizer.

**Table 2** Stationarity and optimality of vertices.

| $\mathbf{Q}$ | $\Lambda = \text{diag}(\bar{\lambda})$ | $n$ | $m = 2^n$ | W | S | G |
|---|---|---|---|---|---|---|
| $\mathbf{Q}_1$ | $\bar{\lambda}$ by (SDP) | 5 | 32 | 32 | 12 | 4 |
| $\mathbf{Q}_1$ | $\bar{\lambda} = \lambda_{\max}(\mathbf{Q}_1)$ | 5 | 32 | 32 | 20 | 4 |
| $\mathbf{Q}_2$ | $\bar{\lambda}$ by (SDP) | 7 | 128 | 124 | 42 | 2 |
| $\mathbf{Q}_2$ | $\bar{\lambda} = \lambda_{\max}(\mathbf{Q}_2)$ | 7 | 128 | 124 | 86 | 2 |

# 4 The Inexact Majorization-Minimization Method

## *4.1 The General Scheme*

We introduce now the main algorithm proposed for solving problem (2). Let $F$ be a directionally differentiable function, and let $h$ be a given consistent majorizer of $F$. For the first variant of the algorithm we need to make the following assumption that is more restrictive than Assumption 2.

**Assumption 3.** *For any* $\mathbf{x} \in \mathbb{R}^n$ *the function* $h_{\mathbf{x}}(\mathbf{y}) \equiv h(\mathbf{y}, \mathbf{x})$ *has at least one global minimizer.*

Whenever Assumption 3 holds, and a minimizer of $h_{\mathbf{x}}$ can be computed **exactly** for any $\mathbf{x} \in \text{dom}(F)$, the general scheme for the so-called *majorization-minimization (MM)* method described below is well-defined.

**Algorithm 1.** *Majorization-Minimization (MM) Algorithm for Solving (2).*

- *Pick an arbitrary* $\mathbf{x}^0 \in \text{dom}(F) \subseteq \mathbb{R}^n$.
- *For* $k = 0, 1, \ldots$ *compute a vector*

$$\mathbf{x}^{k+1} \in \operatorname*{argmin}_{\mathbf{x}} h(\mathbf{x}, \mathbf{x}^k).$$

The choice of the specific minimizer in iterations where more than one minimizer of $h_{\mathbf{x}^k}$ exist can be made arbitrarily, or, in some cases, according to some pre-specified policy. By part 2 of Lemma 3, the sequence generated by Algorithm 1 has a decrease guarantee of

$$F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) \geq S(\mathbf{x}^k) \quad \text{for all } k = 0, 1, \ldots.$$

In many cases, either Assumption 3 does not hold, or, it does, but an exact minimizer of $h_{\mathbf{x}^k}$ cannot be computed. In such cases, we formulate Algorithm 2, which is an *inexact* version of Algorithm 1. Let $\gamma \in (0,1]$ be a given parameter. Algorithm 2 is based on the ability to compute vectors that achieve a decrease of at least $\gamma$ times $S(\mathbf{x}^k)$, which is the decrease that exact minimization of $h_{\mathbf{x}^k}$ would have guaranteed. We still assume that Assumption 2 holds (but not necessarily Assumption 3) whenever we seek to apply Algorithm 2 with $\gamma < 1$. The choice $\gamma := 1$ corresponds to the exact version (Algorithm 1) as the only vectors that satisfy (14) for $\gamma = 1$ are exact minimizers of $h_{\mathbf{x}^k}$. Thus, $\gamma = 1$ requires the validity of Assumption 3.

**Algorithm 2.** *Inexact Majorization-Minimization (IMM) Algorithm for Solving (2).*

- *Input:* $\gamma \in (0,1]$.
- *Pick an arbitrary* $\mathbf{x}^0 \in \mathrm{dom}(F) \subseteq \mathbb{R}^n$.
- *For* $k = 0, 1, \ldots,$ *set* $\mathbf{x}^{k+1}$ *to be any vector satisfying*

$$F(\mathbf{x}^k) - h(\mathbf{x}^{k+1}, \mathbf{x}^k) \geq \gamma \cdot S(\mathbf{x}^k). \tag{14}$$

In the context of the IMM method, for any $\mathbf{x} \in \mathbb{R}^n$, a vector $\mathbf{y}$ satisfying

$$F(\mathbf{x}) - h(\mathbf{y}, \mathbf{x}) \geq \gamma \cdot S(\mathbf{x})$$

is called *an approximate $\gamma$-vector* at $\mathbf{x}$. In this terminology, $\mathbf{x}^{k+1}$ is also chosen as an approximate $\gamma$-vector at $\mathbf{x}^k$. The inexact minimization criterion (14) indeed guarantees a decrease of

$$F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) \geq \gamma \cdot S(\mathbf{x}^k) \quad \text{for all } k = 0, 1, \ldots,$$

as follows by property (A) of consistent majorizers. The following is an example of a simple case where the **exact** method (Algorithm 1) can be implemented. In particular, the constructed consistent majorizer satisfies Assumption 3.

*Example 13.* Let $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(\mathbf{x}) := 2x_1^2 x_2 + 5x_2^3 + 5x_1 x_3^2 + 8x_3^3,$$

and $B := [-100, 1000] \times [-78, 802] \times [-123, 77] \subseteq \mathbb{R}^3$. Then, following Example 8, a consistent majorizer of $f$ can be given by

$$\begin{aligned}h(\mathbf{y},\mathbf{x}) := {} & 5y_2^3+8y_3^3+2x_1^2x_2+4x_1x_2(y_1-x_1)+2x_1^2(y_2-x_2)\\ & +2x_2(y_1-x_1)^2+2|x_1|\cdot\left((y_1-x_1)^2+(y_2-x_2)^2\right)+(y_1-x_1)^4+(y_2-x_2)^2\\ & +5x_1x_3^2+10x_1x_3(y_3-x_3)+5x_3^2(y_1-x_1)\\ & +5x_1(y_3-x_3)^2+5|x_3|\cdot\left((y_1-x_1)^2+(y_3-x_3)^2\right)+2.5(y_3-x_3)^4\\ & +2.5(y_1-x_1)^2.\end{aligned}$$

Consider now the optimization problem of minimizing $f$ over the box-shaped domain $B$. In the setting of Example 1, we set $F := f+g$, where $g : \mathbb{R}^3 \to (-\infty,\infty]$ is the indicator function of $B$, that is,

$$g(\mathbf{x}) := \begin{cases} 0, & \mathbf{x}\in B,\\ \infty, & \mathbf{x}\notin B.\end{cases}$$

The constrained problem can therefore be recast as

$$\min_{\mathbf{x}} F(\mathbf{x}). \tag{15}$$

The function $H(\mathbf{y},\mathbf{x}) := h(\mathbf{y},\mathbf{x})+g(\mathbf{y})$ is a consistent majorizer of $F$. Hence, the exact Algorithm 1 for solving (15) solves at each iteration $k$ the minimization problem

$$\min_{\mathbf{y}\in B} h(\mathbf{y},\mathbf{x}^k) \equiv \min_{\mathbf{y}} H(\mathbf{y},\mathbf{x}^k),$$

which amounts to solving the three univariate minimization problems

$$\begin{aligned}\min\nolimits_{y_1\in[-100,1000]}\ & 4x_1^kx_2^k(y_1-x_1^k)+2x_2^k(y_1-x_1^k)^2+2|x_1^k|(y_1-x_1^k)^2\\ & +(y_1-x_1^k)^4+5(x_3^k)^2(y_1-x_1^k)+5|x_3^k|(y_1-x_1^k)^2+2.5(y_1-x_1^k)^2,\\ \min\nolimits_{y_2\in[-78,802]}\ & 5y_2^3+2(x_1^k)^2(y_2-x_2^k)+2|x_1^k|(y_2-x_2^k)^2+(y_2-x_2^k)^2,\\ \min\nolimits_{y_3\in[-123,77]}\ & 8y_3^3+10x_1^kx_3^k(y_3-x_3^k)+5x_1^k(y_3-x_3^k)^2+5|x_3^k|(y_3-x_3^k)^2\\ & +2.5(y_3-x_3^k)^4.\end{aligned}$$

Each of the above problems can be solved by any solver that calculates roots of univariate polynomials, applied on each derivative. The obtained roots and the edge points of the intervals are the candidates among which the minimizers are those corresponding to the lowest function values.

### *4.2 Convergence Analysis of the IMM method*

We are now able to formulate the main convergence results of Algorithm IMM for a pre-determined fixed parameter $\gamma\in(0,1]$.

**Theorem 3 (Convergence of IMM (Algorithm 2)).** *Let $F$ be a proper, closed, directionally differentiable function. Consider the minimization problem (2) along* *with $h$ being a given consistent majorizer of $F$. Let $\gamma \in (0,1)$ be given. Suppose that Assumption 2 holds, and let $\{\mathbf{x}^k\}_{k\geq 0}$ be the sequence generated by the IMM method (Algorithm 2). Then the following properties hold.*

1. *For any $k = 0,1,\ldots,$*
$$F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) \geq \gamma \cdot S(\mathbf{x}^k).$$
2. *$F(\mathbf{x}^k) \geq F(\mathbf{x}^{k+1})$ for any $k = 0,1,\ldots,$ and $F(\mathbf{x}^k) > F(\mathbf{x}^{k+1})$ if $S(\mathbf{x}^k) > 0$.*
3. *Any accumulation point $\mathbf{x}^*$ of the sequence $\{\mathbf{x}^k\}_{k\geq 0}$ is strongly stationary, that is, $S(\mathbf{x}^*) = 0$.*
4. *For any $K \in \mathbb{N}$ and an accumulation point $\mathbf{x}^*$ of the sequence $\{\mathbf{x}^k\}_{k\geq 0}$ one has*
$$\text{(N)} \quad \min\{S(\mathbf{x}^0), S(\mathbf{x}^1), \ldots, S(\mathbf{x}^{K-1})\} \leq \frac{F(\mathbf{x}^0) - F(\mathbf{x}^*)}{\gamma \cdot K}.$$
5. *If $\gamma = 1$, and Assumption 3 holds, then properties 1-4 remain valid.*

*Proof.* 1. By (14) and property (A) of consistent majorizers,
$$F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) \geq F(\mathbf{x}^k) - h(\mathbf{x}^{k+1}, \mathbf{x}^k) \geq \gamma \cdot S(\mathbf{x}^k).$$

2. By part 1 of Lemma 3 $S(\mathbf{x}^k) \geq 0$, so the monotonicity follows directly by the previous assertion. A strict decrease when $S(\mathbf{x}^k) > 0$ is guaranteed since $\gamma > 0$.
3. Since the sequence $\{F(\mathbf{x}^k)\}$ is non-increasing, it either has a limit $F^* > -\infty$, or it tends to $-\infty$ as $k \to \infty$.
**Case 1.** $\{F(\mathbf{x}^k)\}_{k\geq 0}$ has a finite limit $F^*$. In this case we have $F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) \to F^* - F^* = 0$ as $k \to \infty$, and by the inequalities $F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) \geq \gamma \cdot S(\mathbf{x}^k) \geq 0$ it follows that
$$S(\mathbf{x}^k) \to 0.$$
Let $\{\mathbf{x}^{k_l}\}_{l\geq 1}$ a convergent subsequence of the generated sequence, and denote its limit by $\mathbf{x}^*$. Then, by parts 1 and 3 of Lemma 3 we have
$$0 \leq S(\mathbf{x}^*) \leq \liminf_{l\to\infty} S(\mathbf{x}^{k_l}) = \lim_{k\to\infty} S(\mathbf{x}^k) = 0,$$
and thus, $S(\mathbf{x}^*) = 0$.
**Case 2.** $F(\mathbf{x}^k) \to -\infty$ as $k \to \infty$. We will show by contradiction that the sequence $\{\mathbf{x}^k\}_{k\geq 0}$ has no accumulation points. Let $\{\mathbf{x}^{k_l}\}_{l\geq 1}$ a convergent subsequence, that is, $\mathbf{x}^{k_l} \to \mathbf{x}^*$ as $l \to \infty$. Then since $F$ is closed
$$\liminf_{l\to\infty} F(\mathbf{x}^{k_l}) \geq F(\mathbf{x}^*) > -\infty,$$
contradicting the fact that $F(\mathbf{x}^k) \to -\infty$. Thus, no accumulation points exist in such a case, and the result holds trivially.
4. Again, by part 1 of the current theorem and Lemma 3, part 1,
$$F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) \geq \gamma \cdot S(\mathbf{x}^k) \geq 0 \quad \forall k \geq 0,$$

for any $K \in \mathbb{N}$ we get by summing over $k = 0,\dots,K$ the inequalities

$$F(\mathbf{x}^0) - F(\mathbf{x}^*) \geq F(\mathbf{x}^0) - F(\mathbf{x}^K) = \sum_{k=0}^{K-1} (F(\mathbf{x}^k) - F(\mathbf{x}^{k+1})) \geq \gamma \cdot \sum_{k=0}^{K-1} S(\mathbf{x}^k)$$
$$\geq \gamma \cdot \min_{k \in \{0,\dots,K-1\}} \{S(\mathbf{x}^k)\} \cdot K,$$

where the leftmost inequality $F(\mathbf{x}^K) \geq F(\mathbf{x}^*)$ holds by the monotonicity of $\{F(\mathbf{x}^k)\}_{k \geq 0}$. Thus,

$$\min_{k \in \{0,\dots,K-1\}} \{S(\mathbf{x}^k)\} \cdot \gamma \cdot K \leq F(\mathbf{x}^0) - F(\mathbf{x}^*) \quad \forall K \in \mathbb{N},$$

from which (N) readily follows.

5. Under Assumption 3, the iterates where $\gamma = 1$ (Algorithm 1) are well-defined. The property $F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) \geq S(\mathbf{x}^k)$ is satisfied for any $k = 0, 1, \dots$ by part 2 of Lemma 3. The other properties follow by the same arguments as in the case $0 < \gamma < 1$.

□

At this point, it seems unclear how to verify condition (14) in cases where the inexact method is employed ($\gamma < 1$) since $S(\mathbf{x}^k)$ is not actually computed. In the next section we discuss some specific models on which Algorithm 2 is shown to be implementable. When $\gamma = 1$, assuming that exact minimizers of $h_{\mathbf{x}^k}$ are computable, the implementation of Algorithm 1 is clear, up to properly choosing a stopping criteria, and deciding on a rule for determining which minimizer of $h_{\mathbf{x}^k}$ should be taken when multiple minimizers exist.

*Example 14 (Example 13 revisited).* We implemented the MM method (Algorithm 1) on problem (15) with 100 independent initial guesses $\mathbf{x}^0$ being randomly generated from a uniform distribution in $B$. For the sake of comparison, we also implemented the *gradient projection (GP)* method on the same 100 initial points. The GP is a first-order optimization method, whose accumulation points are guaranteed to be stationary; see e.g., [4, Section 9.4]. If a constant stepsize $t > 0$ is used, the general update step of the GP method is given by

$$\mathbf{x}^{k+1} = P_B(\mathbf{x}^k - t\nabla f(\mathbf{x}^k)),$$

where $P_B$ is the orthogonal projection operator on the box $B$. We roughly pre-estimated the smoothness parameter $L$, which is a positive number satisfying $\|\nabla f(\mathbf{y}) - \nabla f(\mathbf{x})\| \leq L\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in B$. We used the estimate $L \approx 7250$, and then set the constant stepsize $t := 1/L$. We stopped both algorithms (MM and GP) at the first iterate $k$ for which the inequality

$$F(\mathbf{x}^k) - F(\mathbf{x}^{k+1}) < 10^{-7}$$

held true. In the MM method (Algorithm 1), whenever multiple minimizers were found for a univariate subproblem (in a variable $y_i$), we took the minimizer whose distance from $x_i^k$ was maximal.

We compared the results of the two methods. To test the results in terms of the problem's objective, we also computed the global optimal value of (15), by applying the solver SCIP (see [1] and references therein). The solver found the global minimizer $\mathbf{x}^* = (1000, -78, 0)^T$ with an optimal value $F^* = -158372760$. Table 3 presents the following results regarding the 100 runs of each method (with the same 100 initial points).

- P-Glo: number of runs (out of 100) in which the method reached a global optimal solution, that is, with value $F^*$.
- IT-min, IT-max, IT-ave: minimal, maximal and average numbers of iterations (among 100 runs) till the method stopped.
- ITG-min, ITG-max, ITG-ave: minimal, maximal and average iteration numbers only among the runs in which a global solution was reached.

**Table 3** Chances of reaching a global solution and iteration numbers of GP and MM.

| Method | P-Glo | IT-min | IT-max | IT-ave | ITG-min | ITG-max | ITG-ave |
|---|---|---|---|---|---|---|---|
| GP | 56 | 827 | 69079 | 5353.84 | 1360 | 30371 | 4524.3 |
| MM | 75 | 3 | 42 | 18.53 | 4 | 42 | 20.81 |

In addition, in 28 of 100 runs the MM method yielded a final output with a better (lower) objective value than the GP, while in **all** those 100 runs its final output was not worse than GP in objective value. Moreover, for each of the 100 final outputs of the GP method we also tested the performance of MM initialized at that output. In 27 cases we found that a run of MM initialized at those points yielded a better final output (a vector having a lower objective value).

*Remark 2.* It might be possible that the better chances of achieving a global optimum by MM are related to the phenomena demonstrated in Example 12 where it was demonstrated that strongly stationary points can be much less common than standard stationary points. While accumulation points of Algorithm 1 are always strongly stationary by part 3 of Theorem 3, those obtained by first-order methods such as Algorithm GP are only guaranteed to be (standard) stationary points. Since by Lemma 4 global minimizers of (2) must be strongly stationary points, Algorithm 1 seems to be more likely to reach one of them.

# 5 Applying the IMM Method on Compositions of Strongly Convex Majorizers

The subproblem that is being approximately solved at each iteration $k$ of the IMM method for solving (2) is

$$(\mathrm{H}_k) \quad \min_{\mathbf{y}} H(\mathbf{y},\mathbf{x}^k),$$

where $H$ denotes a consistent majorizer of $F$. In this section we treat the case where $F$ is given as a composition $F = \varphi \circ \mathbf{f}$, which is the setting described in Section 2.4. We introduce an algorithm that we relate to as the "inner" method. For each iterate $k$, it computes an approximate minimizer of problem $(\mathrm{H}_k)$ within the tolerance required to ensure the convergence properties established in Theorem 3. Let

$$F(\mathbf{x}) = \varphi(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})), \tag{16}$$

$$H(\mathbf{y},\mathbf{x}) = \varphi(h_1(\mathbf{y},\mathbf{x}), h_2(\mathbf{y},\mathbf{x}), \ldots, h_m(\mathbf{y},\mathbf{x})), \tag{17}$$

where $f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ and $\varphi$ satisfy Assumption 1. For any $i \in \{1,2,\ldots,m\}$ let the function $h_i$ be a consistent majorizer of $f_i$ which satisfies Assumption 2. By Theorem 2, the function $H$ is a consistent majorizer of $F$, and since all the functions $h_i$ satisfy Assumption 2, it follows by the monotonicity of $\varphi$, that $H = \varphi \circ \mathbf{f}$ also satisfies Assumption 2. Recalling that by Assumption 1 we have $\varphi = \sigma_C$ for some convex compact set $C \subseteq \mathbb{R}^m_+$, we further use the following notation. Denote for any given $\lambda \in C$ and $\mathbf{x} \in \mathbb{R}^n$

$$q(\lambda,\mathbf{x}) := \min_{\mathbf{y}} \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x}),$$

where $\mathbf{h}(\mathbf{y},\mathbf{x}) \equiv (h_1(\mathbf{y},\mathbf{x}), h_2(\mathbf{y},\mathbf{x}), \ldots, h_m(\mathbf{y},\mathbf{x}))^T$. It should be noted that the minimum in the definition of $q(\lambda,\mathbf{x})$ is finite for any $\mathbf{x} \in \mathbb{R}^n$ as $h_i$ satisfies Assumption 2 for all $i \in \{1,\ldots,m\}$, and $\lambda \in \mathbb{R}^m_+$.

For any given $\mathbf{x} \in \mathbb{R}^n$ we consider the two functions $H_{\mathbf{x}}(\mathbf{y}) \equiv H(\mathbf{y},\mathbf{x})$ and $q_{\mathbf{x}}(\lambda) \equiv q(\lambda,\mathbf{x})$ as "primal" and "dual", respectively. In addition, we denote

$$Q_{\mathbf{x}} := \max_{\lambda \in C} q_{\mathbf{x}}(\lambda),$$

and recall that by (17) and Assumption 1(A) $H(\mathbf{y},\mathbf{x}) = \max_{\lambda \in C} \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x})$. In the setting of this section, for any $\mathbf{x} \in \mathbb{R}^n$ one has $\mathrm{dom}(H_{\mathbf{x}}) = \mathrm{dom}(F) = \mathbb{R}^n$. The following theorem provides the theoretical basis of the proposed inner method.

**Theorem 4 (Strong duality).** *Let $C \subseteq \mathbb{R}^m_+$ be nonempty, convex and compact, $f_i : \mathbb{R}^n \to \mathbb{R}$ be closed and directionally differentiable for all $i \in \{1,\ldots,m\}$, and let $F$ be defined by (16), with $\varphi \equiv \sigma_C$. Assume that $h_i$ is a consistent majorizer of $f_i$ which satisfies Assumption 2 for any $i \in \{1,\ldots,m\}$. Assume further that for any $i \in \{1,\ldots,m\}$ and $\mathbf{x} \in \mathbb{R}^n$ the function $\mathbf{y} \mapsto h_i(\mathbf{y},\mathbf{x})$ is convex. Let $H$ be defined by (17). Then for any $\mathbf{x} \in \mathbb{R}^n$ it holds that*

$$Q_{\mathbf{x}} = M_{\mathbf{x}} \quad \left[ := \min_{\mathbf{y}} H(\mathbf{y}, \mathbf{x}) \right].$$

*Proof.* Let $\mathbf{x} \in \mathbb{R}^n$ be given. We utilize the classical min-max theorem of Sion [22]. The set $C$ is convex and compact; $\mathbb{R}^n$ is convex and closed. For each $\lambda \in C \subseteq \mathbb{R}^m_+$ the function $\lambda^T \mathbf{h}(\mathbf{y}, \mathbf{x})$ is convex in $\mathbf{y}$ as a nonnegative linear combination of convex functions, and for each $\mathbf{y} \in \mathbb{R}^n$ the function $\lambda^T \mathbf{h}(\mathbf{y}, \mathbf{x})$ is concave in $\lambda$ as an affine function. Thus, by Sion's min-max theorem [22, Theorem 3.4], it follows that

$$M_{\mathbf{x}} = \min_{\mathbf{y}} H(\mathbf{y}, \mathbf{x}) = \min_{\mathbf{y}} \max_{\lambda \in C} \lambda^T \mathbf{h}(\mathbf{y}, \mathbf{x}) = \max_{\lambda \in C} \min_{\mathbf{y}} \lambda^T \mathbf{h}(\mathbf{y}, \mathbf{x}) = \max_{\lambda \in C} q(\lambda, \mathbf{x}) = Q_{\mathbf{x}}.$$

$\square$

The equality $Q_{\mathbf{x}} = M_{\mathbf{x}}$ enables to formulate a criterion ensuring that a tested vector $\tilde{\mathbf{x}} \in \mathbb{R}^n$ is an approximate $\gamma$-vector at $\mathbf{x}$.

**Lemma 5 (stopping criteria).** *Consider problem (2), where $F$ is given by (16) for $f_1, \ldots, f_m$ and $\varphi = \sigma_C$ that satisfy Assumption 1. Let $H$ be defined by (17) for some consistent majorizers $h_1, \ldots, h_m$ of $f_1, \ldots, f_m$, respectively, which satisfy Assumption 2. Let $\mathbf{x} \in \mathbb{R}^n$, and $\gamma \in (0, 1]$. Assume that a vector $\tilde{\mathbf{x}} \in \mathbb{R}^n$ and a vector $\tilde{\lambda} \in C$ satisfy the inequality*

$$H_{\mathbf{x}}(\tilde{\mathbf{x}}) - q_{\mathbf{x}}(\tilde{\lambda}) \leq \frac{1 - \gamma}{\gamma} \left( F(\mathbf{x}) - H_{\mathbf{x}}(\tilde{\mathbf{x}}) \right). \tag{18}$$

*Then $\tilde{\mathbf{x}}$ is a $\gamma$-approximate vector at $\mathbf{x}$:*

$$F(\mathbf{x}) - F(\tilde{\mathbf{x}}) \geq F(\mathbf{x}) - H_{\mathbf{x}}(\tilde{\mathbf{x}}) \geq \gamma \cdot S(\mathbf{x}).$$

*Proof.* By Theorem 4 and the definitions of $M_{\mathbf{x}}, Q_{\mathbf{x}}$, we obtain

$$H_{\mathbf{x}}(\tilde{\mathbf{x}}) \geq M_{\mathbf{x}} = Q_{\mathbf{x}} \geq q_{\mathbf{x}}(\tilde{\lambda}).$$

Thus, along with (18),

$$H_{\mathbf{x}}(\tilde{\mathbf{x}}) - M_{\mathbf{x}} \leq H_{\mathbf{x}}(\tilde{\mathbf{x}}) - q_{\mathbf{x}}(\tilde{\lambda}) \leq \frac{1 - \gamma}{\gamma} \left( F(\mathbf{x}) - H_{\mathbf{x}}(\tilde{\mathbf{x}}) \right).$$

Rearrangement yields

$$-\gamma M_{\mathbf{x}} \leq (1 - \gamma) F(\mathbf{x}) - H_{\mathbf{x}}(\tilde{\mathbf{x}}),$$

or, equivalently,

$$H_{\mathbf{x}}(\tilde{\mathbf{x}}) \leq M_{\mathbf{x}} + (1 - \gamma)(F(\mathbf{x}) - M_{\mathbf{x}}) = M_{\mathbf{x}} + (1 - \gamma) \cdot S(\mathbf{x}).$$

By property (A) of consistent majorizers along with the definition of $S$, it follows that

$$\begin{aligned} F(\mathbf{x}) - F(\tilde{\mathbf{x}}) &\geq F(\mathbf{x}) - H_{\mathbf{x}}(\tilde{\mathbf{x}}) \\ &\geq F(\mathbf{x}) - M_{\mathbf{x}} - (1-\gamma)\cdot S(\mathbf{x}) \\ &= S(\mathbf{x}) - (1-\gamma)\cdot S(\mathbf{x}) = \gamma \cdot S(\mathbf{x}). \end{aligned}$$

□

Lemma 5 covers the result of part 2 of Lemma 3 for $\gamma = 1$; in this case $\tilde{\mathbf{x}}$ is an exact minimizer of $H_{\mathbf{x}}$. Notice that the verification of (18) does not require to know the value $S(\mathbf{x})$. To complete the description of the implementation of Algorithm 2 with $\gamma < 1$ in this case we need to explain how we calculate vectors $\tilde{\lambda} \in C$ and $\tilde{\mathbf{x}} \in \mathbb{R}^n$ satisfying (18). We further assume two additional assumptions on $C$ and on the consistent majorizers $h_1, \ldots, h_m$.

**Assumption 4 (Strongly convex components).** *There exists a number $\sigma > 0$, such that for any $i \in \{1,\ldots,m\}$ and $\mathbf{x} \in \mathbb{R}^n$ the function $\mathbf{y} \mapsto h_i(\mathbf{y},\mathbf{x})$ is $\sigma$-strongly convex.*

**Assumption 5 (Nondegeneracy of the composition).** $\mathbf{0} \notin C$.

Assumptions 4 and 5 guarantee that for each $\lambda \in C$ and $\mathbf{x} \in \mathbb{R}^n$ the function $\mathbf{y} \mapsto \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x})$ is strongly convex with convexity parameter uniformly bounded away from zero over $\lambda \in C$. Indeed, $\lambda^T \mathbf{h}(\mathbf{y},\mathbf{x})$ is a nonnegative linear combination of $\sigma$-strongly convex functions, where at least one coefficient is positive. The convexity parameter is lower bounded by $\sigma \cdot \min_{\lambda \in C} \lambda^T \mathbf{e}$, where the latter quantity is positive since $C \subseteq \mathbb{R}^m_+$ is compact, and $\mathbf{0} \notin C$. In particular, they imply that $H$ satisfies Assumption 2. We further note that Assumption 4 is not very restrictive, as its does not relate for the model itself, but rather only to the constructed majorizers $h_1, \ldots, h_m$.

The above two assumptions are needed for establishing smoothness properties on $q_{\mathbf{x}}$, which in turn, enable to apply some fast first-order optimization methods on the dual problem $\max_{\lambda \in C} q_{\mathbf{x}}(\lambda)$. Such a method can be utilized to compute an approximate minimizer of the primal problem $\min_{\mathbf{y}} H_{\mathbf{x}}(\mathbf{y})$. For $\mathbf{x} = \mathbf{x}^k$ the latter is the subproblem needed to be solved approximately at the $k$th iterate of the IMM method (Algorithm 2). The following shows how two vectors $\tilde{\mathbf{x}}$ and $\tilde{\lambda}$ satisfying (18) can be obtained given a vector $\tilde{\lambda}$ whose corresponding objective value is close in some sense to $Q_{\mathbf{x}}$.

**Proposition 1.** *Let $C \subseteq \mathbb{R}^m_+$ be compact and convex set, and let $f_1, \ldots, f_m$ and $\varphi \equiv \sigma_C$ satisfy Assumption 1. Let $F$ be defined by (16) and $H$ be defined by (17) for some consistent majorizers $h_1, \ldots, h_m$ of $f_1, \ldots, f_m$. Suppose that Assumptions 4 and 5 hold. Let $\mathbf{x} \in \mathbb{R}^n$ be a point satisfying $S(\mathbf{x}) > 0$. Denote $l_C := \min_{\lambda \in C} \lambda^T \mathbf{e}$. For any $\lambda \in C$ let $\mathbf{y}_\lambda = \operatorname*{argmin}_{\mathbf{y}} \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x})$.*

*1. For any $\lambda \in C$ the inequality*

$$\frac{\sigma \cdot l_C}{2} \|\mathbf{y}^* - \mathbf{y}_\lambda\|^2 \leq Q_{\mathbf{x}} - q_{\mathbf{x}}(\lambda)$$

*holds, where $\mathbf{y}^* \in \operatorname*{argmin}_{\mathbf{y}} H_{\mathbf{x}}(\mathbf{y})$.*

*2. For any* $\gamma \in (0,1)$ *there exists some* $\varepsilon_\gamma > 0$ *such that if* $Q_{\mathbf{x}} - q_{\mathbf{x}}(\lambda) < \varepsilon_\gamma$*, then*

$$H_{\mathbf{x}}(\mathbf{y}_\lambda) - q_{\mathbf{x}}(\lambda) < \frac{1-\gamma}{\gamma}(F(\mathbf{x}) - H_{\mathbf{x}}(\mathbf{y}_\lambda)).$$

*Proof.* To show part 1, first we denote for all $\lambda \in C$ and $\mathbf{y} \in \mathbb{R}^n$

$$k_{\mathbf{x},\lambda}(\mathbf{y}) := \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x}).$$

By Assumption 5, $l_C > 0$. The function $k_{\mathbf{x},\lambda}$ is $(\sigma \cdot l_C)$-strongly convex by Assumption 4. Thus, $\mathbf{y}_\lambda$ exists and is unique, and for any $\mathbf{y} \in \mathbb{R}^n$ it holds that

$$k_{\mathbf{x},\lambda}(\mathbf{y}) \geq k_{\mathbf{x},\lambda}(\mathbf{y}_\lambda) + \frac{(\sigma \cdot l_C)}{2}\|\mathbf{y} - \mathbf{y}_\lambda\|^2. \tag{19}$$

In addition,

$$k_{\mathbf{x},\lambda}(\mathbf{y}^*) = \lambda^T \mathbf{h}(\mathbf{y}^*,\mathbf{x}) \leq \max_{\lambda \in C} \lambda^T \mathbf{h}(\mathbf{y}^*,\mathbf{x}) = H_{\mathbf{x}}(\mathbf{y}^*) = Q_{\mathbf{x}}, \tag{20}$$

where the last equality is the result of Theorem 4 (Assumption 2 holds for all $h_i$ by Assumption 4). Therefore, combining (19), (20) and the fact that $q_{\mathbf{x}}(\lambda) = \min_{\mathbf{y}} k_{\mathbf{x},\lambda}(\mathbf{y}) = k_{\mathbf{x},\lambda}(\mathbf{y}_\lambda)$, we conclude that

$$\frac{(\sigma \cdot l_C)}{2}\|\mathbf{y}^* - \mathbf{y}_\lambda\|^2 \leq k_{\mathbf{x},\lambda}(\mathbf{y}^*) - k_{\mathbf{x},\lambda}(\mathbf{y}_\lambda) \leq Q_{\mathbf{x}} - k_{\mathbf{x},\lambda}(\mathbf{y}_\lambda) = Q_{\mathbf{x}} - q_{\mathbf{x}}(\lambda).$$

Let us now show part 2. Since $S(\mathbf{x}) > 0$, the vector $\mathbf{x}$ is not a minimizer of $H_{\mathbf{x}}$, and thus,

$$F(\mathbf{x}) = H_{\mathbf{x}}(\mathbf{x}) > H_{\mathbf{x}}(\mathbf{y}^*).$$

Denote

$$\varepsilon_1 := \frac{1}{2}(F(\mathbf{x}) - H_{\mathbf{x}}(\mathbf{y}^*)).$$

Since $H_{\mathbf{x}}$ is convex with $\mathrm{dom}(H_{\mathbf{x}}) = \mathbb{R}^n$, it is continuous at $\mathbf{y}^*$, so there exists $\delta_H > 0$ such that if $\|\mathbf{y} - \mathbf{y}^*\| < \delta_H$, then

$$|H_{\mathbf{x}}(\mathbf{y}) - H_{\mathbf{x}}(\mathbf{y}^*)| < \varepsilon_1.$$

In particular, for all such $\mathbf{y}$ it holds that

$$\begin{aligned} H_{\mathbf{x}}(\mathbf{y}) &< H_{\mathbf{x}}(\mathbf{y}^*) + \frac{1}{2}(F(\mathbf{x}) - H_{\mathbf{x}}(\mathbf{y}^*)) \\ &= \frac{1}{2}F(\mathbf{x}) + \frac{1}{2}H_{\mathbf{x}}(\mathbf{y}^*) \\ &= \frac{1}{2}F(\mathbf{x}) + \frac{1}{2}F(\mathbf{x}) - \varepsilon_1 = F(\mathbf{x}) - \varepsilon_1, \end{aligned}$$

or, equivalently,

$$F(\mathbf{x}) - H_{\mathbf{x}}(\mathbf{y}) > \varepsilon_1. \tag{21}$$

Denote $\varepsilon_{\gamma,1} := \frac{1-\gamma}{2\gamma}\varepsilon_1$, and let $\delta_{H,\gamma} > 0$ be such that if $\|\mathbf{y}-\mathbf{y}^*\| < \delta_{H,\gamma}$, then

$$|H_{\mathbf{x}}(\mathbf{y}) - H_{\mathbf{x}}(\mathbf{y}^*)| < \frac{\varepsilon_{\gamma,1}}{2}.$$

Now, if $\lambda \in C$ satisfies

$$Q_{\mathbf{x}} - q_{\mathbf{x}}(\lambda) < \varepsilon_\gamma := \min\left\{\frac{\sigma \cdot l_C}{2}\delta_{H,\gamma}^2, \frac{1-\gamma}{2\gamma}\varepsilon_1\right\}, \tag{22}$$

then by part 1 it follows that $\|\mathbf{y}^* - \mathbf{y}_\lambda\| < \delta_{H,\gamma}$, and thus,

$$|H_{\mathbf{x}}(\mathbf{y}_\lambda) - H_{\mathbf{x}}(\mathbf{y}^*)| < \varepsilon_{\gamma,1}.$$

In particular,

$$H_{\mathbf{x}}(\mathbf{y}_\lambda) - Q_{\mathbf{x}} = H_{\mathbf{x}}(\mathbf{y}_\lambda) - H_{\mathbf{x}}(\mathbf{y}^*) < \varepsilon_{\gamma,1} = \frac{1-\gamma}{2\gamma}\varepsilon_1,$$

and by (22),

$$Q_{\mathbf{x}} - q_{\mathbf{x}}(\lambda) < \frac{1-\gamma}{2\gamma}\varepsilon_1.$$

Summation of the above two inequalities yields

$$H_{\mathbf{x}}(\mathbf{y}_\lambda) - q_{\mathbf{x}}(\lambda) < \frac{1-\gamma}{\gamma}\varepsilon_1 \overset{(21)}{<} \frac{1-\gamma}{\gamma}(F(\mathbf{x}) - H_{\mathbf{x}}(\mathbf{y}_\lambda)).$$

□

By Proposition 1, if $\lambda \in C$ satisfies $Q_{\mathbf{x}} - q_{\mathbf{x}}(\lambda) < \varepsilon_\gamma$, one can choose $\tilde{\mathbf{x}} := \mathbf{y}_\lambda$ and $\tilde{\lambda} := \lambda$, and (18) holds. This means in particular that any iterative method for solving the dual problem

$$\max_{\lambda \in C} q_{\mathbf{x}}(\lambda) \tag{23}$$

whose generated sequence $\{\lambda^k\}_{k\geq 0}$ satisfies

$$q_{\mathbf{x}}(\lambda^k) \to Q_{\mathbf{x}} \tag{24}$$

will eventually produce two vectors $\tilde{\mathbf{x}} := \mathbf{y}_{\lambda^k}$ and $\tilde{\lambda} := \lambda^k$ that satisfy the stopping criteria (18).

One possible method that satisfies the convergence condition (24) is a variant of the *fast gradient projection (FGP)* algorithm, described in [24, p.12, eq.37-39] and [2, Section 8]. To apply the FGP method we first need to establish some smoothness properties on the dual function $q_{\mathbf{x}}$ over $C$. Unlike other accelerated gradient projection schemes, e.g., FISTA [6], the proposed FGP algorithm does not evaluate the gradient $\nabla q_{\mathbf{x}}$ on vectors that are not included in $C$. Thus, for the FGP method to be well defined, and for its convergence properties to hold, we need to show that for any $\mathbf{x} \in \mathbb{R}^n$, there exists some $K_{\mathbf{x}} > 0$, such that the function $q_{\mathbf{x}}$ is $K_{\mathbf{x}}$-smooth over $C$. The smoothness of $q_{\mathbf{x}}$ is proven in the following proposition. Here we make an assumption that replaces Assumptions 4 and 5 and is somewhat stronger; we essentially assume strong convexity of the function $\mathbf{y} \mapsto \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x})$ for any $\lambda$ that belongs to an open set containing $C$, and not just to $C$. Note that in cases where this assumption cannot be verified (but rather only Assumptions 4 and 5), some other method should be applied, as $q_{\mathbf{x}}$ might not be smooth at boundary points of $C$.

**Proposition 2.** *Let $F$ be defined by (16) for some $f_1,\dots,f_m$ and $\varphi = \sigma_C$ which satisfy Assumption 1. Let $\mathbf{x} \in \mathbb{R}^n$ be fixed. Let $h_1,\dots,h_m$ be consistent majorizers of $f_1,\dots,f_m$ respectively which satisfy Assumption 2. Assume that for a given $\sigma > 0$ the function $\mathbf{y} \mapsto \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x})$ is $\sigma$-strongly convex for any $\lambda \in \tilde{C}$, where $\tilde{C}$ is an open set satisfying $C \subseteq \tilde{C}$. Then the following properties hold.*

1. *[3, Thm 6.3.3]. For any $\lambda \in \tilde{C}$ the function $q_{\mathbf{x}}$ is differentiable at $\lambda$ and*
$$\nabla q_{\mathbf{x}}(\lambda) = \mathbf{h}(\mathbf{y}_\lambda,\mathbf{x}).$$
2. *If, in addition, the mapping $\mathbf{h}$ is $C^1$, then there exists $L_{\mathbf{x}} > 0$ such that*
$$\|\nabla q_{\mathbf{x}}(\bar{\lambda}) - \nabla q_{\mathbf{x}}(\lambda)\| \le \frac{L_{\mathbf{x}}^2}{\sigma}\|\bar{\lambda} - \lambda\|$$
*for all $\bar{\lambda},\lambda \in C$.*

*Proof.* A full proof of part 1 can be found in [3, p.278-9] where it is assumed that the domain for the minimization in $\mathbf{y}$ is compact, but this assumption is used only for the establishment of the existence of a minimizer $\mathbf{y}_\lambda$ for each $\lambda$. Under the assumption of strong convexity, such a minimizer always exists even over $\mathbb{R}^n$. The uniqueness of such a minimizer also follows by the strong convexity of $\mathbf{y} \mapsto \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x})$.

Let us prove the second part. Since for any $\lambda \in C$ the function $\mathbf{y} \mapsto k_{\mathbf{x},\lambda}(\mathbf{y}) := \lambda^T \mathbf{h}(\mathbf{y},\mathbf{x})$ is $\sigma$-strongly convex, it follows that for all $\lambda,\bar{\lambda} \in C$ one has

$$\begin{aligned} k_{\mathbf{x},\lambda}(\mathbf{y}_{\bar{\lambda}}) &\ge k_{\mathbf{x},\lambda}(\mathbf{y}_\lambda) + \frac{\sigma}{2}\|\mathbf{y}_{\bar{\lambda}} - \mathbf{y}_\lambda\|^2 \\ &= \lambda^T \mathbf{h}(\mathbf{y}_\lambda,\mathbf{x}) + \frac{\sigma}{2}\|\mathbf{y}_{\bar{\lambda}} - \mathbf{y}_\lambda\|^2 \\ &= \lambda^T \nabla q_{\mathbf{x}}(\lambda) + \frac{\sigma}{2}\|\mathbf{y}_{\bar{\lambda}} - \mathbf{y}_\lambda\|^2, \end{aligned}$$

where the last equality is valid by part 1, stating that $q_{\mathbf{x}}$ is differentiable in $\tilde{C} \supseteq C$ with $\nabla q_{\mathbf{x}}(\lambda) = \mathbf{h}(\mathbf{y}_\lambda,\mathbf{x})$. In addition, as

$$k_{\mathbf{x},\lambda}(\mathbf{y}_{\bar{\lambda}}) = \lambda^T \mathbf{h}(\mathbf{y}_{\bar{\lambda}},\mathbf{x}) = \lambda^T \nabla q_{\mathbf{x}}(\bar{\lambda}),$$

we get

$$\lambda^T \nabla q_{\mathbf{x}}(\bar{\lambda}) \ge \lambda^T \nabla q_{\mathbf{x}}(\lambda) + \frac{\sigma}{2}\|\mathbf{y}_{\bar{\lambda}} - \mathbf{y}_\lambda\|^2.$$

Similarly, by changing roles of $\lambda$ and $\bar{\lambda}$ we also get

$$\bar{\lambda}^T \nabla q_{\mathbf{x}}(\lambda) \geq \bar{\lambda}^T \nabla q_{\mathbf{x}}(\bar{\lambda}) + \frac{\sigma}{2}\|\mathbf{y}_{\bar{\lambda}} - \mathbf{y}_{\lambda}\|^2.$$

Summing the last two inequalities yields (after rearrangement)

$$(\lambda - \bar{\lambda})^T (\nabla q_{\mathbf{x}}(\bar{\lambda}) - \nabla q_{\mathbf{x}}(\lambda)) \geq \sigma \|\mathbf{y}_{\bar{\lambda}} - \mathbf{y}_{\lambda}\|^2. \tag{25}$$

In addition, since $\mathbf{h} \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, its Jacobian matrix is continuous, and by Weierstrass theorem its norm is bounded on compact sets. We will now show that the set $U := \{\mathbf{y}_\lambda : \lambda \in C\} \subseteq \mathbb{R}^n$ is bounded. By Assumption 2, the monotonicity of $\varphi$ and Theorem 4, $Q_{\mathbf{x}} = M_{\mathbf{x}}$ is finite. In addition, by part 1, $q_{\mathbf{x}}$ is differentiable over $\tilde{C}$ and thus, also continuous over $\tilde{C}$. Thus, the term $Q_{\mathbf{x}} - q_{\mathbf{x}}(\lambda)$ is bounded, and by Proposition 1, so is $\|\mathbf{y}^* - \mathbf{y}_\lambda\|$, and hence, $U$ is bounded. In particular, $\bar{U} := \mathrm{cl}(U)$ is compact. Denote

$$L_{\mathbf{x}} := \max_{\mathbf{y} \in \bar{U}} \|\mathbf{J}_{\mathbf{h}}(\mathbf{y})\| < \infty,$$

where $\mathbf{J}_{\mathbf{h}}(\mathbf{y}, \mathbf{x})$ denotes the Jacobian matrix of the mapping $\mathbf{y} \mapsto \mathbf{h}(\mathbf{y}, \mathbf{x})$. Then, for any $\lambda, \bar{\lambda} \in C$ we have

$$\|\nabla q_{\mathbf{x}}(\bar{\lambda}) - \nabla q_{\mathbf{x}}(\lambda)\| = \|\mathbf{h}(\mathbf{y}_{\bar{\lambda}}) - \mathbf{h}(\mathbf{y}_\lambda)\| \leq L_{\mathbf{x}} \|\mathbf{y}_{\bar{\lambda}} - \mathbf{y}_\lambda\|. \tag{26}$$

By (25), (26) and the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}
\|\bar{\lambda} - \lambda\| \cdot \|\nabla q_{\mathbf{x}}(\bar{\lambda}) - \nabla q_{\mathbf{x}}(\lambda)\| &\geq (\lambda - \bar{\lambda})^T (\nabla q_{\mathbf{x}}(\bar{\lambda}) - \nabla q_{\mathbf{x}}(\lambda)) \\
&\geq \sigma \|\mathbf{y}_{\bar{\lambda}} - \mathbf{y}_\lambda\|^2 \\
&\geq \frac{\sigma}{L_{\mathbf{x}}^2} \|\nabla q_{\mathbf{x}}(\bar{\lambda}) - \nabla q_{\mathbf{x}}(\lambda)\|^2,
\end{aligned}$$

or, equivalently,

$$\|\nabla q_{\mathbf{x}}(\bar{\lambda}) - \nabla q_{\mathbf{x}}(\lambda)\| \leq \frac{L_{\mathbf{x}}^2}{\sigma} \|\bar{\lambda} - \lambda\|.$$

□

For any $\mathbf{x} \in \mathbb{R}^n$, the function $q_{\mathbf{x}}$ is $(L_{\mathbf{x}}^2/\sigma)$-smooth over $C$ by Proposition 2, and concave as a minimum of linear functions. Following [24], Algorithm 3 below explicitly describes the FGP method with a constant stepsize setting, applied on the dual problem $\max_{\lambda \in C} q_{\mathbf{x}}(\lambda)$ for a given $\mathbf{x}$, where for simplicity we denote $L_q := \frac{L_{\mathbf{x}}^2}{\sigma}$. Algorithm 3 is therefore guaranteed to generate a sequence $\{\lambda_l\}_{l \geq 0}$ such that $q_{\mathbf{x}}(\lambda_l) \to Q_{\mathbf{x}}$ as $l \to \infty$. See [24] for the complete details. As a result, for a large enough $l$, it reaches vectors $\tilde{\lambda} := \lambda_l \in C$ and $\tilde{\mathbf{x}} := \mathbf{y}_{\lambda_l}$, such that (18) is satisfied.

**Algorithm 3.** *FGP for finding an approximate solution $\tilde{\lambda}$ for the dual problem.*

- *Pick arbitrary $\lambda^0, \eta^0 \in C$ and $\theta_0 = 1$.*
- *For $l = 0, 1, \ldots,$ until a stopping criterion (18) holds for $\tilde{\lambda} := \lambda^l$ and $\tilde{\mathbf{x}} := \mathbf{y}_{\tilde{\lambda}}$, compute*

$$\begin{aligned}
\mu^l &= (1-\theta_l)\lambda^l + \theta_l \eta^l, \\
\eta^{l+1} &= P_C\left(\eta^l + \frac{1}{\theta_l L_q}\nabla q_{\mathbf{x}}(\mu^l)\right), \\
\lambda^{l+1} &= (1-\theta_l)\lambda^l + \theta_l \eta^{l+1}, \\
\theta_{l+1} &= \frac{1}{2}\left(\sqrt{\theta_l^4 + 4\theta_l^2} - \theta_l^2\right).
\end{aligned}$$

*Example 15 (sparse source localization).* Consider a scenario where one seeks to find the best approximate solution for the system

$$\|\mathbf{x} - \mathbf{a}_i\| \approx d_i, \quad i = 1, \ldots, m,$$

where $\mathbf{x} \in \mathbb{R}^n$ is the unknown location of a radiating source, and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n$ are $m$ different known anchor points in $\mathbb{R}^n$. At each anchor point there exists a sensor that measures the distance from the source, but returns just a noisy measurement $d_i$, where in most applications $n = 2$ or $n = 3$. The system can be reformulated as an optimization problem called *sparse source localization (SSL)*, where the approximation is in terms of the minimum sum of absolute values of the errors in the **squared** measurements:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \left\{ F(\mathbf{x}) := \sum_{i=1}^m \left| \|\mathbf{x} - \mathbf{a}_i\|^2 - d_i^2 \right| \right\}.$$

As in Example 11, the objective $F$ can be rewritten by $F = \varphi \circ \mathbf{f}$, where $\varphi \equiv \sigma_C$ with $C := (\Delta_2)^m \subseteq \mathbb{R}_+^{2m}$, and for all $i \in \{1, \ldots, m\}$

$$f_{2i-1}(\mathbf{x}) := \|\mathbf{x} - \mathbf{a}_i\|^2 - d_i^2, \qquad f_{2i}(\mathbf{x}) := -\|\mathbf{x} - \mathbf{a}_i\|^2 + d_i^2.$$

Note that $f_{2i-1}$ is strongly convex, and $f_{2i}$ is concave. Thus, a consistent majorizer of $F$ can be defined by $H(\mathbf{y}, \mathbf{x}) := \varphi \circ \mathbf{h}(\mathbf{y}, \mathbf{x})$, where for all $i \in \{1, \ldots, m\}$ we define

$$h_{2i-1}(\mathbf{y}, \mathbf{x}) := f_{2i-1}(\mathbf{y}),$$

and

$$h_{2i}(\mathbf{y}, \mathbf{x}) := f_{2i}(\mathbf{x}) + \nabla f_{2i}(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \eta \|\mathbf{y} - \mathbf{x}\|^2$$

for some $\eta > 0$.

For all $i$, the functions $h_{2i-1}$ and $h_{2i}$ are both strongly convex in $\mathbf{y}$ for any $\mathbf{x} \in \mathbb{R}^n$ with parameters 2 and $2\eta$, respectively, and thus, Assumption 4 holds with the choice $\sigma := \min\{2, 2\eta\}$. In addition, Assumption 5 also holds, as $\mathbf{0} \notin C$. In this case, even the assumption required for the smoothness of the dual function in Proposition 2 is satisfied. Indeed, one can take

$$\tilde{C} := \bigcup_{\lambda \in C} \{\tilde{\lambda} \in \mathbb{R}^{2m} : \|\tilde{\lambda} - \lambda\|_\infty < \varepsilon\}$$

for some number $\varepsilon$ satisfying $0 < \varepsilon < \min\left\{\frac{1}{2+\eta}, \frac{\eta}{1+2\eta}\right\}$. Some algebraic manipulations can show that indeed, for any $\tilde{\lambda} \in \tilde{C}$ the function $\mathbf{y} \mapsto \tilde{\lambda}^T \mathbf{h}(\mathbf{y}, \mathbf{x})$ is strongly convex with parameter bounded below by the positive number $2m\tilde{\sigma}$, where

$$\tilde{\sigma} := \min\{1 - \varepsilon(2+\eta), \eta - \varepsilon(1+2\eta)\}.$$

We now show how at each iteration of Algorithm 2 one can apply the FGP method (Algorithm 3) on the dual problem. Denote by $P_C : \mathbb{R}^{2m} \to C$ the orthogonal projection on $C$. As $C$ is a cartesian product of $m$ copies of the two-dimensional unit simplex, $P_C(\lambda)$ can be calculated as $m$ independent projections $P_{\Delta_2} : \mathbb{R}^2 \to \Delta_2$. Each of those projections is given by

$$P_{\Delta_2}(\lambda_1, \lambda_2) = \begin{cases} (1,0)^T, & \lambda_2 < \lambda_1 - 1, \\ (0,1)^T, & \lambda_2 > \lambda_1 + 1, \\ 0.5(1 - \lambda_2 + \lambda_1, 1 + \lambda_2 - \lambda_1)^T, & |\lambda_2 - \lambda_1| \le 1, \end{cases}$$

and thus, $P_C(\lambda) = \left(P_{\Delta_2}(\lambda_1, \lambda_2)^T, \ldots, P_{\Delta_2}(\lambda_{2m-1}, \lambda_{2m})^T\right)^T$. We denote by $L_q := \frac{L_{\mathbf{x}}^2}{2m\tilde{\sigma}}$ the smoothness parameter of $q_{\mathbf{x}}$ guaranteed by Proposition 2. Since $L_q$ is not known in general, it can be estimated through a backtracking procedure.

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## Appendix - a Proof of Lemma 2

We provide a proof of Lemma 2. The necessity is proven very similarly to the proof given in [4, Theorems 9.2] for the case where $F$ is continuously differentiable.

*Proof.* Let $\mathbf{x}^*$ be a local minimizer of problem (2). Assume to the contrary that $\mathbf{x}^*$ is not a stationary point of (2). Then, recalling that $F$ is directionally differentiable (dd), there exists $\mathbf{y} \in \mathrm{dom}(F)$ such that $F'(\mathbf{x}^*; \mathbf{y} - \mathbf{x}^*) < 0$. By the definition of a directional derivative, it follows that there exists a number $0 < \delta < 1$ such that $F(\mathbf{x}^* + t(\mathbf{y} - \mathbf{x}^*)) < F(\mathbf{x}^*)$ for all $0 < t < \delta$. Since $\mathrm{dom}(F)$ is convex (as $F$ is dd), we have $\mathbf{x}^* + t(\mathbf{y} - \mathbf{x}^*) = (1-t)\mathbf{x}^* + t\mathbf{y} \in \mathrm{dom}(F)$ for all $0 < t < \delta$, contradicting the local minimality of $\mathbf{x}^*$.

As for the sufficiency part when $F$ is convex, let $\mathbf{x}^*$ be a stationary point of (2), and assume to the contrary that $\mathbf{x}^*$ is not a global minimizer of (2). Then there exists

$\mathbf{y} \in \mathrm{dom}(F)$ such that $F(\mathbf{y}) < F(\mathbf{x}^*)$. By the stationarity of $\mathbf{x}^*$ and the convexity of $F$, we obtain

$$
\begin{aligned}
0 \le F'(\mathbf{x}^*;\mathbf{y}-\mathbf{x}^*) &= \lim_{t\to 0^+} \frac{F(\mathbf{x}^*+t(\mathbf{y}-\mathbf{x}^*))-F(\mathbf{x}^*)}{t} \\
&= \lim_{t\to 0^+} \frac{F(t\mathbf{y}+(1-t)\mathbf{x}^*)-F(\mathbf{x}^*)}{t} \le \lim_{t\to 0^+} \frac{tF(\mathbf{y})+(1-t)F(\mathbf{x}^*)-F(\mathbf{x}^*)}{t} \\
&= \lim_{t\to 0^+} \frac{t(F(\mathbf{y})-F(\mathbf{x}^*))}{t} = F(\mathbf{y})-F(\mathbf{x}^*) < 0,
\end{aligned}
$$

which is a contradiction. □

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