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CONVERGENCE RATE ANALYSIS OF GRADIENT BASED ALGORITHMS

Thesis submitted for the degree of "Doctor of Philosophy"
by
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Contents

Abstract

The focus of this thesis is on the rate of convergence analysis and efficiency of gradient based methods for convex optimization problems (COP) and the related convex feasibility problem (CFP). The main advantage of these methods is their simplicity. In most cases, these methods consists only of matrix/vector multiplications at each iteration of the algorithm. A main drawback of these methods is that without making strong assumptions on the problem's data, they suffer in general, from relatively low convergence rate. Our main objective and contribution is to derive simple algorithms with improved rate of convergence. We analyze various classes of problems, including quadratically constrained convex problems, conic linear systems and nonsmooth convex constrained minimization. For each of these classes we develop and analyze specific and simple algorithms. We prove that under suitable assumptions on the problem's data, the resulting methods are either linearly convergent or exhibit rates which are almost independent of the problem's dimension. Our methodology relies on convex analytic tools combined with the theory of error bounds within which we develop new concepts and approaches to achieve the desired stated goals.

Chapter 1

Introduction

This thesis concentrates on the rate of convergence analysis and efficiency of gradient based methods for convex optimization problems (COP) and the related convex feasibility problem (CFP). While each of the following four chapters of the thesis can be read independently as being essentially self contained, there is much in common with respect to the mathematical tools we use and to the kind of results we derive. The unified line of analysis in all of the chapters relies on the fact that we are considering first order methods in order to solve (COP) and (CFP). First order methods are methods that use at each iteration of the algorithm only information on the function values and the gradients (or subgradients), at some given points, but not the the information on the Hessian. The main advantage of these methods is their simplicity. In most cases, these methods consists only of matrix/vector multiplications at each iteration of the algorithm. A main drawback of first order methods is that they suffer in general, from relatively low convergence rate and produce low accuracy optimal solutions. This is in sharp contrast with the more sophisticated interior point methods that exhibit very good theoretical rate of convergence, but often require heavy computational tasks, such as matrix factorizations and solutions of linear systems at each iteration. Therefore, the simplicity of first order methods may have an edge over the interior point based algorithms for solving very large scale problems, where it is sometimes impossible to apply even one iteration of an interior point method.

Most of the analysis in the thesis is devoted to the investigation of non-asymptotic rate of convergence analysis for first order methods, that is we are interested to know the order of magnitude on the number of iterations necessary to produce an approximate optimal solution.

In general, without strong assumptions on the problem's data, the rate of convergence of first order methods can in fact be very poor. The natural question which emerges is then: can we derive better rates of convergence, under reasonable assumptions on the problem's data? The answer to this question will be the main preoccupation of this thesis and is developed along the following four chapters, for various classes of gradient-based algorithms to solve (COP) and related projection algorithms for solving (CFP).

A larger introductory chapter was considered, but such a chapter would have included too much repetition of context to be attractive. Indeed, each of the following chapter contains its own detailed introduction that describes the problem, the motivation behind it, a literature overview, and our contribution. We thus give below just a brief summary of the main results. Here is how the rest of this thesis is organized.

- In Chapter 2, the gradient projection algorithm is considered for solving problem (COP). We introduce an error bound called the Gradient Error Bound (GREB) which is proven to imply linear rate of convergence of the algorithm. We then consider the class of quadratically constrained convex programs and derive a dual formulation on which the projected gradient method can be applied to produce a simple algorithm. We then develop the mathematical tools necessary to prove that (GREB) is satisfied for the resulting dual, which thus establish a provably linear convergent and simple algorithm for solving this class of problems.

- Chapter 3 is concerned with the convex feasibility problem where we present several projection type algorithms for its solution. We prove that the rate of convergence of the known projection algorithms is not bounded. However, if a Local Error Bound (LEB) is satisfied then linear rate of convergence is proven. Moreover, using elementary convexity arguments, we show that the standard Slater condition implies LEB. A connection between (LEB) and the error bound derived for the gradient projection algorithm (GREB) is established and used to show further convergence rate results.

- Chapter 4 studies a specific case of the convex feasibility problem. More specifically, we consider the problem of finding a point in the intersection of an affine set and a compact set. The approach in this chapter is to transform the problem to a convex optimization problem and solve it with the conditional gradient method. The rate of convergence of the function values of the sequence generated by the conditional gradient method is known to be sublinear. However, we prove that if the Slater condition on the original convex feasibility is satisfied then the function values converge to an optimal point with a linear rate.

- In Chapter 5 we analyze the Mirror Descent Algorithm, which is shown to be a generalization of the subgradient projection method and thus is also applicable to non-differentiable optimization problems. As a consequence of the developed analysis, we present a simple algorithm for solving convex problems over the unit simplex which has an efficiency estimate proven to be almost independent in the dimension of the problem.

To make the thesis completely self-contained and to separate known results from our contribution, we have also included four appendices. Appendix A includes all the basic classical results on projections. Appendix B contains some classical mathematical results that are used throughout the thesis. At the end of Chapter 2 we added an appendix that includes the classical results on the gradient projection algorithm. Chapter 5 is ended with an appendix that analyzes the rate of convergence of the conditional gradient method. All of the appendices serve us throughout the thesis and can be used as a reference tool.

Chapter 2

The Gradient Error Bound

2.1 Introduction

This chapter considers the convex optimization problem:

$$\min_{x \in S} f(x)$$

where f is a continuously differentiable convex function with Lipschitz gradient with Lipschitz constant L and S is a closed convex set. A well known and simple algorithm to solve this problem is the Gradient Projection Algorithm (in short, GPA). It starts with any arbitrary point $x^0 \in \Re^n$ and generates a sequence $\{x^k\}$ via the iteration: $x^{k+1} = P_S(x^k - t\nabla f(x^k))$, where $t > 0$ is a stepsize and P_S denotes the orthogonal projection on S. The GPA has been studied extensively in the literature, see for example [23],[25],[17],[7],[9], [18],[28],[41].
More details and further references on the GPA can be found in the book of Bertsekas [6]. In [25] it is proven that if the step size is a positive constant less than $\frac{2}{L}$ then the function values converge to the minimum value with a sublinear rate. Also, it is proven that if f is twice differentiable and strongly convex then the sequence generated by GPA converges to an optimal point with a linear rate. Further results in that direction can be found in Dunn [17].

The main advantage of GPA is its simplicity, provided that the orthogonal projection on the set S and the gradient of f can be easily computed. For example, if S is an affine space then the projection on S at each iteration of GPA involves only matrix/vector multiplications. (Other cases where the projections can be computed analytically are given in Appendix A).

Its main drawback, as just mentioned above, is that the convergence rate of GPA is in general only sublinear, unless some further and often restrictive assumptions on the problem's data are made. A natural question is thus to identify classes of problems for which: on one hand the rate of convergence can be improved, say to linear, under weaker or/and reasonable assumptions, while on the other, the simplicity of the algorithm of GPA will be preserved, namely the projections and gradients can be computed efficiently/analytically. As we shall see, these two requirements often lead to some conflicting situations.

The first part of this chapter will study general conditions under which the rate of convergence of GPA can be guaranteed to be *linear*. More specifically, we discuss the conditions which insure linear convergence rate of the the sequence of the distances from the optimal set to zero and linear convergence rate of the function values to an optimal value. At this point, we would like to emphasize that we are interested here in *nonasymptotic* rate of convergence, namely we are looking for results of the type:

$$f(x^k) - f^* \leq C\gamma^k,$$

for some $\gamma \in (0, 1)$ and $C > 0$.

For that purpose, we introduce the Gradient Error Bound (in short, GREB) which reads as follows: *For every closed bounded set $B \subset \Re^n$ there exists $\sigma_B > 0$ such that:*

$$\forall x \in B \cap S \quad d(x, X^*) \leq \sigma_B T(x).$$

where X^* is the optimal set, $T(x) = \|x - P_S(x - t\nabla f(x))\|$, and t satisfies $0 < t < \frac{2}{L}$. Notice that $T(x)$ is an easily computable quantity and satisfies the property that it is nonnegative and $T(x) = 0$ is if and only if $x \in S$. One important application of the GREB assumption is that we can bound the distance of a point to the optimal set in terms of an easily computable quantity. Thus, we can use this bound to define stopping rules in iterative algorithms solving the optimization problem.

Besides the above theoretical implication, it is widely known that existence of error bounds is a key ingredient in proving convergence rates of iterative methods. For a comprehensive survey on Error Bounds, their applications and references we refer the reader to Pang [34].

The concept of GREB considered here is a slightly different version of an error bound recently introduced by Luo in [26] (For convenience, this assumption will be called here LGREB).

Assumption LGEEB(Luo,[26]): For every $v \geq \inf_{x \in S} f(x)$ there exists scalars $\gamma > 0, \tau > 0$ such that,

$$d(x, X^*) \leq \tau T(x)^{\frac{1}{\gamma}},$$

for all $x \in S$ with $f(x) \leq \nu$ and $T(x) \leq \delta$.

It was proven in [26] that the existence the LGREB assumption with $\gamma = 1$ together with an assumption of proper separation of isocost surfaces implies an *asymptotic* linear rate of convergence of the corresponding sequences of function values generated by GPA, even in the nonconvex case for f. The GREB assumption was not investigated in the literature. However, because of the apparent similarity between the two assumptions, most of the known results on LGREB can be easily transformed to results on GREB. Moreover, the form of the GREB assumption is much more consistent to error bounds defined on convex feasibility problems (see chapter 3).

Unfortunately, it is in general a very hard task to prove the validity of the LGREB assumption. As we shall see, in most cases this requires to admit the existence of a unique optimal solution for (P) and the constraints set S to be polyhedral. In fact, the LGREB assumption was proven to hold for only very few instances of problem (P) which we now review. The first case is the well known situation when f is assumed strictly convex and S is a polyhedral set. Note that this result was recently recovered by Luo [26] as a special case of a more general framework. In the same paper, convergence rate results are established under the weaker assumption of γ strict convexity for the objective f, where $\gamma \geq 1$ (the case $\gamma = 1$ corresponding to strong convexity) and S is either polyhedral or consist of convex inequalities described by differentiable subanalytic functions ([14]) satisfying the Slater's condition. However, only sublinear rate of convergence is established in these cases whenever $\gamma > 1$ and thus these results are not be applicable to our declared task of proving linear convergence of GPA.

The second case is when f is assumed quadratic (possibly nonconvex) and S is a polyhedral set, see Luo and Tseng [27]). The third case is the case where the constraint set S is polyhedral and the objective is given in the composite form

$$f(x) = \langle q, x \rangle + g(Ex),$$

where E is a given $m \times n$ matrix with no zero column, $q \in \Re^n$ and g is a strongly convex differentiable function on $\Re^m$ with ∇g Lipschitz continuous in $\Re^m$. The later can be relaxed at the price of further assumptions on g, see Luo and Tseng [28] for further details. Finally the last case is for the dual functional case (see [29]) where S is a polyhedral set and f has the following form:

$$f(x) = \langle q, x \rangle + \max_{y \in Y} \{ \langle Ex, y \rangle - g(y) \},$$

where Y is a polyhedral set in $\Re^m$, E is an $m \times n$ matrix with no zero column, q is a vector in $\Re^n$, and g is a strongly convex differentiable function in $\Re^m$ with ∇g Lipschitz continuous in $\Re^m$.

Thus, it appears that all known results in the literature on the application of LGREB (and thus GREB) to GPA type algorithms with easy/computable projections have been restricted to consider constrained problems only with *polyhedral constraints*. Not too surprisingly, the polyhedral structure indeed plays a central role in the analysis and results developed in [26], [27], [28].

We thus consider the question whether it is possible to extend the application of GREB either to other and more general structures for the constraints set S or to other classes of objective functions than the ones discussed above, when solving (P) through GPA.

The first aim of the chapter is to further investigate the GREB assumption. We will prove that whenever the GREB assumption holds in the convex case, it implies linear rate of convergence of the distances from the optimal set of the sequence generated by GPA. We will also show that if f is a strongly convex function then the GREB assumption is valid. Thus, the GREB assumption is in some sense a generalization of the more restrictive notion of strong convexity (which as already mentioned also implies linear rate of convergence of the sequence generated by GPA).

The second aim, and main contribution of this chapter is to prove that for the important class of convex quadratically constrained quadratic problems (for short QCQP), one can derive a GPA-based algorithm which involves matrix/vector multiplications and satisfies GREB and thus is provably linearly convergent algorithm. (QCQP) is the next natural generic class of constraints one might think of after polyhedral constrained problems. (QCQP) is the problem of minimizing a quadratic function subject to convex quadratic inequality constraints. Besides the theoretical interest in convex quadratically constrained quadratic problems ((QCQP) for short), we note that the class (QCQP) can be used to model many important Engineering problems, see for example the recent book of Ben-Tal and Nemirovsky [5] which includes a wealth of engineering problems that can be formulated as (QCQP). The class (QCQP) can also be formulated as a second order cone problem and either be solved directly or through their conic formulations via interior point methods (see e.g. Nesterov-Nemirovsky [33], Ben-Tal-Nemirovsky [5], and [30]). These methods are proven to be theoretically efficient, i.e., with polynomial complexity bounds. However, there are instances of QCQP where these sophisticated methods might not be a good choice. In particular, whenever the problems are very large scale, e.g., with the dimension of the decision variables or the number of constraints or both is extremely large. Indeed, these methods require heavy computations at each iteration and thus are in general impractical in these cases, unless very special structures are identifiable in the problems under consideration. This motivated us to develop a simple algorithm that involve elementary/matrix vector multiplication at each iteration, and a natural candidate to achieve this task with a linear rate of convergence properties is the GPA, provided that one can establish the validity of the GREB assumption. To achieve this goal requires a novel way to adequately formulate the QCQP and a new line of analysis for proving the validity of GREB. Interestingly, the class QCQP enlightens well the difficulties encounter in the task of deriving simple algorithms with linear rate of convergence. Indeed, none of the known results available in the literature and described above are directly applicable to our problem, as we explain now.

We first note that GPA cannot be applied to (QCQP) directly since it is impossible to calculate analytically the orthogonal projection onto an intersection of convex quadratic constraints. Now, we suppose that the QCQP is a strongly convex problem, namely both the objective and constraints are strongly convex functions. Then, the lack of polyhedrality of the constraints set precludes the use of the results describe in the first case above to derive a linear rate of convergence for the sequence of function values.

Thus, we propose to study a *dual approach* to (QCQP). As shown in the next section, the standard dual problem consists of minimizing a strictly convex over nonnegative constraints. In that case we do obtain the desired polyhedral constraints, but GREB does not hold since the objective is only strictly convex. In fact, even if we could prove strong convexity, the simplicity of GPA would anyway be lost, and required the inversion of all the matrices *at each iteration* of the algorithm. To overcome this difficulty, our first task is to construct a new dual problem on which the GPA can be applied, namely where the projections can be computed explicitly and with an objective with Lipschitz continuous gradient. It turns out that one can construct a dual problem with the desirable affine constraints, rendering also the computation of the projections a trivial task. The dual objective function we derive possesses an interesting structure in its own, and is proven to be continuously differentiable with a Lipschitz continuous gradient and with a computable Lipschitz constant. However, the dual functional does not belong to any of the class of functions alluded above for which known results could be applicable to verify the validity of the GREB assumption. This is the price to pay for reducing the QCQP to a problem with two of the three desirable properties (here we get affine constraints and easy computations of the projections) but no strong convexity (or even strict convexity of the dual objective). Thus, the second task will be to prove that the GREB holds for the obtained dual objective. The later task required to develop a rather involved analysis which will be one of the main preoccupation of this chapter.

The chapter is organized as follows. Section 2.2 presents some notations and briefly recall well known results concerning the gradient projection algorithm. In section 2.3 we present the GREB assumption and recall that the validity of the GREB assumption implies asymptotic linear convergence of the function values. In the convex case we prove that the sequence generated by GPA converges to an optimal point at a linear rate. In section 2.4 we present a new dual formulation of (QCQP) which is derived through the use of a decomposition technique. We establish explicit relations between the primal-dual optimal solutions and study the properties of the dual objective. The following section develop the machinery needed to prove that GREB is satisfied for the derived dual formulation. To make the whole chapter self-contained and for references purposes, we end this chapter with an appendix including compact proofs of the basic and well known results on the gradient projection algorithm.

Part 1: Convergence Rate Analysis of the Gradient Projection Algorithm

2.2 General Results on GPA

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Consider the following general optimization problem:

$$(P)\ \inf\{f(x) : x \in S\}.$$

Throughout the chapter we make the following three assumptions:

Assumption 1 *$S \subseteq \Re^n$ is a closed convex set.*

Assumption 2 *$f : \Re^n \to \Re$ is continuously differentiable and ∇f is Lipschitz continuous on S, that is,*

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \ \ \forall x, y \in S,$$

where $L > 0$ is the Lipschitz constant.

Assumption 3

$$\inf\{f(x) : x \in S\} > -\infty,$$

We use the following notations: P_S is the orthogonal projection onto S. N_S is the normal cone to S. X^* is the set of all stationary points of (P) which we assume always to be nonempty. A stationary point of (P) is a point x^* that satisfies $x^* = P_S(x^* - t\nabla f(x^*))$ for some $t > 0$. This is equivalent to the condition $-\nabla f(x^*) \in N_S(x^*)$. A local minimum of (P) is necessarily a stationary point and in the case where f is convex a stationary point is also a local minimum (thus, in the convex case X^* is the optimal set). In the convex case we denote $f^* = \inf_{x \in S} f(x)$.
The GPA algorithm (GPA is a shortcut for Gradient Projection Algorithm) is defined as follows:

GPA

Initial step: take $x^0 \in S$
general step: $x^{k+1} = P_S(x^k - t\nabla f(x^k)),\ t > 0,\ \ k = 1, 2, \ldots$

Notice that GPA does not include any line search and instead uses a constant stepsize. Later on, we will find suitable choices for t.
Other possible assumptions that will be made on the problem (in some sections) are:

Assumption 4 *f is a convex function.*

Assumption 5 *f is strongly convex with parameter $m > 0$.*

The convergence results for GPA are scattered throughout the literature and thus we included an appendix at the end of this chapter which gives all the proofs of the main and known results in a self contained manner. Here we will state the main results. The convergence results will be given in the following cases:

- the general case (we assume only assumptions 1,2,3).
- f is convex (assumptions 1,2,3,4).
- f is strongly convex (assumptions 1,2,3,5).

Theorem 2.2.1 (Convergence of GPA in the non convex case) *Suppose that assumptions 1,2,3 are fulfilled. Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$. Then,*

1. *$\{f(x^k)\}$ is monotone decreasing.*
2. *Every accumulation point of $\{x^k\}$ is a stationary point of (P).*

Theorem 2.2.2 (Convergence of GPA in the convex case) *Suppose that assumptions 1,2,3,4 are fulfilled. Let $\{x^k\}$ be the sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$ then:*

1. *x^k converges to some $x^* \in X^*$.*
2. *$f(x^k) - f^* \leq \frac{C}{k}$ for every $k \geq 1$ and some constant $C > 0$.*

Theorem 2.2.3 (Linear Rate of Convergence Under Strong Convexity) *Suppose that assumptions 1,2,3,5 are fulfilled. Let $\{x^k\}$ be the sequence generated by GPA with a constant stepsize $0 < t < \frac{2m}{L}$ then it converges to the unique minimum x^* with a linear rate. In fact,*

$$\|x^{k+1} - x^*\| \leq \theta \|x^k - x^*\|,$$

where

$$\theta = \sqrt{1 - 2tm + t^2 L^2} < 1.$$

2.3 The Gradient Error Bound

In this section we assume assumptions 1,2,3. We present the gradient error bound (shortcut - GREB) which is closely related to the operator T:

$$T(x) = \|P_S(x - t\nabla f(x)) - x\|.$$

Assumption 6 (GREB) *For every closed bounded set $B \subseteq \Re^n$ there exists a $\sigma_B > 0$ such that:*

$$\forall x \in B \cap S \quad d(x, X^*) \leq \sigma_B T(x).$$

We have already seen that under strong convexity GPA has a linear rate of convergence to an optimal point. We will prove that strong convexity is not necessary for deriving linear rate of convergence. Indeed, GREB is a weaker condition which implies the linear rate of convergence. Before proving that, we will prove that strong convexity implies GREB.

Lemma 2.3.1 *Let $S \subseteq \Re^n$ be a closed convex set and let $f : \Re^n \to \Re$ be strongly convex function with a parameter $m > 0$. Then $\exists C > 0$ such that,*

$$\|x - x^*\| \leq CT(x) \;\; \forall x \in S.$$

In particular, GREB is satisfied.

Proof: The strong convexity of f implies that $X^* = \{x^*\}$. Recall that by the definition of X^* the point x^* satisfies:

$$x^* = P_S(x^* - t\nabla f(x^*)). \tag{2.1}$$

Also note that by Theorem 2.6.4 one has:

$$\|P_S(x - t\nabla f(x)) - P_S(x^* - t\nabla f(x^*))\| \leq \sqrt{1 - 2tm + t^2L^2}\|x - x^*\|. \tag{2.2}$$

Now,

$$\begin{aligned}
d(x, X^*) &= \|x - x^*\| \\
&\overset{(2.1)}{=} \|x - P_S(x^* - t\nabla f(x^*))\| \\
&= \|x - P_S(x - t\nabla f(x)) + P_S(x - t\nabla f(x)) - P_S(x^* - t\nabla f(x^*))\| \\
&\leq \|x - P_S(x - t\nabla f(x))\| + \|P_S(x - t\nabla f(x)) - P_S(x^* - t\nabla f(x^*))\| \\
&\overset{(2.2)}{\leq} T(x) + \sqrt{1 - 2tm + t^2L^2}\|x - x^*\|.
\end{aligned}$$

Thus,

$$\|x - x^*\| \leq T(x) + \sqrt{1 - 2tm + t^2L^2}\|x - x^*\|,$$

which is equivalent to:

$$\|x - x^*\| \leq \frac{1}{1 - \sqrt{1 - 2tm + t^2L^2}}T(x).$$

To summarize, we have found that GREB is fulfilled with $\sigma_B = \frac{1}{1-\sqrt{1-2tm+t^2L^2}}$. □

In the next two subsections we will derive convergence properties of GPA under the GREB assumption in the non-convex case and in the convex case.

2.3.1 The Non Convex Case

In this subsection we assume only assumptions 1,2,3,6.

Let $\{x^k\}$ be a sequence generated by GPA. As proved in the appendix (Theorem 2.6.1, Corollary 2.6.1) every accumulation point x^* of the sequence is also a stationary point and $f(x^k) \to f(x^*)$. In [26] it was proven that if in addition LGREB is satisfied, $\{x^k\}$ is a bounded sequence and assumption 7 (which follows) is satisfied, then $f(x^k)$ converges to f^* with an asymptotic linear rate of convergence where $f^* = f(x^*)$ for some stationary point x^* of (P). From completeness reasons, the result is stated here. The proof is given in the appendix and adapted to our definition of GREB.

Assumption 7 (Proper separation of isocost surfaces) *There exists a scalar* $\epsilon > 0$ *such that*

$$x, y \in X^*, f(x) \neq f(y) \quad \Rightarrow \quad \|x - y\| \geq \epsilon$$

Assumption 7 is satisfied for instance when X^* is finite.

Theorem 2.3.1 (Asymptotic Linear Convergence Rate of the Function Values) *Let f be a function with Lipschitz continuous gradient with Lipschitz constant L. Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$. Suppose that $\{x^k\}$ is bounded and that GREB and assumption 7 are satisfied. Then, $\{f(x^k)\}$ converge to f^* where $f^* = f(x^*)$ for some stationary point x^*. Furthermore, there is $0 < \beta < 1$ and $K > 0$ such that,*

$$\forall k > K \qquad f(x^{k+1}) - f^* < \beta(f(x^k) - f^*),$$

where $f^* = f(x^*)$.

2.3.2 The Convex Case

In this subsection we assume only assumptions 1,2,3,4,6. We prove that in the convex case GREB implies linear rate of convergence of the sequence generated by GPA. First, we prove a technical lemma that investigates the operator ρ defined by:

$$\rho(x) = x - P_S(x - t\nabla f(x)).$$

Remark: In our notations, $T(x) = \|\rho(x)\|$.

Lemma 2.3.2

$$\langle \rho(x) - \rho(y), x - y \rangle \geq \theta \|\rho(x) - \rho(y)\|^2 \quad \forall x, y \in \Re^n,$$

where

$$\theta = 1 - \frac{Lt}{4}.$$

Proof: Let $x, y \in \Re^n$. By Theorem A.1.6:

$$\langle P_S(x - t\nabla f(x)) - P_S(y - t\nabla f(y)), x - t\nabla f(x) - y + t\nabla f(y) \rangle \geq \|P_S(x - t\nabla f(x)) - P_S(y - t\nabla f(y))\|^2, \tag{2.3}$$

By using the ρ notation we have that (2.3) is equivalent to:

$$\langle x - \rho(x) - y + \rho(y), x - t\nabla f(x) - y + t\nabla f(y)\rangle \geq \|x - \rho(x) - y + \rho(y)\|^2.$$

After subtracting the RHS of the equation from the LHS of the equation, the equation becomes:

$$\langle x - \rho(x) - y + \rho(y), \rho(x) - t\nabla f(x) - \rho(y) + t\nabla f(y)\rangle \geq 0,$$

which is equivalent to,

$$\begin{aligned}
\langle \rho(x) - \rho(y), x - y\rangle \quad & \geq \quad \|\rho(x) - \rho(y)\|^2 + t\langle \nabla f(x) - \nabla f(y), x - y\rangle \\
& \quad - t\langle \rho(x) - \rho(y), \nabla f(x) - \nabla f(y)\rangle \\
& \overset{\text{Theorem } B.0.5}{\geq} \quad \|\rho(x) - \rho(y)\|^2 + \frac{t}{L}\|\nabla f(x) - \nabla f(y)\|^2 \\
& \quad - t\langle \rho(x) - \rho(y), \nabla f(x) - \nabla f(y)\rangle \\
& \overset{\text{Cauchy Schwartz inequality}}{\geq} \quad \|\rho(x) - \rho(y)\|^2 + \frac{t}{L}\|\nabla f(x) - \nabla f(y)\|^2 \\
& \quad - t\|\rho(x) - \rho(y)\| \cdot \|\nabla f(x) - \nabla f(y)\|.
\end{aligned}$$

Denote $\alpha = \|\rho(x) - \rho(y)\|, \beta = \|\nabla f(x) - \nabla f(y)\|$ then we obtain,

$$\begin{aligned}
\langle \rho(x) - \rho(y), x - y\rangle \quad & \geq \quad \alpha^2 + \frac{t}{L}\beta^2 - t\alpha\beta \\
& = \quad \alpha^2\left(1 - \frac{Lt}{4}\right) + \frac{t}{L}\left(\beta - \frac{L\alpha}{2}\right)^2 \\
& \geq \quad \alpha^2\left(1 - \frac{Lt}{4}\right) \\
& \geq \quad \left(1 - \frac{Lt}{4}\right)\|\rho(x) - \rho(y)\|^2.
\end{aligned}$$

□

Theorem 2.3.2 (Linear Rate of Convergence of $d(x^k, X^*)$) *Let f be a convex function with Lipschitz continuous gradient. Suppose that GREB is satisfied. Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$. Then there is $0 < \eta < 1$ such that,*

$$d(x^{k+1}, X^*) \leq \eta d(x^k, X^*).$$

Proof: By the second part of Lemma 2.6.3 we have that $\{x^k\}$ is bounded and by GREB we obtain that there is a $\sigma > 0$ such that:

$$d(x^k, X^*) \leq \sigma \|x^{k+1} - x^k\|,$$

or,

$$d(x^k, X^*) \leq \sigma \|\rho(x^k)\|. \tag{2.4}$$

Let $x^* \in X^*$ then applying Lemma 2.3.2 with $x = x^k, y = x^*$ we obtain:

$$\langle \rho(x^k), x^k - x^* \rangle \geq \theta \|\rho(x^k)\|^2. \tag{2.5}$$

Now,

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|x^k - x^*\|^2 + 2\langle x^{k+1} - x^k, x^k - x^* \rangle + \|x^{k+1} - x^k\|^2 \\
&= \|x^k - x^*\|^2 - 2\langle \rho(x^k), x^k - x^* \rangle + \|\rho(x^k)\|^2 \\
&\overset{(2.5)}{\leq} \|x^k - x^*\|^2 - 2\theta \|\rho(x^k)\|^2 + \|\rho(x^k)\|^2 \\
&= \|x^k - x^*\|^2 - (2\theta - 1)\|\rho(x^k)\|^2 \\
&\overset{(2.4)}{\leq} \|x^k - x^*\|^2 - \frac{2\theta - 1}{\sigma} d^2(x^k, X^*).
\end{aligned}$$

Therefore,

$$\begin{aligned}
d^2(x^{k+1}, X^*) &\leq d(x^k, X^*) - \frac{2\theta - 1}{\sigma} d(x^k, X^*) \\
&= \left(1 - \frac{2\theta - 1}{\sigma}\right) d^2(x^k, X^*),
\end{aligned}$$

and hence,

$$d(x^{k+1}, X^*) \leq \sqrt{1 - \frac{2\theta - 1}{\sigma}} d(x^k, X^*)$$

□

The linear rate of convergence of the distance of the sequence from the optimal set implies also the linear rate of convergence of the function values of the sequence as stated in the following theorem:

Theorem 2.3.3 (Linear rate of convergence of the function values) *Let f be a convex function with Lipschitz continuous gradient. Suppose that GREB is satisfied. Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$ and denote the optimal value by f^*. Then there is $0 < \gamma < 1$ and $C > 0$ such that,*

$$f(x^k) - f^* \leq C\gamma^k.$$

Proof: Since $\{x^k\} \subseteq S$ is bounded, invoking Lemma B.0.2 one obtains,

$$f(x^k) - f^* \leq \|\nabla f(x^k)\| d(x^k, X^*) \leq l d(x^k, X^*),$$

for some $l > 0$. Applying Theorem 2.3.2 the result immediately follows. □

Part 2: A Dual-Based Gradient Projection Algorithm for Quadratically Constrained Convex Problems with a Linear Rate of Convergence

2.4 A dual approach to Convex Quadratic Programming

2.4.1 The Problem

In this section we consider the minimization of a strictly convex quadratic function under strictly convex quadratic inequalities,

$$(QCQP) \qquad \begin{array}{ll} \text{minimize} & x^T Q_0 x + 2b_0^T x \\ \text{s.t.} & x^T Q_i x + 2b_i^T x \leq c_i \;\; \forall i = 1, 2, \ldots, m, \end{array}$$

where $Q_0, Q_1, \ldots, Q_m$ are $n \times n$ positive definite matrices, $b_0, b_1, \ldots, b_m \in \Re^n$ and $c_1, \ldots, c_m \in \Re$. Throughout we assume that (QCQP) is feasible i.e. $c_i + b_i^T Q_i^{-1} b_i > 0 \;\; \forall i$. Our approach will consist of solving (QCQP) through a dual problem that can be easily solved via the gradient projection algorithm. The main task of this section is to derive a new dual formulation of (QCQP). Throught this chapter we assume that the Slater condition is satisfied and thus strong duality is satisfied and the optimal value of (QCQP) is equal to the optimal value of (DQCQP).

2.4.2 Standard Dual Formulations

First, we shall see that the standard dual formulations of (QCQP) are rather complicated and thus cannot be used in order to construct efficient algorithms. A standard dual formulation of (QCQP) can be easily shown to be given by:

$$\begin{array}{ll} \text{maximize} & -\left(b_0 + \sum_{i=1}^m b_i\lambda_i\right)^T \left(Q_0 + \sum_{i=1}^m \lambda_i Q_i\right)^{-1} \left(b_0 + \sum_{i=1}^m b_i\lambda_i\right) - \sum_{i=1}^m \lambda_i c_i \\ \text{s.t.} & \lambda_i \geq 0 \ \ \forall i = 1, 2, \ldots, m, \end{array}$$

The trouble with this formulation is that each gradient calculation of the dual objective function consist of inverting a matrix. Thus, for very large scale problems in the design variables n (even with small m), a gradient based algorithm would require *at each iteration* the computation of inverting a huge (and in general with no structure) matrix, a task which is practically intractable. Our goal is to find an algorithm that consists only of matrix vector multiplication and does not involve any matrix inversion at each iteration. To achieve this task we use the decomposition approach. Here, we duplicate the variables so that we will obtain simpler expressions for the dual problem. The equivalent primal problem is as follows

$$\begin{array}{ll} \text{minimize} & x^T Q_0 x + 2b_0^T x \\ \text{s.t.} & x_i^T Q_i x_i + 2b_i^T x_i \leq c_i \ \ \forall i = 1, 2, \ldots m \\ & x_i = x \ \ \forall i = 1, \ldots, m \end{array}$$

We can, for instance, write this problem in the following way:

$$\begin{array}{ll} \text{minimize} & \frac{1}{m}\sum_{i=1}^m x_i^T Q_0 x_i + 2b_0^T x \\ \text{s.t.} & x_i^T Q_i x_i + 2b_i^T x_i \leq c_i \ \ \forall i = 1, 2, \ldots m \\ & x_i = x \ \ \forall i = 1, \ldots, m \end{array}$$

Assigning a lagrange multiplier $\lambda_i \in \Re^n$ for each linear equality constraint we obtain the following dual:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^m g_i(\lambda_i) \\ \text{s.t.} & \sum_{i=1}^m \lambda_i = b_0, \end{array}$$

where $g_i(\lambda_i) = \min_{x_i^T Q_i x_i + 2b_i^T x_i \leq c_i} \frac{1}{m} x_i^T Q_0 x_i + 2\lambda_i^T x_i$.

The problem here is that the functions g_i do not have an explicit expression. The only case where it is possible to find an explicit expression for g_i is the case where $Q_0 = Q_i$. The motivation behind the new formulation is to enforce the situation where the matrices in the definition of g_i (Q_0, Q_i) are the same. It turns out that this can be done by adding a redundant constraint.

2.4.3 The New Dual Formulation

One of the key arguments in establishing the new dual formulation is to guarantee that we can write Q_0 as a positive linear combination of the matrices Q_i i.e., that $Q_0 = \sum_{i=1}^{m+1} \beta_i Q_i$ where $\beta_1, \beta_2, \ldots, \beta_{m+1} > 0$ (Q_{m+1} will be constructed later). Of course, there is no guaranty that there is such linear combination. This is the reason why we will add a "redundant" constraint to the original problem (QCQP) which will enforce such linear combination. The following two lemmas allow us to do that. In the sequel $\lambda_{min}(Q)$ ($\lambda_{max}(Q)$) denotes the minimum (maximum) eigenvalue of Q.

Lemma 2.4.1 *Let Q be a positive definite matrix, $b \in \Re^n, c \in \Re$. If x satisfies the following quadratic inequality*

$$x^T Q x + 2b^T x \leq c, \tag{2.6}$$

then

$$\|x\|^2 \leq a,$$

where

$$a = \left(\frac{1}{\sqrt{\lambda_{min}(Q)}} \sqrt{c + b^T Q^{-1} b} + \|Q^{-1} b\| \right)^2.$$

Proof: First, since $Q \succ 0$, the inverse Q^{-1} exists and we can write,

$$x^T Q x + 2b^T x = (x + Q^{-1} b)^T Q (x + Q^{-1} b) - b^T Q^{-1} b.$$

Therefore, (2.6) is equivalent to:

$$(x + Q^{-1} b)^T Q (x + Q^{-1} b) \leq c + b^T Q^{-1} b. \tag{2.7}$$

Suppose that x satisfies (2.6) then:

$$\begin{aligned}
\|x\| &= \|x + Q^{-1}b - Q^{-1}b\| \\
&\leq \|x + Q^{-1}b\| + \|Q^{-1}b\| \\
&= \frac{1}{\sqrt{\lambda_{min}(Q)}}\sqrt{\lambda_{min}(Q)\|x + Q^{-1}b\|^2} + \|Q^{-1}b\| \\
&\leq \frac{1}{\sqrt{\lambda_{min}(Q)}}\sqrt{(x + Q^{-1}b)^T Q(x + Q^{-1}b)} + \|Q^{-1}b\| \\
&\overset{(2.7)}{\leq} \frac{1}{\sqrt{\lambda_{min}(Q)}}\sqrt{c + b^T Q^{-1}b} + \|Q^{-1}b\| = \sqrt{a},
\end{aligned}$$

where in the second inequality above we use $z^T Q z \geq \lambda_{min}(Q)\|z\|^2\ \forall z$. $\square$

Lemma 2.4.2 *Let $Q_0, \ldots, Q_m$ be $n \times n$ positive definite matrices, $b_1, \ldots, b_m \in \Re^n$ and $c_1, \ldots, c_m \in \Re$. Let $\beta_1, \ldots, \beta_m$ be m positive real numbers that satisfy the following inequality:*

$$\sum_{i=1}^{m} \beta_i \lambda_{max}(Q_i) < \lambda_{min}(Q_0). \tag{2.8}$$

Then the following set of quadratic inequalities

$$x^T Q_i x + 2b_i^T x \leq c_i \quad \forall i = 1, 2, \ldots m, \tag{2.9}$$

imply the inequality

$$x^T Q_{m+1} x \leq c_{m+1},$$

where

$$Q_{m+1} = Q_0 - \sum_{i=1}^{m} \beta_i Q_i, \tag{2.10}$$

$$c_{m+1} = \lambda_{max}(Q_{m+1}) \min_{i=1,\ldots,m} \left(\frac{1}{\sqrt{\lambda_{min}(Q_i)}}\sqrt{c_i + b^T Q_i^{-1} b_i} + \|Q_i^{-1} b_i\| \right)^2.$$

Proof: By Lemma 2.4.1 we have that the set of inequalities (2.9) imply that

$$\|x\|^2 \leq \alpha \tag{2.11}$$

where

$$\alpha = \min_{i=1,\ldots,m} \left(\frac{1}{\sqrt{\lambda_{min}(Q_i)}} \sqrt{c_i + b^T Q_i^{-1} b_i} + \|Q_i^{-1} b_i\| \right)^2$$

Let $\beta_1, \ldots, \beta_m$ be m positive real numbers such that

$$\sum_{i=1}^{m} \beta_i \lambda_{max}(Q_i) < \lambda_{min}(Q_0)$$

This inequality implies that $Q_{m+1} = Q_0 - \sum_{i=1}^{m} \beta_i Q_i$ is a positive definite matrix. Thus, a consequence of (2.11) is that

$$x^T Q_{m+1} x \leq \lambda_{max}(Q_{m+1}) \|x\|^2 \overset{(2.11)}{\leq} \lambda_{max}(Q_{m+1}) \alpha$$

□

An immediate consequence of Lemma 2.4.2 is that (QCQP) is equivalent to:

$$\begin{array}{ll} \text{minimize} & x^T Q_0 x + 2b_0^T x \\ \text{s.t.} & x^T Q_i x + 2b_i^T x \leq c_i \ \ \forall i = 1, 2, \ldots m+1 \end{array}$$

where Q_{m+1}, c_{m+1} are as defined in Lemma 2.4.2 and $b_{m+1} = 0$. Note that by the construction of Q_{m+1} it follows that there are positive numbers $\beta_1, \ldots, \beta_{m+1}$ such that:

$$Q_0 = \sum_{i=1}^{m+1} \beta_i Q_i \tag{2.12}$$

where $\beta_1, \ldots, \beta_m > 0$ are chosen to satisfy (2.8) and $\beta_{m+1} = 1$ (given the eigenvalues of the matrices, finding such parameters is a trivial task). We can now use the decomposition technique in order to find the desired dual problem. The decomposition is obtained by duplicating the variables $x \in \Re^n$, so that the resulting problem is equivalent to (QCQP) in the variables (x, x_i) $i = 1, 2, \ldots, m+1$.

$$\begin{array}{ll} \text{minimize} & x^T Q_0 x + 2b_0^T x \\ \text{s.t.} & x_i^T Q_i x_i + 2b_i^T x_i \leq c_i \ \ \forall i = 1, 2, \ldots m+1 \\ & x_i = x \ \ \forall i = 1, \ldots, m+1 \end{array} \tag{2.13}$$

Substituting (2.12) we have that (2.13) is equivalent to:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m+1} \beta_i x_i^T Q_i x_i + 2b_0^T x \\ \text{s.t.} & x_i^T Q_i x_i + 2b_i^T x_i \leq c_i \ \ \forall i = 1, 2, \ldots m+1 \\ & x_i = x \ \ \forall i = 1, \ldots, m+1 \end{array}$$

where $x_1, \ldots, x_{m+1}$ are vectors in $\Re^n$. We associate a Lagrange multiplier $\lambda_i \in \Re^n$ for every constraint $x_i = x$ and form the Lagrangian:

$$\begin{aligned} L(x, x_1, \ldots, x_{m+1}, \lambda_1, \ldots, \lambda_{m+1}) &= \sum_{i=1}^{m+1} \beta_i x_i^T Q_i x_i + 2b_0^T x + \sum_{i=1}^{m+1} 2\lambda_i^T (x_i - x) \\ &= \sum_{i=1}^{m+1} (\beta_i x_i^T Q_i x_i + 2\lambda_i^T x_i) + 2 \left(b_0 - \sum_{i=1}^{m+1} \lambda_i \right)^T x. \end{aligned}$$

Consequently, the dual problem of (QCQP) is

$$\max\{h(\lambda_1, \ldots \lambda_{m+1})\}.$$

where

$$\begin{aligned} h(\lambda_1, \ldots \lambda_{m+1}) &= \inf_{x_i^T Q_i x_i + 2b_i^T x_i \leq c_i \ \forall i=1,\ldots,m+1} L(x, x_1, \ldots, x_{m+1}, \lambda_1, \ldots, \lambda_{m+1}) \\ &= \sum_{i=1}^{m+1} \inf_{x_i^T Q_i x_i + 2b_i^T x_i \leq c_i} (\beta_i x_i^T Q_i x_i + 2\lambda_i^T x_i) + \inf_x \left(2(b_0 - \sum_{i=1}^{m+1} \lambda_i)^T x \right) \end{aligned}$$

Notice that a direct consequence of the last expression is that $h(\lambda_1, \ldots \lambda_{m+1}) > -\infty$ iff $\sum_{i=1}^{m+1} \lambda_i = b_0$. Thus,

$$h(\lambda_1, \ldots, \lambda_{m+1}) = \begin{cases} \sum_{i=1}^{m+1} \inf_{x_i^T Q_i x_i + 2b_i^T x_i \leq c_i} (\beta_i x_i^T Q_i x_i + 2\lambda_i^T x_i) & \sum_{i=1}^{m+1} \lambda_i = b_0 \\ -\infty & \sum_{i=1}^{m+1} \lambda_i \neq b_0 \end{cases} \tag{2.14}$$

In order to find an explicit expression for $h(\lambda_1, \ldots, \lambda_{m+1})$ we will solve each of the minimization problems in (2.14). The next lemma enables us to find the required expression.

Lemma 2.4.3 *Let Q be an $n \times n$ positive definite matrix, let $b, \lambda \in \Re^n, c \in \Re$ s.t. $c + b^T Q^{-1} b > 0$. Then,*

$$\min_{x^T Qx + 2b^T x \leq c} (x^T Qx + 2\lambda^T x) = \gamma q_Q(z) - 2\sqrt{\gamma} z^T Q^{-1} b - b^T Q^{-1} b$$

where,

$$\begin{aligned}
q_Q(z) &= \begin{cases} -z^T Q^{-1} z & \text{if } z^T Q^{-1} z \leq 1 \\ -2\sqrt{z^T Q^{-1} z} + 1 & \text{if } z^T Q^{-1} z > 1 \end{cases} \\
z &= \frac{\lambda - b}{\sqrt{\gamma}}, \\
\gamma &= c + b^T Q^{-1} b.
\end{aligned}$$

Proof: First, we make the following change of variables:

$$y = x + Q^{-1} b,$$

and we obtain the equivalent minimization problem

$$\min_{y^T Qy \leq c + b^T Q^{-1} b} (y^T Qy + 2(\lambda - b)^T y - 2\lambda^T Q^{-1} b + b^T Q^{-1} b)$$

Define η, γ as follows

$$\begin{aligned}
\eta &= \lambda - b, \\
\gamma &= c + b^T Q^{-1} b,
\end{aligned}$$

we obtain the following equivalent minimization problem

$$R(\eta) = \min_{y^T Qy \leq \gamma} (y^T Qy + 2\eta^T y). \tag{2.15}$$

Solving the later via KKT one has:

$$\begin{aligned}
Qy + \eta + \mu Qy &= 0 \Rightarrow (\mu + 1) Qy = -\eta \\
\mu(y^T Qy - \gamma) &= 0 \\
\mu &\geq 0
\end{aligned}$$

Therefore, if $y^TQy < \gamma$ then $\mu = 0$ and we have $y = -Q^{-1}\eta$ so we obtain $R(\eta) = -\eta^TQ^{-1}\eta$ in the case $y^TQy < \gamma$.
Otherwise, $y^TQy = \gamma$ which implies $(\mu+1)^2 = \frac{\eta^TQ^{-1}\eta}{\gamma}$ and thus $\mu = \sqrt{\frac{\eta^TQ^{-1}\eta}{\gamma}} - 1$. Substituting this in the objective function in (2.15) we get $R(\eta) = -2\sqrt{\gamma}\sqrt{\eta^TQ^{-1}\eta} + \gamma$.
Make the change of variables $\eta = \sqrt{\gamma}z$ and the lemma is proved. □

Now, we will use the separable structure of the minimization problem (2.14). We use the following notations:

$$\begin{aligned} z_i &= \frac{\frac{1}{\beta_i}\lambda_i - b_i}{\sqrt{\gamma_i}} \;\; \forall i = 1,\ldots,m+1, \\ \gamma_i &= c_i + b_i^TQ_i^{-1}b_i \;\; \forall i = 1,\ldots,m+1. \end{aligned}$$

Then,

$$\begin{aligned} h(\lambda_1,\ldots\lambda_{m+1}) &= \sum_{i=1}^{m+1} \inf_{x_i^TQ_ix_i+2b_i^Tx_i\le c_i} (\beta_i x_i^TQ_ix_i + 2\lambda_i^Tx_i) + \inf_x \left(2(b_0 - \sum_{i=1}^{m+1}\lambda_i)^Tx\right) \\ &= \sum_{i=1}^{m+1} \beta_i \inf_{x_i^TQ_ix_i+2b_i^Tx_i\le c_i} (x_i^TQ_ix_i + 2\frac{1}{\beta_i}\lambda_i^Tx_i) + \inf_x \left(2(b_0 - \sum_{i=1}^{m+1}\lambda_i)^Tx\right) \\ &= \sum_{i=1}^{m+1} (\beta_i\gamma_i q_{Q_i}(z_i) - 2\sqrt{\gamma_i}z_i^TQ_i^{-1}b_i - b_i^TQ_i^{-1}b_i) + \inf_x \left(2(b_0 - \sum_{i=1}^{m+1}\lambda_i)^Tx\right) \\ &= \begin{cases} \sum_{i=1}^{m+1}(\beta_i\gamma_i q_{Q_i}(z_i) - 2\sqrt{\gamma_i}z_i^TQ_i^{-1}b_i - b_i^TQ_i^{-1}b_i) & \sum_{i=1}^{m+1}(\sqrt{\gamma_i}\beta_i z_i + \beta_i b_i) = b_0 \\ -\infty & \text{else} \end{cases} \end{aligned}$$

Thus, a dual to problem to (QCQP) is:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{m+1} (\beta_i\gamma_i q_{Q_i}(z_i) - 2\sqrt{\gamma_i}z_i^TQ_i^{-1}b_i - b_i^TQ_i^{-1}b_i) \\ \text{s.t.} \quad & \sum_{i=1}^{m+1} \sqrt{\gamma_i}\beta_i z_i = b_0 - \sum_{i=1}^{m+1}\beta_i b_i. \end{aligned}$$

Denote for $i = 1,2,\ldots,m+1$,

$$\begin{aligned}\delta_i &= \beta_i \gamma_i, \\ h_i &= -\sqrt{\gamma_i} Q_i^{-1} b_i, \\ \alpha_i &= \sqrt{\gamma_i} \beta_i,\end{aligned}$$

and

$$\begin{aligned}f &= -\sum_{i=1}^{m+1} b_i^T Q_i^{-1} b_i, \\ e &= b_0 - \sum_{i=1}^{m+1} \beta_i b_i.\end{aligned}$$

We summarize our analysis in the following

Theorem 2.4.1 (A Dual Problem for (QCQP)) *A dual problem for (QCQP)is given by (DQCQP) defined by:*

$$(DQCQP) \quad \begin{aligned} &\textit{maximize} \quad \sum_{i=1}^{m+1} (\delta_i q_{Q_i}(z_i) + h_i^T z_i) + f \\ &s.t. \qquad \sum_{i=1}^{m+1} \alpha_i z_i = e \end{aligned}$$

where

- $$q_Q(z) = \begin{cases} -z^T Q^{-1} z & z^T Q^{-1} z \leq 1 \\ -2\sqrt{z^T Q^{-1} z} + 1 & else \end{cases} \tag{2.16}$$
- $\beta_1, \ldots, \beta_m$ *are* m *positive real numbers such that*

$$\sum_{i=1}^{m} \beta_i \lambda_{max}(Q_i) < \lambda_{min}(Q_0)$$

- $$\begin{aligned} Q_{m+1} &= Q_0 - \sum_{i=1}^{m} \beta_i Q_i, \\ c_{m+1} &= \lambda_{max}(Q_{m+1}) \min_{i=1,\ldots,m} \left(\frac{1}{\sqrt{\lambda_{min}(Q_i)}} \sqrt{c_i + b^T Q_i^{-1} b_i} + \|Q_i^{-1} b_i\| \right)^2, \\ b_{m+1} &= 0, \\ \beta_{m+1} &= 1, \\ \delta_i &= \beta_i (c_i + b_i^T Q_i^{-1} b_i), \quad i = 1, \ldots, m+1 \\ f &= -\sum_{i=1}^{m+1} b_i^T Q_i^{-1} b_i, \\ e &= b_0 - \sum_{i=1}^{m+1} \beta_i b_i. \end{aligned}$$

- $$\begin{aligned} h_i &= -\sqrt{c_i + b_i^T Q_i^{-1} b_i} Q_i^{-1} b_i, \\ \alpha_i &= \sqrt{c_i + b_i^T Q_i^{-1} b_i} \beta_i. \end{aligned}$$

By following the analysis of derivation of the dual problem, we can easily obtain the connection between the optimal solution of (QCQP) and the optimal solution of (DQCQP). This connection is presented in the following lemma:

Lemma 2.4.4 *Suppose that $(z_1, z_2, \ldots, z_{m+1})$ is the solution of (DQCQP). Define the following variables for $i = 1, 2, \ldots, m+1$:*

$$x_i = \begin{cases} -\sqrt{\frac{\gamma_i}{z_i^T Q_i^{-1} z_i}} Q_i^{-1} z_i - Q_i^{-1} b_i & \text{if } z_i^T Q_i^{-1} z_i \geq 1 \\ -Q_i^{-1}(\sqrt{\gamma_i} z_i + b_i) & \text{if } z_i^T Q_i^{-1} z_i < 1 \end{cases}.$$

Then, $x_1 = x_2 = \ldots = x_{m+1}$ and their common value x is the solution to (QCQP).

We will now show that the objective function in (DQCQP) $h(z_1, \ldots, z_{m+1}) = \sum_{i=1}^{m+1} (\delta_i q_{Q_i}(z_i) + h_i^T z_i) + f$ is a concave function with Lipschitz continuous gradient.

Theorem 2.4.2 *The objective function of (DQCQP) satisfies the following properties:*

1. *h is concave.*

2. *h is continuously differentiable and has a Lipschitz continuous gradient with Lipschitz constant* $L_h = \max_{1 \le i \le m+1} \left\{ \frac{\delta_i}{\lambda_{min}(Q_i)} \right\}$.

Proof:

1. A direct result of duality theory.

2. Clearly, from the separable structure of the function h, we have that it is sufficient to show that:

$$g_Q(\eta) = \begin{cases} \eta^T Q^{-1} \eta & \eta^T Q^{-1} \eta \le 1 \\ 2\sqrt{\eta^T Q^{-1} \eta} - 1 & \text{else} \end{cases} . \tag{2.17}$$

where $Q \succ 0$ has a Lipschitz continuous gradient ∇g_Q (Note that $g_Q = -q_Q$). In fact, this property follows directly from the general result on proximal regularization on convex functions (Lemma B.0.1). Indeed, we will now show that g_Q is a proximal regularization of the l_2 norm. More specifically, we will prove that:

$$g_Q(u) = 2 \inf_{v \in \Re^n} \left\{ \|v\|_{Q^{-1}} + \frac{1}{2} \|v - u\|^2_{Q^{-1}} \right\}, \tag{2.18}$$

where $\|z\|_{Q^{-1}} := \sqrt{z^T Q^{-1} z}$, $Q \succ 0$, Lemma B.0.1 then implies that g_Q is differentiable and finite everywhere and has a Lipschitz gradient with Lipschitz constant $2\lambda_{max}(Q^{-1})$. Thus, let us prove (2.18). Let $h_1(v) = \frac{1}{2}\|v - u\|^2_{Q^{-1}}$ and $h_2(v) = \|v\|_{Q^{-1}}$. Then, $g_Q(u) = 2 \inf_{v \in \Re^n} \{h_1(v) + h_2(v)\}$. Invoking Fenchel Duality Theorem we have:

$$g_Q(u) = 2 \sup_z \{-h_1^*(z) - h_2^*(z)\}.$$

But the conjugates are:

$$\begin{aligned} h_1^*(z) &= \frac{1}{2} z^T Q z + z^T u \\ h_2^*(z) &= \begin{cases} 0 & \text{if } \|z\|_Q \le 1 \\ +\infty & \text{otherwise} \end{cases} . \end{aligned}$$

Therefore, $g_Q(u) = \sup\{-z^T Qz - 2z^T u : \|z\|_Q \le 1\}$. Invoking Lemma 2.4.3 (see (2.15)) it follows that (2.19) and (2.18) coincide, and a simple computation shows that ∇h has a Lipschitz constant $L_h = \max_{1\le i\le m+1}\left\{\frac{\delta_i}{\lambda_{min}(Q_i)}\right\}$.

□

2.5 Linear Rate Of Convergence of GPA for QCQP

In the previous section we have proven that the dual of the strictly convex quadratic program (QCQP) is (DQCQP) which is the problem of maximizing a concave function with Lipschitz continuous gradient subject to affine constraints. The implementation of GPA to (DQCQP) is trivial because the orthogonal projection on an affine set is just a linear transformation. Our main task which thus remains is to prove that GREB is fulfilled for (DQCQP) so that by Theorem 2.3.2 the linear rate of convergence of distances of from the optimal set of the sequence produced by GPA on (DQCQP) will follow. Furthermore, since the strong duality holds for the pair (QCQP) and (DQCQP) then as a consequence Theorem 2.3.3 this will prove the linear convergence of the sequence of the function values for both problems.
¿From convenience reasons we will describe (DQCQP) as a minimization problem and omit the constant f in the objective. Thus by Theorem 2.4.1 the dual is equivalent to:

$$(DQCQP)\qquad \begin{array}{ll} \text{minimize} & \sum_{i=1}^{m+1} \delta_i g_{Q_i}(\eta_i) - h_i^T \eta_i \\ \text{s.t.} & \sum_{i=1}^{m+1} \alpha_i \eta_i = e, \end{array}$$

where,

$$g_Q(\eta) = \begin{cases} \eta^T Q^{-1}\eta & \eta^T Q^{-1}\eta \le 1 \\ 2\sqrt{\eta^T Q^{-1}\eta} - 1 & \text{else} \end{cases}. \tag{2.19}$$

¿From now on, the term (DQCQP) will refer to the minimization problem and not to the maximization problem defined in the previous section. Denote,

$$f(\eta) = \sum_{i=1}^{m+1} (\delta_i g_{Q_i}(\eta_i) - h_i^T \eta_i), \tag{2.20}$$

where $\eta = (\eta_1, \ldots, \eta_{m+1})$ and $\eta_i \in \Re^n \quad \forall i = 1, \ldots, m+1$. f is the objective function of (DQCQP) and it was proved in the previous section that f has Lipschitz continuous gradient. The feasible set is denoted by S and defined by:

$$S = \left\{ \eta : \sum_{i=1}^{m+1} \alpha_i \eta_i = e \right\}.$$

S is, of course, an affine set.
Note that S has a very special structure which enables us to find a simple and explicit expression for the projection P_S

Lemma 2.5.1 *Let $y = (y_1, \ldots, y_{m+1})$ then,*

$$P_S y = (y_j - \alpha_j \eta)_{j=1}^{m+1},$$

where,

$$\eta = \frac{\sum_{j=1}^{m+1} \alpha_j y_j - e}{\sum_{j=1}^{m+1} \alpha_j^2}.$$

Proof: A direct result of Theorem A.2.3.
The linear space associated with S is denoted by W:

$$W = \left\{ \eta : \sum_{i=1}^{m+1} \alpha_i \eta_i = 0 \right\}.$$

Terminology: a vector $d \in W$ is called a *feasible direction.*

The following technical lemma analyzes the relation between the projection on S and the projection on W.

Lemma 2.5.2 *There exists $b \in S$ such that*

$$P_S \eta = P_W \eta + b \quad \forall \eta \in \Re^{(m+1)n}.$$

Proof: Let s be some point in S. Let $\eta \in \Re^{(m+1)n}$ then,

$$\begin{aligned} P_S(\eta) &= \operatorname*{argmin}_{x\in S} \|x-\eta\| \\ &= \operatorname*{argmin}_{x\in S} \|x-s+s-\eta\| \\ &\overset{y=x-s}{=} \operatorname*{argmin}_{y\in W} \|y-\eta+s\|+s \\ &= \operatorname*{argmin}_{y\in W} \|y-(\eta-s)\|+s \\ &= P_W(\eta-s)+s \\ &= P_W(\eta)-P_W s+s. \end{aligned}$$

Denote $b = s - P_W s \in S$ and the lemma is proved. $\square$

The GREB assumption is usually very hard to prove in a direct way. It is very useful to note that the following assumption implies GREB:

Assumption 8 *There is $\epsilon > 0$ such that for every bounded set B, there is $\sigma_B > 0$ such that:*

$$d(x, X^*) \leq \sigma_B T(x) \quad \forall x \in B \cap X^*_\epsilon \cap S,$$

where $X^*_\epsilon = \{x \ : \ d(x, X^*) \leq \epsilon\}$.

The following lemma states that GREB is equivalent to assumption 8.

Lemma 2.5.3 *Let f be a function with Lipschitz continuous gradient. Then, assumption 8 is equivalent to GREB.*

Proof: First, assume that GREB is fulfilled. Then, assumption 8 is true because $B \cap X^*_\epsilon$ is also a bounded set. Now, assume that assumption 8 is fulfilled. Let B be a bounded set. Define the function $h(x) = \frac{d(x,X^*)}{T(x)}$. By assumption 8 we have a positive number $\sigma_B > 0$ such that:

$$h(x) \leq \sigma_B \ \ \forall x \in B \cap X^*_\epsilon \cap S.$$

But, $h(x)$ is continuous on the closed bounded set $cl(B - X^*_\epsilon) \cap S$. Thus, by Weierstrass' theorem, $h(x)$ is bounded over $cl(B - X^*_\epsilon) \cap S$ so that there is a $\tau > 0$ such that:

$$h(x) \leq \tau \quad \forall x \in cl(B - X_\epsilon^*) \cap S,$$

and hence it follows that,

$$h(x) \leq \max\{\sigma_B, \tau\} \quad \forall x \in B \cap S.$$

$\square$

Another possible assumption that *implies assumption 8* is the following:

Assumption 9 *There is a $\epsilon > 0$ and $\sigma > 0$ such that:*

$$\forall x \in X_\epsilon^* \cap S \quad d(x, X^*) \leq \sigma T(x),$$

where $X_\epsilon^* = \{x \; : \; d(x, X^*) \leq \epsilon\}$.

Assumption 9 will be used in the analysis of subsection 2.5.1, while assumption 8 will be used in the analysis of subsection 2.5.2.

Remark: If X^* is bounded then GREB is equivalent to Assumption 9. This is true for instance in the case where $h_i = 0$ (the pure quadratic case) where the objective function is coercive which enforces the optimal set X^* to be bounded (and also to be non empty).

The proof that assumption 8 or assumption 9 is fulfilled for (DQCQP) is rather involved, and thus will be separated into two cases: the case where X^* (the set of optimal points) is a singleton and then the general case. Furthermore, the analysis of the case $X^* = \{\eta^*\}$ will pave the way to prove the more general case.

2.5.1 The First Case: $X^* = \{\eta^*\}$.

A Sufficient Condition For GREB

In the first case we assume that (DQCQP) has a unique minimizer η^*. First, for every feasible direction $d \in W$, we investigate the following scalar function:

$$h_d(\beta) = f(\eta^* + \beta d) \quad \forall\, 0 \leq \beta \leq 1,$$

and find a condition in terms of the function $h_d(\beta)$ that implies GREB. The following technical lemma is a key argument in proving the sufficient condition for GREB.

Lemma 2.5.4 *Assumption 9 is equivalent to the following condition: There exists* $\epsilon > 0$ *and* $\sigma > 0$ *such that,*

$$\frac{\|P_W \nabla f(\eta^* + \beta d)\|}{\beta} \geq \frac{\epsilon}{t\sigma} \quad \forall \, \|d\| = \epsilon, d \in W, \forall \, 0 < \beta \leq 1, \forall \, t > 0.$$

Proof: By assumption 9 there is a $\sigma > 0$ such that,

$$\|\eta - \eta^*\| \leq \sigma \|\eta - P_S(\eta - t\nabla f(\eta))\| \quad \forall \eta \in S, \|\eta - \eta^*\| \leq \epsilon. \tag{2.21}$$

Now, for every $\eta \in S$,

$$\begin{aligned}
\eta - P_S(\eta - t\nabla f(\eta)) &\overset{\text{Lemma 2.5.2}}{=} \eta - P_W(\eta - t\nabla f(\eta)) - b \\
&\overset{P_W \text{ is a linear operator}}{=} \eta - P_W\eta + tP_W\nabla f(\eta) - b \\
&\overset{\text{Lemma 2.5.2}}{=} \eta - P_S\eta + b + tP_W\nabla f(\eta) - b \\
&\overset{\eta \in S}{=} \eta - \eta + b + tP_W\nabla f(\eta) - b \\
&= tP_W\nabla f(\eta).
\end{aligned}$$

Substituting this in (2.21) we have that assumption 9 is equivalent to:

$$\|\eta - \eta^*\| \leq \sigma t \|P_W \nabla f(\eta)\| \quad \forall \eta \in S, \|\eta - \eta^*\| \leq \epsilon.$$

Substituting $\eta = \eta^* + \beta d$ where $d \in W, \|d\| = \epsilon$ and $0 \leq \beta \leq 1$, we obtain that assumption 9 is equivalent to:

$$\|\beta d\| \leq \sigma t \|P_W \nabla f(\eta^* + \beta d)\| \quad \forall d \in W, \|d\| = \epsilon, 0 \leq \beta \leq 1.$$

Since the inequality is trivial for $\beta = 0$ we can dismiss the case $\beta = 0$ and hence we have the following equivalent inequality:

$$\beta\|d\| = \|\beta d\| \leq \sigma t \|P_W \nabla f(\eta^* + \beta d)\| \quad \forall d \in W, \|d\| = \epsilon, 0 < \beta \leq 1.$$

Dividing by β yields:

$$\frac{\|P_W \nabla f(\eta^* + \beta d)\|}{\beta} \geq \frac{\epsilon}{t\sigma} \quad \forall d \in W, \|d\| = \epsilon, 0 < \beta \leq 1.$$

□

It is now possible to state a condition that implies assumption 9 (and thus also implies GREB).

Lemma 2.5.5 (A Sufficient Condition for GREB) *The following condition implies GREB: There is $\epsilon > 0$ and $s > 0$ (which possibly depends on ϵ) such that*

$$\frac{h'_d(\beta)}{\beta} \geq s \quad \forall d \in W, \|d\| = \epsilon, 0 < \beta \leq 1,$$

Proof: Let $\{\psi_1, \psi_2, \ldots, \psi_k\}$ be an orthonormal basis for W, then $d \in W$ implies that:

$$d = \sum_{j=1}^{k} \langle d, \psi_j \rangle \psi_j. \tag{2.22}$$

Recall that P_W is a projection on the linear space W. Thus, for all η,

$$P_W \eta = \sum_{j=1}^{k} \langle \eta, \psi_j \rangle \psi_j.$$

So that,

$$\|P_W \eta\|^2 = \sum_{j=1}^{k} \langle \eta, \psi_j \rangle^2.$$

Now, compute $h'_d(\beta)$ using the directional derivative formula:

$$\begin{aligned} h'_d(\beta) &= \langle d, \nabla f(\eta^* + \beta d) \rangle \\ &\overset{(2.22)}{=} \sum_{j=1}^{k} \langle d, \psi_j \rangle \langle \psi_j, \nabla f(\eta^* + \beta d) \rangle \\ &\leq \sum_{j=1}^{k} |\langle d, \psi_j \rangle| \cdot |\langle \psi_j, \nabla f(\eta^* + \beta d) \rangle|. \end{aligned}$$

By Cauchy Schwartz inequality one has for all $j = 1, 2, \ldots, k$, $|\langle d, \psi_j \rangle| \leq \overbrace{\|d\|}^{\epsilon} \cdot \overbrace{\|\psi_j\|}^{1} = \epsilon$. Also, from the equivalence of norms in finite dimension spaces we obtain that there is a $N > 0$ such that $\|x\|_1 \leq N\|x\|$, where $\|\cdot\|_1$ is the usual l_1 norm. Therefore,

$$\begin{aligned} h_d'(\beta) &\leq \sum_{j=1}^{k} |\langle d, \psi_j \rangle| \cdot |\langle \psi_j, \nabla f(\eta^* + \beta d) \rangle| \\ &\leq \epsilon \sum_{j=1}^{k} |\langle \psi_j, \nabla f(\eta^* + \beta d) \rangle| \\ &\leq \epsilon N \sqrt{\sum_{j=1}^{k} \langle \psi_j, \nabla f(\eta^* + \beta d) \rangle^2} \\ &= \epsilon N \| P_W \nabla f(\eta^* + \beta d) \|. \end{aligned} \tag{2.23}$$

Under the hypothesis of the lemma one has:

$$\exists s > 0,\ \exists \epsilon > 0 : \quad \frac{h_d'(\beta)}{\beta} \geq s \quad \forall d \in W, \|d\| = \epsilon, 0 < \beta \leq 1,$$

and hence with (2.23) one obtains:

$$\frac{\|P_W \nabla f(\eta^* + \beta d)\|}{\beta} \geq \frac{s}{\epsilon N} \quad \forall \|d\| = \epsilon, d \in W, 0 < \beta \leq 1.$$

Invoking Lemma 2.5.4, the latter relation is equivalent to Assumption 9 which thus implies that GREB holds. □

A Necessary Condition For An Optimum

In this Section we find a necessary condition for an optimum of (DQCQP). This condition on η^* will be the core of the proof of the GREB assumption for (DQCQP). The uniqueness of the optimum is a crucial part of the proof and thus there is no simple generalization of the condition to the general case (where the optimum is not necessarily unique). The essence of the condition is that $h_d(\beta)$ can not be linear on an interval containing zero because the objective function is differentiable.

Lemma 2.5.6 *Let $0 \neq d \in W$ be some feasible direction and let $s > 0$. Then, $h_d(\beta)$ is not linear on the interval $[0, s]$.*

Proof: Suppose, in contradiction, that there is a direction $d \in W$ and a $s > 0$ such that $h_d(\beta)$ is linear on $[0, s]$. There are two cases:

1. **The slope of the line is zero.** In this case, all the points $\eta = \eta^* + \beta d$ are in the feasible set S and are minimizers of (DQCQP). Thus, we have a contradiction to the uniqueness of the minimizer.

2. **The slope of the line is not zero.** η^* is the minimizer of (DQCQP) and h_d is a differentiable function thus $h'_d(0) = 0$ by fermat's theorem. On the other hand, $h_d(\beta)$ is linear on $[0, s]$ with nonzero slope which yields $h'_{d_+}(0) \neq 0$, and so we have a contradiction to the differentiability of the function f.

□

We will use the following notation. For every positive definite matrix Q:

$$\|\eta\|_Q = \sqrt{\eta^T Q \eta}.$$

The next theorem states that a certain linear system admits only the trivial solution. This property will be the most important argument in proving the necessary condition on η^*.

Theorem 2.5.1 *Let η^* be the unique minimizer of (DQCQP). Then, the following linear system of equalities and inequalities doesn't have any solution other than the trivial one (i.e. $\theta_j = 0 \ \ \forall j = 1, 2, \ldots, m+1$):*

$$(LS) \begin{cases} \sum_{j \in J \cup K} \theta_j \alpha_j \eta_j^* = 0 & \\ & \\ \theta_j \geq 0 & \forall j \in K \\ \theta_j = 0 & \forall j \in I \end{cases}$$

where,

$$I = \left\{ j : \|\eta_j^*\|_{Q_j^{-1}} < 1 \right\} \ , \ J = \left\{ j : \|\eta_j^*\|_{Q_j^{-1}} > 1 \right\} \ , \ K = \left\{ j : \|\eta_j^*\|_{Q_j^{-1}} = 1 \right\} \qquad (2.24)$$

Remark: I, J, K is a partition of the index set $\{1, 2, \ldots, m+1\}$.
Proof: Suppose in contradiction that (LS) *does* have a nonzero solution. Define:

$$d_j = \begin{cases} \theta_j \eta_j^* & j \in J \cup K \\ 0 & j \in I \end{cases} . \tag{2.25}$$

Then, $d = (d_1, \ldots, d_{m+1})$ is a feasible direction (i.e. $\sum_{j=1}^{m+1} \alpha_j d_j = 0$ so $d \in W$). Also, for $\beta \in [0, B]$ where,

$$B = \min_{j:\theta_j < 0} \left\{ \frac{1}{\theta_j} \left(\frac{1}{\|\eta_j^*\|_{Q_j^{-1}}} - 1 \right) \right\} > 0, \tag{2.26}$$

the following is satisfied:

$$\begin{aligned} \|\eta_j^* + \beta d_j\|_{Q_j^{-1}} &\geq 1 \quad \forall j \in J \cup K & (2.27) \\ 1 + \beta\theta_j &\geq 0 \quad \forall j \in J \cup K & (2.28) \end{aligned}$$

Thus, for any $d \in W$ as defined in (2.25) and any $\beta \in [0, B]$ one has using the definition of g_Q and f (c.f. (2.19),(2.20)):

$$\begin{aligned} h_d(\beta) &= f(\eta^* + \beta d) \\ &= \sum_{j=1}^{m+1} (\delta_j g_{Q_j}(\eta_j^* + \beta d_j) - h_j^T(\eta_j^* + \beta d_j)) \\ &= \sum_{j \in J \cup K} (\delta_j g_{Q_j}(\eta_j^* + \beta d_j) - h_j^T(\eta_j^* + \beta d_j)) + \underbrace{\sum_{j \in I} (\delta_j g_{Q_j}(\eta_j^*) - h_j^T \eta_j^*)}_{constant} \\ &= \sum_{j \in J \cup K} (\delta_j g_{Q_j}(\eta_j^* + \beta\theta_j\eta_j^*) - h_j^T(\eta_j^* + \beta\theta_j\eta_j^*)) + \text{constant} \\ &\overset{(2.27)}{=} \sum_{j \in J \cup K} (2\delta_j |1 + \beta\theta_j| \sqrt{(\eta^*)^T Q_j^{-1} \eta_j^*} - \beta\theta_j h_j^T \eta_j^* - h_j^T \eta_j^*) + \text{constant} \\ &\overset{(2.28)}{=} \underbrace{\sum_{j \in J \cup K} (2\delta_j (1 + \beta\theta_j) \sqrt{(\eta^*)^T Q_j^{-1} \eta_j^*} - \beta\theta_j h_j^T \eta_j^* - h_j^T \eta_j^*) + \text{constant}}_{\text{linear in } \beta} . \end{aligned}$$

To summarize, we have obtained that $h_d(\beta)$ is a linear function of β on $[0, B]$ in contradiction to Lemma 2.5.6. Thus, (LS) admits only the trivial solution $\theta_j = 0 \;\; \forall j$. $\square$

Theorem 2.5.2 (Necessary Condition for an Optimum) *Let η^* be the unique minimizer of (DQCQP). Then, for every $\epsilon > 0$ there exists $\xi > 0$ such that the following system of inequalities:*

$$\begin{cases} 2\epsilon^2 \geq \sum_{j \in J \cup K} \|\theta_j \eta_j^*\|^2 \geq \frac{\epsilon^2}{4} & \\ & \\ \theta_j \geq 0 & \forall j \in K = \{j : \|\eta_j^*\|_{Q_j^{-1}} = 1\} \\ \theta_j = 0 & \forall j \in I \end{cases} .$$

implies

$$\left\| \sum_{j \in J \cup K} \alpha_j \theta_j \eta_j^* \right\| \geq \xi > 0.$$

Proof: Consider the following minimization problem in the variables θ_j:

$$\begin{aligned} \text{minimize} \quad & \left\| \sum_{j \in J \cup K} \alpha_j \theta_j \eta_j^* \right\| \\ \text{s.t.} \quad & 2\epsilon^2 \geq \sum_{j \in J \cup K} \|\theta_j \eta_j^*\|^2 \geq \frac{\epsilon^2}{4} \\ & \theta_j \geq 0 \quad \forall j \in K. \end{aligned}$$

The feasible set of this problem is closed, bounded and nonempty. Thus, by Weierstrass theorem the minimum is attained. By Theorem 2.5.1, the minimum can not be zero and thus denoting the value of the minimum by $\xi > 0$ the desired result follows. □

Proving GREB for the First Case $X^* = \{\eta^*\}$

We will need some technical lemmas before proving GREB in the first case.

Lemma 2.5.7 *For any $d \in \Re^{(m+1)n}$ define $z(\alpha) = 2\|\eta^* + \alpha d\|_{Q^{-1}} - 1$. Then,*

$$\frac{d^2 z}{d\alpha^2} := z''(\alpha) = 2\frac{\|d\|_{Q^{-1}}^2 \|\eta^*\|_{Q^{-1}}^2 - (d^T Q^{-1} \eta^*)^2}{\|\eta^* + \alpha d\|_{Q^{-1}}^3},$$

for every α such that $\|\eta^* + \alpha d\|_{Q^{-1}} > 0$.

Proof: First, we will show the result for $Q = I$ and the conclusion will follow by an elementary argument. Consider the function $t_{v,w}(\alpha) = \|w + \alpha v\|$. For every α such that $\|w + \alpha v\| > 0$,

$$t_{v,w}(\alpha) = \|w + \alpha v\| = \sqrt{\|w\|^2 + 2\alpha v^T w + \alpha^2\|v\|^2}.$$

Thus,

$$t'_{v,w}(\alpha) = \frac{\alpha\|v\|^2 + v^T w}{\|w + \alpha v\|}.$$

So,

$$\begin{aligned} t''_{v,w}(\alpha) &= \frac{\|v\|^2\|w+\alpha v\| - (\alpha\|v\|^2 + v^T w)^2 \cdot \frac{1}{\|w+\alpha v\|}}{\|w+\alpha v\|^2} \\ &= \frac{\|v\|^2\|w+\alpha v\|^2 - (\alpha\|v\|^2 + v^T w)^2}{\|w+\alpha v\|^3} \\ &= \frac{\|v\|^2(\|w\|^2 + 2\alpha v^T w + \alpha^2\|v\|^2) - \alpha^2\|v\|^4 - 2\alpha(v^T w)\|v\|^2 - (v^T w)^2}{\|w+\alpha v\|^3} \\ &= \frac{\|v\|^2\|w\|^2 - (v^T w)^2}{\|w+\alpha v\|^3}. \end{aligned} \tag{2.29}$$

Since Q^{-1} is a positive definite matrix, there is a matrix A s.t. $Q^{-1} = A^2$. Then, $z(\alpha) = 2\|\eta^* + \alpha d\|_{Q^{-1}} - 1 = 2t_{v,w}(\alpha) - 1$ where $w = A\eta^*, v = Ad$. The result follows by substituting this in (2.29) . $\square$

Lemma 2.5.8 *Let Q be a positive definite matrix and let $u, v \in \Re^n$. Then,*

$$\min_{\delta\in\Re} \|u - \delta v\|_Q^2 = \|u\|_Q^2 - \frac{(u^T Q v)^2}{\|v\|_Q^2},$$

where the minimum is attained at

$$\delta^* = \frac{u^T Q v}{\|v\|_Q^2}.$$

Proof: One has

$$\|u - \delta v\|_Q^2 = \|u\|_Q^2 - 2(u^T Q v)\delta + \|v\|_Q^2\delta^2,$$

and the result follows immediately by minimizing the resulting one dimensional quadratic function. □

We are now ready to prove our main result.

Theorem 2.5.3 (GREB is Fulfilled for (DQCQP)) *There exists* $\epsilon > 0$ *and* $\gamma > 0$ *such that,*

$$\frac{h'_d(\beta)}{\beta} \geq \gamma \quad \forall d \in W, \|d\| = \epsilon, \beta \in (0,1],$$

where

$$\begin{aligned} \epsilon &< \min_{j \notin K} \left\{ \frac{\left|1 - \|\eta_j^*\|_{Q_j^{-1}}\right|}{\|Q_j^{-1}\|^{\frac{1}{2}}}, \frac{1}{\|Q_j^{-1}\|^{\frac{1}{2}}} \right\} \\ \gamma &< \min_{j=1,\ldots,m+1} \left\{ 1, \frac{1}{4M_j}, \left(\frac{\epsilon}{4C}\right)^4, \left(\frac{\xi}{D}\right)^4 \right\}. \end{aligned} \tag{2.30}$$

ξ *is as defined in Theorem 2.5.2.* C, D *are defined by:*

$$C = \sqrt{(m+1) \cdot \max_{j=1,\ldots,m+1} \lambda_{max}(Q_j) \cdot \max_{j=1,\ldots,m+1} \left\{ \sqrt{M_j}, M_j, \frac{1}{2\delta_j} \right\}},$$

$$D = \sqrt{m+1} \cdot \max_{j=1,\ldots,m+1} \{|\alpha_j| \lambda_{max}(Q_j)\} \cdot \max_{j=1,\ldots,m+1} \left\{ \sqrt[4]{M_j}, \sqrt{M_j}, \frac{1}{\sqrt{2\delta_j}} \right\}.$$

where

$$M_j = \frac{(\|\eta_j^*\|_{Q_j^{-1}} + \|Q_j^{-1}\|^{\frac{1}{2}}\epsilon)^3}{2\delta_j} \quad \forall j = 1, \ldots, m+1. \tag{2.31}$$

Proof: We use the definitions of the index sets defined in (2.24).

For every feasible direction $d \in W$, we partition K into two disjoint sets: $K = K_1^d \cup K_2^d$ where,

$$K_1^d = \{j : \|\eta_j^*\|_{Q_j^{-1}} = 1 \ , \ d_j^T Q^{-1} \eta_j^* > 0\}, \tag{2.32}$$

$$K_2^d = \{j : \|\eta_j^*\|_{Q_j^{-1}} = 1 \ , \ d_j^T Q^{-1} \eta_j^* \leq 0\}. \tag{2.33}$$

We will show that the sufficient condition for GREB is fulfilled (Lemma 2.5.5). Assume otherwise that there exists a $d \in W$ such that $\|d\| = \epsilon$ and a $0 < \beta \leq 1$ such that,

$$\frac{h'_d(\beta)}{\beta} < \gamma.$$

We will now prove that this is a contradiction to Theorem 2.5.2. Note that by the choice of ϵ we have that

$$\begin{aligned} j \in I \quad &\Rightarrow \quad \|\eta_j^* + \beta d_j\|_{Q_j^{-1}} < 1 \ \ \forall \beta \in [0,1], \\ j \in J \quad &\Rightarrow \quad \|\eta_j^* + \beta d_j\|_{Q_j^{-1}} > 1 \ \ \forall \beta \in [0,1]. \end{aligned}$$

By the optimality of η^* we have $h'_d(0) = 0$. Now, $h'_d(\beta)$ is a continuous function with directional derivatives for every $\beta \in (0,1]$. Thus, by the mean value theorem (see Theorem B.0.3) there exists a $0 < c < \beta$ such that,

$$\gamma > \frac{h'_d(\beta)}{\beta} = \frac{h'_d(\beta) - h'_d(0)}{\beta - 0} \in [h''_{d_-}(c), h''_{d_+}(c)].$$

It is not known apriori if $h''_{d_-}(c) < h''_{d_+}(c)$ or $h''_{d_-}(c) \geq h''_{d_+}(c)$ so $[h''_{d_-}(c), h''_{d_+}(c)]$ is in fact the interval $[\min\{h''_{d_-}(c), h''_{d_+}(c)\}, \max\{h''_{d_-}(c), h''_{d_+}(c)\}]$.
In particular, we have that $h''_{d_-}(c) < \gamma$ or $h''_{d_+}(c) < \gamma$ (or both). Without loss of generality we assume that $h''_{d_+}(c) < \gamma$. Recall that

$$h_d(c) = \sum_{j=1}^{m+1} (\delta_j g_{Q_j}(\eta_j^* + cd_j) - h_j^T(\eta_j^* + cd_j)).$$

Define,

$$z_j(c) = g_{Q_j}(\eta_j^* + cd_j) \qquad \forall c \in [0,1], i = 1, \ldots, m+1.$$

With this definition, we have $h_d(c) = \sum_{j=1}^{m+1}(\delta_j z_j(c) - h_j^T(\eta_j^* + cd_j))$. Differentiating twice one obtains:

$$h''_{d_+}(c) = \sum_{i=1}^{m+1} \delta_i z''_{i_+}(c).$$

¿From the convexity of z_j we have that $z''_{j_+}(c) \geq 0$ for all $j = 1, 2, \ldots, m+1$ and $0 < c < 1$. As a consequence (recall that $\delta_j > 0$ for every j),

$$\gamma > h''_{d_+}(c) = \sum_{i=1}^{m+1} \delta_i z''_{i_+}(c) \geq \delta_j z''_{j_+}(c) \quad \forall j = 1, 2, \ldots, m+1.$$

We divide the investigation of the inequality $\frac{\gamma}{\delta_j} > z''_{j_+}(c)$ to several cases,

- $\boldsymbol{j \in I}$. In this case recall that $g_{Q_j}(u) = \|u\|^2_{Q_j^{-1}}$ and thus for all $\beta \in [0,1]$,

$$z_j(\beta) = \|\eta_j^* + \beta d_j\|^2_{Q_j^{-1}} = \|\eta_j^*\|^2_{Q_j^{-1}} + 2\beta d_j^T Q_j^{-1} \eta_j^* + \beta^2 \|d_j\|^2_{Q_j^{-1}}.$$

  Thus,

$$\frac{\gamma}{\delta_j} > z''_{j_+}(c) = 2\|d_j\|^2_{Q_j^{-1}}.$$

- $\boldsymbol{j \in J}$. In this case recall that $g_{Q_j}(u) = 2\|u\|_{Q_j^{-1}} - 1$ and thus $z_j(\beta) = 2\|\eta_j^* + \beta d_j\|_{Q_j^{-1}} - 1 \quad \forall \beta \in [0,1]$ and thus,

$$\frac{\gamma}{\delta_j} > z''_{j_+}(c) \overset{\text{Lemma 2.5.7}}{=} 2\frac{\|d_j\|^2_{Q_j^{-1}}\|\eta_j^*\|^2_{Q_j^{-1}} - (d_j^T Q_j^{-1}\eta_j^*)^2}{\|\eta_j^* + cd_j\|^3_{Q_j^{-1}}} \tag{2.34}$$

$$= \frac{2\|\eta_j^*\|^2_{Q_j^{-1}}}{\|\eta_j^* + cd_j\|^3_{Q_j^{-1}}}\left(\|d_j\|^2_{Q_j^{-1}} - \frac{(d_j^T Q_j^{-1}\eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}}}\right)$$

$$\overset{(2.31)}{>} \frac{1}{M_j\delta_j}\left(\|d_j\|^2_{Q_j^{-1}} - \frac{(d_j^T Q_j^{-1}\eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}}}\right). \tag{2.35}$$

  The last inequality is true because $\|\eta_j^*\|_{Q_j^{-1}} > 1 \; \forall j \in J$ and

$$\begin{aligned}
\|\eta_j^* + cd_j\|_{Q_j^{-1}} \leq \|\eta_j^*\|_{Q_j^{-1}} + c\|d_j\|_{Q_j^{-1}} &\leq \|\eta_j^*\|_{Q_j^{-1}} + c\|Q_j^{-1}\|^{\frac{1}{2}}\|d_j\| \\
&\leq \|\eta_j^*\|_{Q_j^{-1}} + c\|Q_j^{-1}\|^{\frac{1}{2}}\|d\| \\
&= \|\eta_j^*\|_{Q_j^{-1}} + c\|Q_j^{-1}\|^{\frac{1}{2}}\epsilon \\
&\overset{c<1}{<} \|\eta_j^*\|_{Q_j^{-1}} + \|Q_j^{-1}\|^{\frac{1}{2}}\epsilon \\
&\overset{(2.31)}{=} \sqrt[3]{2M_j\delta_j}.
\end{aligned}$$

By Lemma 2.5.8 there is $\theta_j \in \Re$ such that,

$$\begin{aligned}\|d_j - \theta_j \eta_j^*\|^2_{Q_j^{-1}} &= \|d_j\|^2_{Q_j^{-1}} - \frac{(d_j^T Q_j^{-1} \eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}}} \\ &\overset{(2.35)}{<} M_j \gamma.\end{aligned}$$

- $\boldsymbol{j \in K_1^d}$ In this case $\|\eta_j^* + c d_j\|_{Q_j^{-1}} > 1$ and thus $z''_{j+}(c)$ has the same form as in (2.34) and so there is a $\theta_j \in \Re$ such that,

$$\|d_j - \theta_j \eta_j^*\|^2_{Q_j^{-1}} \leq M_j \gamma.$$

  We also have that $d_j^T Q_j^{-1} \eta_j^* > 0$ which implies that (see Lemma 2.5.8) $\theta_j = \frac{d_j^T Q_j^{-1} \eta_j^*}{\|\eta_j^*\|_{Q_j^{-1}}} > 0$.

- $\boldsymbol{j \in K_2^d}$ Here we have two possibilities:

  Case 1: $z''_{j+}(c) = 2\|d_j\|^2_{Q_j^{-1}}$. Thus, (as in the case $j \in I$) one has:

$$\|d_j\|^2_{Q_j^{-1}} < \frac{\gamma}{2\delta_j}.$$

  Case 2: $z''_{j+}(c) = 2\frac{\|d_j\|^2_{Q_j^{-1}} \|\eta_j^*\|^2_{Q_j^{-1}} - (d_j^T Q_j^{-1} \eta_j^*)^2}{\|\eta_j^* + c d_j\|^3_{Q_j^{-1}}}$ (in particular, $\|\eta_j^* + c d_j\| \geq 1$). By (2.35) we have

$$\|d_j\|^2_{Q_j^{-1}} \left(1 - \frac{(d_j^T Q_j^{-1} \eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}} \cdot \|d_j\|^2_{Q_j^{-1}}}\right) < M_j \gamma.$$

  As a result, at least one of the following two inequalities *must* be satisfied:

$$\begin{aligned}\|d_j\|^2_{Q_j^{-1}} &< \sqrt{M_j \gamma}, \\ 1 - \frac{(d_j^T Q_j^{-1} \eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}} \cdot \|d_j\|^2_{Q_j^{-1}}} &< \sqrt{M_j \gamma}.\end{aligned}$$

We will show that the second inequality is impossible. Suppose otherwise that the second inequality is valid. By the definition of γ (c.f. (2.30)), one has $\gamma < \frac{1}{4M_j}\ \ \forall j$ and as a result we have $\sqrt{M_j\gamma} < \frac{1}{2}$. Thus,

$$\frac{(d_j^T Q_j^{-1}\eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}} \cdot \|d_j\|^2_{Q_j^{-1}}} > 1 - \sqrt{M_j\gamma} > \frac{1}{2}. \tag{2.36}$$

Recall that for $j \in K_2^d$, $\|\eta_j^*\|_{Q_j^{-1}} = 1$ and $d_j^T Q_j^{-1}\eta_j^* \leq 0$, and so by substituting this in (2.36) we obtain:

$$d_j^T Q_j^{-1}\eta_j^* < -\frac{\|d_j\|_{Q_j^{-1}}}{\sqrt{2}} < -\frac{\|d_j\|_{Q_j^{-1}}}{2}. \tag{2.37}$$

¿From this we have that for all $\beta \in (0,1]$ and $j \in K_2^d$:

$$\begin{aligned}
\|\eta_j^* + \beta d_j\|^2_{Q_j^{-1}} &= \|\eta_j^*\|^2_{Q_j^{-1}} + 2\beta d_j^T Q_j^{-1}\eta_j^* + \beta^2\|d_j\|^2_{Q_j^{-1}} \\
&\overset{(2.37), j\in K}{<} 1 - \beta\|d_j\|_{Q_j^{-1}} + \beta^2\|d_j\|^2_{Q_j^{-1}} \\
&= 1 + \beta\|d_j\|_{Q_j^{-1}}(-1 + \beta\|d_j\|_{Q_j^{-1}}) \\
&\overset{0<\beta\leq 1}{<} 1 + \beta\|d_j\|_{Q_j^{-1}}(-1 + \|d_j\|_{Q_j^{-1}}) \\
&< 1 + \beta\|d_j\|_{Q_j^{-1}}(-1 + \|Q_j^{-1}\|^{\frac{1}{2}}\epsilon) \\
&\overset{\epsilon < \frac{1}{\|Q_j^{-1}\|^{\frac{1}{2}}}}{<} 1.
\end{aligned}$$

This is a contradiction to the assumption that $\|\eta_j^* + cd_j\| \geq 1$. Thus, in this case we have:

$$\|d_j\|^2_{Q_j^{-1}} < \sqrt{M_j\gamma}.$$

We summarize all the obtained cases for the inequality $\gamma > \delta_j z''_{j+}(c)$ for $j = 1, \ldots, m+1$:

Define a vector $u \in \Re^{(m+1)n}$ by

$$u_j = \begin{cases} \theta_j\eta_j^* & j \in J \cup K_1^d \\ 0 & j \notin J \cup K_1^d \end{cases}. \tag{2.38}$$

$$\forall j \in I \quad \|d_j\|^2_{Q_j^{-1}} < \tfrac{\gamma}{2\delta_j},$$
$$\forall j \in J \quad \exists \theta_j: \quad \|d_j - \theta_j \eta_j^*\|^2_{Q_j^{-1}} \le M_j \gamma,$$
$$\forall j \in K_1^d \quad \exists \theta_j > 0: \quad \|d_j - \theta_j \eta_j^*\|^2_{Q_j^{-1}} \le M_j \gamma,$$
$$\forall j \in K_2^d \quad \|d_j\|^2_{Q_j^{-1}} < \sqrt{M_j \gamma}.$$

Figure 2.1: Summary

Now, define the following norm on vectors in $\Re^{(m+1)n}$ by:

$$\|v\|_\alpha^2 := \sum_{j=1}^{m+1} \|v_j\|^2_{Q_j^{-1}},$$

and denote by $|I|$ the cardinality of an index set I. Then,

$$\begin{aligned}
\|d-u\|_\alpha^2 &= \sum_{j=1}^{m+1} \|d_j - u_j\|^2_{Q_j^{-1}} \\
&= \sum_{j \in J \cup K_1^d} \|d_j - u_j\|^2_{Q_j^{-1}} + \sum_{j \in I \cup K_2^d} \|d_j - u_j\|^2_{Q_j^{-1}} \\
&= \sum_{j \in J \cup K_1^d} \|d_j - \theta_j \eta_j^*\|^2_{Q_j^{-1}} + \sum_{j \in I \cup K_2^d} \|d_j\|^2_{Q_j^{-1}} \\
&\le \sum_{j \in J \cup K_1^d} M_j \gamma + \sum_{j \in I \cup K_2^d} \max\left\{\frac{\gamma}{2\delta_j}, \sqrt{M_j \gamma}\right\} \\
&\le |J \cup K_1^d| \max_j \{M_j \gamma\} + |I \cup K_2^d| \max_j \left\{\frac{\gamma}{2\delta_j}, \sqrt{M_j \gamma}\right\} \\
&\overset{|J \cup K_1^d|, |I \cup K_2^d| \le m+1}{\le} (m+1) M_j \gamma + (m+1) \max\left\{\frac{\gamma}{2\delta_j}, \sqrt{M_j \gamma}\right\} \\
&\overset{0<\gamma<1}{<} \sqrt{\gamma}(m+1) \max_{j=1,\ldots,m+1} \left\{\sqrt{M_j}, M_j, \frac{1}{2\delta_j}\right\}.
\end{aligned}$$

Notice that,

$$\|v\|^2 \le \max_{j=1,\ldots,m+1} \lambda_{max}(Q_j) \|v\|_\alpha^2 \quad \forall v \in \Re^{(m+1)n}$$

Define,

$$C = \sqrt{(m+1) \cdot \max_{j=1,\dots,m+1} \lambda_{max}(Q_j) \cdot \max_{j=1,\dots,m+1} \left\{ \sqrt{M_j}, M_j, \frac{1}{2\delta_j} \right\}},$$

and so with these notations we have obtained,

$$\|d - u\| \leq C \sqrt[4]{\gamma}.$$

Remember that by the definition of γ (2.30) we have that $C\sqrt[4]{\gamma} < \frac{\epsilon}{4}$. As a result, since $\|d\| = \epsilon$ one has:

$$\begin{aligned} \|u\| &= \|u - d + d\| \leq \|u - d\| + \|d\| \leq \frac{5\epsilon}{4} \Rightarrow \|u\|^2 \leq \frac{25\epsilon^2}{16} < 2\epsilon^2. \\ \|u\| &= \|u - d + d\| \geq \|d\| - \|u - d\| \geq \frac{3\epsilon}{4} \Rightarrow \|u\|^2 \geq \frac{9\epsilon^2}{16} > \frac{\epsilon^2}{4}. \end{aligned}$$

Thus, recalling the definition of u in (2.38), we have found real numbers $\{\theta_j\}_{j \in J \cup K_1^d}$ such that $\theta_j \geq 0, \quad \forall j \in K_1^d$ that satisfies,

$$\frac{\epsilon^2}{4} < \sum_{j \in J \cup K_1^d} \|\theta_j \eta_j^*\|^2 < 2\epsilon^2.$$

According to Theorem 2.5.2, in order to get a contradiction to the optimality of η^* it is sufficient to prove that:

$$\left\| \sum_{j \in J \cup K_1^d} \theta_j \alpha_j \eta_j^* \right\| < \xi.$$

And in fact,

$$\begin{aligned} \left\| \sum_{j \in J \cup K_1^d} \theta_j \alpha_j \eta_j^* \right\| &\overset{d \in W}{=} \left\| \sum_{j \in J \cup K_1^d} \theta_j \alpha_j \eta_j^* - \sum_{j=1}^{m+1} \alpha_j d_j \right\| \\ &= \left\| \sum_{j \in J \cup K_1^d} \alpha_j (\theta_j \eta_j^* - d_j) - \sum_{j \in I \cup K_2^d} \alpha_j d_j \right\| \\ &\leq \sum_{j \in J \cup K_1^d} |\alpha_j| \|\theta_j \eta_j^* - d_j\| + \sum_{j \in I \cup K_2^d} |\alpha_j| \|d_j\| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j\in J\cup K_1^d} |\alpha_j|\lambda_{max}(Q_j)\|\theta_j\eta_j^* - d_j\|_{Q_j^{-1}} + \sum_{j\in I\cup K_2^d} |\alpha_j|\lambda_{max}(Q_j)\|d_j\|_{Q_j^{-1}} \\
&\leq \sqrt[4]{\gamma}D,
\end{aligned}
$$

where

$$
D = \sqrt{m+1}\cdot \max_{j=1,\ldots,m+1}\{|\alpha_j|\lambda_{max}(Q_j)\}\cdot \max_{j=1,\ldots,m+1}\left\{\sqrt[4]{M_j}, \sqrt{M_j}, \frac{1}{\sqrt{2\delta_j}}\right\}.
$$

But by the definition of γ we have that $\sqrt[4]{\gamma}D < \xi$ and thus we have obtained the desired contradiction to Theorem 2.5.2 and the theorem is proved. □

2.5.2 The Second Case: The General Case.

X^* is a polyhedral set

First, we will show that the optimal solution set X^* of (DQCQP) is a polyhedral set.

Theorem 2.5.4 *X^* is a polyhedral set.*

Proof: Let $\eta^* \in X^*$ and let I, J, K be the index sets defined by (2.24). Then, $\eta \in X^*$ iff the following three conditions are satisfied:

1. $\eta \in S$. i.e.,

$$
\sum_{j=1}^{m+1} \alpha_j\eta_j = e. \tag{2.39}
$$

2. $h_d(\beta) = f(\eta^* + \beta d)$ is a linear function on $[0,1]$with $d = \eta - \eta^*$.

3. The slope of the linear function $h_d(\beta)$ is zero.

We will prove that the last two conditions can be written as linear inequality constraints:
Condition 2: Let us show that this condition is equivalent to the following set of linear equalities and inequalities.

$$\begin{cases} d_j = 0 & \forall j \in I \\ d_j = a_j\eta_j^*, a_j \geq \frac{1}{\|\eta_j^*\|_{Q_j^{-1}}} - 1 & \forall j \in J \\ d_j = a_j\eta_j^*, a_j \geq 0 & \forall j \in K \end{cases}. \tag{2.40}$$

Let $j \in I$. Let us suppose otherwise that $d_j \neq 0$. Then, the function $h_d(\beta)$ is a quadratic function of β in contradiction to the linearity of $h_d(\beta)$.
$j \in J$: $h_d(\beta)$ is a linear function and thus $h''_d(\beta) = 0$ for all $\beta \in (0,1)$. Recall that $h_d(\beta) = \sum_{i=1}^{m+1}(\delta_i z_i(\beta) - h_i^T(\eta_i^* + \beta d_i))$ where $z_i(\beta) = g_{Q_i}(\eta_i^* + \beta d_i)$. Thus,

$$h''_d(\beta) = \sum_{i=1}^{m+1} \delta_i z''_{i_+}(\beta) = 0.$$

From the convexity of z_i we have $z''_{i_+}(\beta) \geq 0$ and thus,

$$z''_{j_+}(\beta) = 0 \quad \forall 1 \leq j \leq m+1, \ \ \forall \beta \in (0,1).$$

For $j \in J$ we have that this equality is just:

$$z''_{j_+}(\beta) = 2\frac{\|d_j\|^2_{Q_j^{-1}}\|\eta_j^*\|^2_{Q_j^{-1}} - (d_j^T Q_j^{-1}\eta_j^*)^2}{\|\eta_j^* + \beta d_j\|^3_{Q_j^{-1}}} = 0$$

Thus,

$$\|d_j\|^2_{Q_j^{-1}}\|\eta_j^*\|^2_{Q_j^{-1}} - (d_j^T Q_j^{-1}\eta_j^*)^2 = 0.$$

By the Cauchy-Schwartz inequality we have that there is $a_j \in \Re$ such that $d_j = a_j\eta_j^*$. Denote by X_j^* the set of all the j-th components of the optimal set X^*. X_j^* is a closed convex set and satisfies the following property:

$$X_j^* \subseteq \mathcal{B}_j^< \text{ or } X_j^* \subseteq \mathcal{B}_j^> \tag{2.41}$$

where,

$$\mathcal{B}^< = \{x : \|x\|_{Q_j^{-1}} \leq 1\}, \mathcal{B}^> = \{x : \|x\|_{Q_j^{-1}} \geq 1\}.$$

Now, observe that $a_j \geq -1$, because otherwise, $(1 + a_j)\eta_j^* = \eta_j^* + d_j \in X_j^*$ and thus

$$0 = -\frac{1}{a_j}(1+a_j)\eta_j^* + \left(1+\frac{1}{a_j}\right)\eta_j^* \in X_j^*,$$

in contradiction to (2.41). Also, $\|\eta_j\|_{Q_j^{-1}} \geq 1$ and so since $d_j = a_j\eta_j^*$ one has,

$$\begin{array}{rcl} 1 & \leq & \|\eta_j\|_{Q_j^{-1}} \\ & = & \|\eta_j^* + d_j\|_{Q_j^{-1}} \\ & = & \|\eta_j^* + a_j\eta_j^*\|_{Q_j^{-1}} \\ & = & \|\eta_j^*\|_{Q_j^{-1}}|1+a_j| \\ & \overset{a_j \geq -1}{=} & \|\eta_j^*\|_{Q_j^{-1}}(1+a_j), \end{array}$$

and thus $a_j \geq \frac{1}{\|\eta_j^*\|_{Q_j^{-1}}} - 1$.

For $j \in K$ we have in a similar fashion that $d_j = a_j\eta_j^*$ but here $a_j \geq 0$ which is a necessary condition for the inequality $\|\eta_j\|_{Q_j^{-1}} \geq 1$ to be satisfied.

Condition 3: In order to find an explicit equation describing the condition that the slope of $h_d(\beta)$ is zero we will analyze the function $h_d(\beta)$ for $\beta \in [0, B]$ where B is defined by (2.26):

$$\begin{array}{rcl} h_d(\beta) & = & f(\eta^* + \beta d) \\ & = & \sum_{j=1}^{m+1}(\delta_j g_{Q_j}(\eta_j^* + \beta d_j) - h_j^T(\eta_j^* + \beta d_j)) \\ & = & \sum_{j\in I}\delta_j g_{Q_j}(\eta_j^*) + \sum_{j\in J\cup K}\delta_j(2\|\eta_j^* + \beta d_j\|_{Q_j^{-1}} - 1) - \sum_{j=1}^{m+1}(h_j^T\eta_j^* + \beta h_j^T d_j) \\ & = & \sum_{j\in I}\delta_j g_{Q_j}(\eta_j^*) + \sum_{j\in J\cup K}\delta_j(2\|\eta_j^* + \beta a_j\eta_j^*\|_{Q_j^{-1}} - 1) - \sum_{j=1}^{m+1}(h_j^T\eta_j^* + \beta h_j^T d_j) \\ & = & 2\sum_{j\in J\cup K}\delta_j|1+\beta a_j|\cdot\|\eta_j^*\|_{Q_j^{-1}} - \beta\sum_{j=1}^{m+1}h_j^T d_j + \text{constant} \\ & = & 2\sum_{j\in J\cup K}\delta_j(1+\beta a_j)\cdot\|\eta_j^*\|_{Q_j^{-1}} - \beta\sum_{j=1}^{m+1}h_j^T d_j + \text{constant} \\ & = & \beta\underbrace{\left(2\sum_{j\in J\cup K}\delta_j a_j\|\eta_j^*\|_{Q_j^{-1}} - \sum_{j=1}^{m+1}h_j^T d_j\right)}_{\text{slope of } h_d(\beta)} + \text{constant.} \end{array}$$

Thus, the condition that the slope of $h_d(\beta)$ is zero means,

$$2\sum_{j\in J\cup K}\delta_j a_j\|\eta_j^*\|_{Q_j^{-1}} - \sum_{j=1}^{m+1} h_j^T d_j = 0. \tag{2.42}$$

This shows that, X^* is characterized by a set of linear equalities and inequalities in the variables $\{a_j\}_{j\in J\cup K}, \{\eta_j\}_{j=1}^{m+1}$ ((2.39)-(2.42)) which yields the result. $\square$

Some facts about the faces of a convex set

Now, we need to introduce the concept of a *face* and derive some basic properties of faces of convex sets needed for our analysis.

Definition 2.5.1 *Let C be a closed convex set. A closed convex set $F \subseteq C$ is called a* face *if there is a supporting hyperplane H of C such that $H \cap C = F$.*

Example: A set that contains one extreme point of C is a face.
We use the notation $ri(S)$ for the relative interior of a set S.

Lemma 2.5.9 *Let C be a closed convex set and let F be a face of C. If $x, y \in ri\, F$ then,*

$$N_C(x) = N_C(y).$$

Proof: It is sufficient to show that $N_C(x) \subseteq N_C(y)$ and thus the result will follow by symmetry. Let $d \in N_C(x)$, by the definition of the normal cone we have that

$$\langle d, s - x\rangle \leq 0 \quad \forall s \in C.$$

Let $z \in C$, we will prove that $d^T(z-y) \leq 0$ and the theorem will be proved.
Let $x \in \mathrm{ri}\, F$ and thus there exists a (small enough) $t > 0$ such that,

$$w = x + t(x - y) \in F.$$

Since $w \in F$ and $z \in F \subseteq C$, for $t > 0$ the convex combination of the points $w, z \in C$ defined by $s := \frac{1}{t+1}w + \frac{t}{t+1}z$ is in C. Then using the definition of w one has:

$$0 \geq \langle d, s - x\rangle = \left\langle d, \frac{1}{t+1}w + \frac{t}{t+1}z - x\right\rangle = \left\langle d, x - \frac{t}{t+1}y + \frac{t}{t+1}z - x\right\rangle = \frac{t}{t+1}\langle d, z - y\rangle,$$

proving that $d \in N_C(y)$. $\square$

Proving GREB for the general case

First, we present a condition equivalent to assumption 8. This condition, necessarily, implies GREB.

Lemma 2.5.10 *Assumption 8 is equivalent to the following condition: For every* $\eta^* \in X^*$ *and a bounded set* B *there exists* $\epsilon > 0$ *and* $\sigma_B > 0$ *such that* $\forall \|d\| = \epsilon, d \in W \cap N_{X^*}(\eta^*), 0 < \beta \leq 1, \eta^* \in bd(X^*), \eta^* + \beta d \in B$ *one has,*

$$\frac{\|P_W \nabla f(\eta^* + \beta d)\|}{\beta} \geq \frac{\epsilon}{t\sigma_B}$$

Proof: By assumption 8 there exists $\sigma_B > 0$ such that,

$$d(\eta, X^*) \leq \sigma_B \|\eta - P_S(\eta - t\nabla f(\eta))\| \quad \forall \eta \in S \cap B, d(\eta, X^*) \leq \epsilon.$$

Similarly to the proof of Lemma 2.5.4 we have that,

$$\eta - P_S(\eta - t\nabla f(\eta)) = tP_W \nabla f(\eta).$$

Thus, we have that assumption 8 is equivalent to:

$$d(\eta, X^*) \leq \sigma_B t \|P_W \nabla f(\eta)\| \quad \forall \eta \in S \cap B, d(\eta, X^*) \leq \epsilon.$$

Denote $\eta^* = P_{X^*}(\eta)$ and make the following change of variables:

$$\eta = \eta^* + \beta d.$$

Notice that as a consequence of the relation $\eta^* = P_{X^*}(\eta)$ we have that $d \in N_{X^*}(\eta^*)$. Thus, assumption 8 is equivalent to say that $\forall \|d\| = \epsilon, d \in W \cap N_{X^*}(\eta^*), 0 < \beta \leq 1, \eta^* \in bd(X^*), \eta^* + \beta d \in B$ one has,

$$\|\beta d\| \leq \sigma_B t \|P_W \nabla f(\eta^* + \beta d)\|.$$

Dividing by β yields the desired result. □

Now, proceeding in a similar manner as the previous subsection we find a sufficient condition of GREB. The condition will be in terms of the following function in one variable β:

$$h_{d,\eta^*} = f(\eta^* + \beta d)$$

This function is defined for every $\eta^* \in bd(X^*)$ (Here, there is no unique minimizer) and for every direction $d \in W \cap N_{X^*}(\eta^*)$.

Lemma 2.5.11 (A Sufficient Condition for GREB) *The following condition implies GREB: For every bounded set B there exists $\epsilon > 0$ and $s_B > 0$ (which depends on ϵ) such that*

$$\frac{h'_{d,\eta^*}(\beta)}{\beta} \geq s_B \quad \forall d \in W \cap N_{X^*}(\eta^*), \|d\| = \epsilon, 0 < \beta \leq 1, \eta^* \in bd(X^*), \eta^* + \beta d \in B.$$

Proof: Exactly the same as the proof of Lemma 2.5.5. □
We define the following sets of indices:

$$J_{\eta^*} = \left\{ j : \|\eta_j^*\|_{Q_j^{-1}} > 1 \right\} \quad , \quad K_{\eta^*} = \left\{ j : \|\eta_j^*\|_{Q_j^{-1}} = 1 \right\} \quad , \quad I_{\eta^*} = \left\{ j : \|\eta_j^*\|_{Q_j^{-1}} < 1 \right\}.$$

Following the layout of the previous analysis in subsection 2.5.1 we prove a necessary condition that must be satisfied by the optimum set. First, we remember that X^* is a polyhedral set and thus has only a finite number of faces. Denote the faces of X^* by $F_1, F_2, \ldots, F_k$ and let $v^1, \ldots, v^k$ be arbitrary chosen representatives of the relative interiors of the faces, i.e.,

$$v^i \in riF_i \quad i = 1, 2, \ldots, k.$$

We can assume that every two points in the same relative interior of some face have the same set of active constraints in the linear system defining the optimal set. Otherwise, we might take several more representatives of the relative interior. This process does not ruin the finiteness of the representatives set.

Definition 2.5.2 *Let C be a closed convex set. A direction d is an* exterior direction *of C at a point $x \in C$ if $x + \beta d \notin C$ for all $\beta > 0$. The set of all exterior directions of C at x is denoted by $E_C(x)$.*

Remarks:

- For every $x \in bd(C) : N_C(x) \subseteq E_C(x) \cup \{0\}$.

- For every $x \in int(C)$: $E_C(x) = \emptyset, N_C(x) = \{0\}$.

The following lemma is a natural generalization of Lemma 2.5.6:

Lemma 2.5.12 *Let $\eta^* \in X^*$, $d \in W \cap E_{X^*}(\eta^*)$ and let $s > 0$. Then, $h_{d,\eta^*}(\beta)$ is not linear on the interval $[0, s]$.*

Proof: Exactly follows the proof of Lemma 2.5.6. □
Similarly to the analysis of subsection 2.5.1, our next result describes the appropriate non consistent system.

Theorem 2.5.5 *For every $i = 1, 2, \dots, k$ the following system does not have a solution:*

$$(NLS_i) \begin{cases} \sum_{j \in J_{v^i} \cup K_{v^i}} \theta_j \alpha_j v_j^i = 0 & \\ & \\ \theta_j \geq 0 & \forall j \in K_{v^i} \\ \theta_j = 0 & \forall j \in I_{v^i} \\ (\theta_j v_j^i)_{j=1}^{m+1} \in E_{X^*}(v_i) & \end{cases}$$

Proof: Fix some i and suppose that the (NLS_i) *does* have a solution, and define:

$$d_j = \theta_j v_j^i \;\; j = 1, 2, \dots, m+1$$

Thus, $d = (d_1, \dots, d_{m+1})$ is a feasible direction (i.e. $\sum_{j=1}^{m+1} \alpha_j d_j = 0$ so $d \in W$). Also, one has $d \in E_{X^*}(v^i)$. Define $B = \min_{j:\theta_j<0} \left\{ \frac{1}{\theta_j} \left(\frac{1}{\|v_j^i\|_{Q_j^{-1}}} - 1 \right) \right\}$. Proceeding as in the proof of Theorem 2.5.1 one can verify that the function $h_{d,v^i}(\beta) = f(v^i + \beta d)$ is linear on $[0, B]$ in contradiction to Lemma 2.5.12. Thus, (NLS_i) does not have a solution. □
Notation: $B_\epsilon = \{x : \|x\| = \epsilon\}$.

Theorem 2.5.6 (Necessary Condition on the representative points of X^*) *Let $1 \leq i \leq k$, then for every $\epsilon > 0$ there exists $\xi > 0$ such that the following system of inequalities:*

$$\begin{cases} d((\theta_j v_j^i)_{j=1}^{m+1}, N_X^*(v^i) \cap B_\epsilon) \leq \frac{\epsilon}{2} & \\ \theta_j \geq 0 & \forall j \in K_{v^i} \\ \theta_j = 0 & \forall j \notin J_{v^i} \cup K_{v^i} \end{cases} \tag{2.43}$$

implies

$$\left\| \sum_{j \in J_{v^i} \cup K_{v^i}} \alpha_j \theta_j v_j^i \right\| \geq \xi$$

Proof: Suppose in contradiction that there are variables $\theta_1, \ldots, \theta_{m+1}$ that satisfy (2.43). We will show that $(\theta_j v_j^i)_{j=1}^{m+1} \in E_{X^*}(v^i)$. Denote $w = (\theta_j v_j^i)_{j=1}^{m+1}$. Now, $d((\theta_j v_j^i)_{j=1}^{m+1}, N_X^*(v^i) \cap B_\epsilon) \leq \frac{\epsilon}{2}$ and thus there is a direction $d \in N_{X^*}(v^i)$ such that $\|d\| = \epsilon$ and

$$\|d - w\| \leq \frac{\epsilon}{2}.$$

Therefore, with $\|d\| = \epsilon$ we obtain:

$$\begin{aligned} \|w\| &\leq \|d\| + \|d - w\| = \frac{3}{2}\epsilon, \\ \|w\| &\geq \|d\| - \|d - w\| = \frac{\epsilon}{2}. \end{aligned}$$

The inequality $\|d-w\| \leq \frac{\epsilon}{2}$ is equivalent to $\|d-w\|^2 \leq \frac{\epsilon^2}{4}$, which after some algebra together with the bounds on $\|w\|$ implies:

$$\langle d, w \rangle \geq \frac{\|d\|^2 + \|w\|^2 - \frac{\epsilon^2}{4}}{2} = \frac{\frac{3}{4}\epsilon^2 + \|w\|^2}{2} \geq \frac{\frac{3}{4}\epsilon^2 + \frac{1}{4}\epsilon^2}{2} = \frac{1}{2}\epsilon^2 > 0. \tag{2.44}$$

To summarize, we have that $\langle d, w \rangle > 0$ for some $d \in N_{X^*}(v^i)$. Furthermore, $w \in E_{X^*}(v^i)$ because otherwise there would exist $\beta > 0$ such that $\bar{x} = v^i + \beta w \in X^*$. But from the definition of the normal cone we have,

$$d \in N_{X^*}(v^i) \Rightarrow \langle \bar{x} - v^i, d \rangle \leq 0.$$

Substituting $\bar{x} = v^i + \beta w$ we derive that

$$\beta \langle w, d \rangle \leq 0,$$

in contradiction to (2.44).

Consider the following minimization problem:

$$\text{minimize} \quad \left\| \sum_{j \in J_{v^i} \cup K_{v^i}} \alpha_j \theta_j v_j^i \right\|$$

$$\begin{aligned} \text{s.t.} \qquad & d((\theta_j v_j^i)_{j=1}^{m+1}, N_X^*(v^i) \cap B_\epsilon) \leq \frac{\epsilon}{2} \\ & \theta_j \geq 0 \quad \forall j \in K_{v^i} \\ & \theta_j = 0 \quad \forall j \notin J_{v^i} \cup K_{v^i} \end{aligned}$$

Here we minimize a continuous function on a closed bounded set. Thus, the minimum is attained. Denoting the value of the minimum by ξ, one has $\xi > 0$ since otherwise the minimizing vector is a solution for (NLS_i) which is a contradiction to Theorem 2.5.5. □

Theorem 2.5.7 (GREB is Fulfilled for (DQCQP)) *For every bounded set B, there exists $\epsilon > 0$ and $\gamma_B > 0$ such that $\forall d \in W \cap N_{X^*}(\eta^*), \|d\| = \epsilon, 0 < \beta \leq 1, \eta^* \in bd(X^*), \eta^* + \beta d \in B$, one has*

$$\frac{h'_{d,\eta^*}(\beta)}{\beta} \geq \gamma_B,$$

where

$$\begin{aligned} \epsilon &< \min_{j \in I} \left\{ \frac{\left|1 - \|\eta_j^*\|_{Q_j^{-1}}\right|}{\|Q_j^{-1}\|^{\frac{1}{2}}}, \frac{1}{\|Q_j^{-1}\|^{\frac{1}{2}}} \right\} \\ \gamma &< \min_{j=1,\ldots,m+1} \left\{ 1, \frac{1}{4N_j}, \left(\frac{\epsilon}{4C}\right)^4, \left(\frac{\xi}{D}\right)^4 \right\}. \end{aligned}$$

Here, ξ is as defined in Theorem 2.5.2 and C, D are defined by:

$$C = \sqrt{(m+1) \cdot \max_{j=1,\ldots,m+1} \lambda_{max}(Q_j) \cdot \max_{j=1,\ldots,m+1} \left\{ \sqrt{N_j}, N_j, \frac{1}{2\delta_j} \right\}},$$

$$D = \sqrt{m+1} \cdot \max_{j=1,\ldots,m+1} \{|\alpha_j| \lambda_{max}(Q_j)\} \cdot \max_{j=1,\ldots,m+1} \left\{ \sqrt[4]{N_j}, \sqrt{N_j}, \frac{1}{\sqrt{2\delta_j}} \right\}.$$

$$N_j = \|Q_j^{-1}\|^{\frac{3}{2}} \frac{(N+\epsilon)^3}{2\delta_j}$$

Proof: Let B be some bounded set. Suppose, in contrary that there exists a $\eta^* \in bd(X^*)$ and a direction d such that

$$\frac{h'_{d,\eta^*}(\beta)}{\beta} < \gamma_B, d \in W \cap N_{X^*}(\eta^*), \|d\| = \epsilon, 0 < \beta \leq 1, \eta^* \in bd(X^*), \eta^* + \beta d \in B.$$

$\eta^* \in bd(X^*)$ and thus it is contained in a relative interior of a face of X^*. Suppose that

$$\eta^* \in ri(F_p),$$

for some $p \in [1, k]$. Then, we have that v^p is also a member of $ri(F_p)$. By Lemma 2.5.9 we have that

$$N_{X^*}(\eta^*) = N_{X^*}(v^p),$$

and thus $d \in N_{X^*}(v^p)$. Just like in the proof of Theorem 2.5.3 we define the function $z_j(\beta) = g_{Q_j}(\eta_j^* + \beta d_j)$ and obtain that there is a $0 < c < 1$ such that for all $1 \leq j \leq m+1$:

$$z''_{j_+}(c) < \gamma_B.$$

For convenience reasons we denote $I = I_{\eta^*} = I_{v^p}, J = J_{\eta^*} = J_{v^p}, K = K_{\eta^*} = K_{v^p}$ and define K_1^d, K_2^d exactly like in (2.32)-(2.33). By Theorem 2.5.3 we have that there is a direction d such that:

$$\begin{aligned}
\|d_j\|_{Q_j^{-1}} &< \sqrt{\frac{\gamma}{2\delta_j}} \quad \forall j \in I, \\
\|d_j - \theta_j \eta_j^*\|_{Q_j^{-1}} &< \sqrt{M_j \gamma} \quad \forall j \in J, \\
\|d_j - \theta_j \eta_j^*\|_{Q_j^{-1}} &< \sqrt{M_j \gamma}, \theta_j > 0 \quad \forall j \in K_1^d, \\
\|d_j\|^2_{Q_j^{-1}} &< \sqrt{M_j \gamma} \quad \forall j \in K_2^d.
\end{aligned}$$

By (2.40) we obtain that there are numbers $\{\lambda_j\}_{j \in J \cup K_1^d}$ such that $\lambda_j \geq 0$ for all $j \in K_1^d$ and satisfy,

$$\eta_j^* = \lambda_j v_j^p \quad \forall j \in J \cup K_1^d.$$

Define $\tilde{\theta_j} = \lambda_j \theta_j$ and obtain:

$$
\begin{aligned}
\|d_j\|_{Q_j^{-1}} &< \sqrt{\frac{\gamma}{2\delta_j}} \quad \forall j \in I, \\
\|d_j - \tilde{\theta}_j v_j^p\|_{Q_j^{-1}} &< \sqrt{M_j\gamma} \quad \forall j \in J, \\
\|d_j - \tilde{\theta}_j v_j^p\|_{Q_j^{-1}} &< \sqrt{M_j\gamma}, \theta_j > 0 \quad \forall j \in K_1^d, \\
\|d_j\|^2_{Q_j^{-1}} &< \sqrt{M_j\gamma} \quad \forall j \in K_2^d.
\end{aligned}
$$

All is left to prove is that $M_j = \frac{(\|\eta_j^*\|_{Q_j^{-1}} + \|Q_j^{-1}\|^{\frac{1}{2}}\epsilon)^3}{2\delta_j}$ is bounded above for all $j = 1, \ldots, m+1$. Indeed, we have that $\eta^* + \beta d \in B$. Since B is supposed to be bounded, $\exists N > 0 \ : \ B \subseteq \{x : \|x\| \leq N\}$. Then,

$$
\|\eta_j^*\| \leq \|\eta^*\| \leq N,
$$

and thus,

$$
\begin{aligned}
M_j &= \frac{(\|\eta_j^*\|_{Q_j^{-1}} + \|Q_j^{-1}\|^{\frac{1}{2}}\epsilon)^3}{2\delta_j} \\
&\leq \frac{(\|Q_j^{-1}\|^{\frac{1}{2}}\|\eta_j^*\| + \|Q_j^{-1}\|^{\frac{1}{2}}\epsilon)^3}{2\delta_j} \\
&\leq \frac{(\|Q_j^{-1}\|^{\frac{1}{2}}N + \|Q_j^{-1}\|^{\frac{1}{2}}\epsilon)^3}{2\delta_j} \\
&\leq \|Q_j^{-1}\|^{\frac{3}{2}}\frac{(N+\epsilon)^3}{2\delta_j} \\
&= N_j
\end{aligned}
$$

Now like in Theorem 2.5.3, we end the proof by verifying that the above estimation leads to a contradiction to Theorem 2.5.6.

□

2.6 Appendix: Proof of the classical results on GPA

These results are known in the literature see e.g. [6],[25]. However, the proofs of all the results below cannot be found in a single reference, and thus we provide here in a compact way, often with simplified proofs, all the known results on GPA.

2.6.1 The Non Convex Case

In this subsection we assume only assumptions 1,2,3.

Lemma 2.6.1 *Let $\{x^k\}$ be a sequence generated by GPA then:*

$$\langle \nabla f(x^k), x^{k+1} - x^k \rangle \leq -\frac{1}{t}\|x^k - x^{k+1}\|^2, \ \ \forall t > 0. \tag{2.45}$$

Proof: By Theorem A.1.3, we have for any $t > 0$:

$$\langle x^k - t\nabla f(x^k) - x^{k+1}, x - x^{k+1} \rangle \leq 0 \quad \forall x \in S.$$

Substituting $x = x^k$ we have:

$$\|x^k - x^{k+1}\|^2 - t\langle \nabla f(x^k), x^k - x^{k+1} \rangle \leq 0,$$

from which it follows,

$$\langle \nabla f(x^k), x^{k+1} - x^k \rangle \leq -\frac{1}{t}\|x^k - x^{k+1}\|^2, \ \ \forall t > 0.$$

□

Lemma 2.6.2 *Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize t that satisfies $0 < t < \frac{2}{L}$. Then,*

$$f(x^{k+1}) - f(x^k) \leq \left(\frac{L}{2} - \frac{1}{t}\right) \|x^{k+1} - x^k\|^2, \tag{2.46}$$

where L is the Lipschitz constant of $\nabla f(x)$.

Proof: From the descent lemma (see Theorem B.0.2) it follows that:

$$\begin{aligned} f(x^{k+1}) - f(x^k) &\leq \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\ &\overset{\text{Lemma 2.6.1}}{\leq} -\frac{1}{t}\|x^{k+1} - x^k\|^2 + \frac{L}{2}\|x^{k+1} - x^k\|^2 \\ &= \left(\frac{L}{2} - \frac{1}{t}\right) \|x^{k+1} - x^k\|^2. \end{aligned}$$

□

The main convergence result of GPA under assumptions 1,2,3 is:

Theorem 2.6.1 *Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$ and $T : \Re^n \to \Re$ be the operator defined by:*

$$T(x) = \|P_S(x - t\nabla f(x)) - x\|.$$

Then,

1. *$\{f(x^k)\}$ is monotone decreasing.*
2. *$T(x^k) \to 0$.*
3. *Every accumulation point of $\{x^k\}$ is a stationary point of (P).*

Proof:

1. Obvious from Lemma 2.6.2 and the fact that we have chosen t satisfying $0 < t < \frac{2}{L}$.

2. Using the definition of the operator T one has:

$$\begin{aligned} 0 < \sum_{k=1}^{n} T^2(x^k) &\overset{\text{Lemma 2.6.2}}{\leq} \frac{1}{\frac{1}{t} - \frac{L}{2}} \sum_{k=1}^{n} (f(x^k) - f(x^{k+1})) \\ &= \frac{f(x^1) - f(x^{n+1})}{\frac{1}{t} - \frac{L}{2}} \\ &\leq \frac{f(x^1) - f^*}{\frac{1}{t} - \frac{L}{2}}, \end{aligned}$$

   where f^* is the optimal value of (P) (by assumption 3, $f^* \neq -\infty$). Therefore, $\sum_{k=1}^{\infty} T^2(x^k)$ is finite and so the general element of the sequence converges to zero. i.e.,

$$T(x^k) \to 0.$$

3. Let x^* be an accumulation point of $\{x^k\}$. Let $\{x^{k_l}\}_{l=1}^{\infty}$ be a subsequence that converges to x^*:

$$x^{k_l} \overset{l\to\infty}{\longrightarrow} x^*. \tag{2.47}$$

Note that:

$$x^{k_l+1} = P_S(x^{k_l} - t\nabla f(x^{k_l})).$$

$\nabla f, P_S$ are continuous operators and thus:

$$x^{k_l+1} = P_S(x^{k_l} - t\nabla f(x^{k_l})) \overset{l\to\infty}{\longrightarrow} P_S(x^* - t\nabla f(x^*)). \tag{2.48}$$

But, as already proved, $T(x^k) \to 0$. In particular, $T(x^{k_l}) \overset{l\to\infty}{\longrightarrow} 0$, that is,

$$\|x^{k_l+1} - x^{k_l}\|^2 \overset{l\to\infty}{\longrightarrow} 0. \tag{2.49}$$

On the other hand, by (2.47)-(2.48) we have:

$$\|x^{k_l+1} - x^{k_l}\|^2 \overset{l\to\infty}{\longrightarrow} \|P_S(x^* - t\nabla f(x^*)) - x^*\|^2. \tag{2.50}$$

Combining equations (2.49)-(2.50) we then obtain:

$$\|P_S(x^* - t\nabla f(x^*)) - x^*\|^2 = 0,$$

which is equivalent to:

$$P_S(x^* - t\nabla f(x^*)) = x^*,$$

that is, x^* is a stationary point of (P).

□

Corollary 2.6.1 *Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$. If $\{x^k\}$ has an accumulation point x^* then:*

$$f(x^k) \to f(x^*).$$

Proof: By Theorem 2.6.1 part 1, $\{f(x^k)\}$ has a limit (finite or equal to $-\infty$). This is the limit of every subsequence of $\{f(x^k)\}$. x^* is an accumulation point so there is a subsequence $x^{n_k} \to x^*$. By the continuity of f we have $f(x^{n_k}) \to f(x^*)$, and thus $f(x^k) \to f(x^*)$.

□

2.6.2 The Convex Case

In this section we also assume that f is a convex function. So, the assumptions that are enforced in this subsection are 1,2,3, 4.

Lemma 2.6.3 *Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$. Then,*

1. $\|x^{k+1} - x^*\| \leq \|x^k - x^*\| \quad \forall x^* \in X^*$.

2. *The sequence $\{x^k\}$ is bounded.*

3. $T(x^k) \to 0$.

Proof: By definition of X^* we have that:

$$x^* = P_S(x^* - t\nabla f(x^*)).$$

Thus,

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|P_S(x^k - t\nabla f(x^k)) - P_S(x^* - t\nabla f(x^*))\|^2 \\
&\overset{\text{Theorem } A.1.4}{\leq} \|x^k - t\nabla f(x^k) - x^* + t\nabla f(x^*)\|^2 \\
&= \|x^k - x^*\|^2 - 2t\langle x^k - x^*, \nabla f(x^k) - \nabla f(x^*)\rangle \\
&\quad + t^2\|\nabla f(x^k) - \nabla f(x^*\|^2 \\
&\overset{\text{Theorem } B.0.5}{\leq} \|x^k - x^*\|^2 - 2\frac{t}{L}\|\nabla f(x^k) - \nabla f(x^*)\|^2 + t^2\|\nabla f(x^k) - \nabla f(x^*\|^2 \\
&= \|x^k - x^*\|^2 + t\left(t - \frac{2}{L}\right)\|\nabla f(x^k) - \nabla f(x^*)\|^2 \\
&\overset{0<t<\frac{2}{L}}{\leq} \|x^k - x^*\|^2,
\end{aligned}$$

which proves the first part of the theorem.
The boundedness of $\{x^k\}$ follows directly from the first part. Let x^* be a point in X^*. Then one obtains

$$\|x^k - x^*\| \leq \|x^{k-1} - x^*\| \leq \ldots \leq \|x^0 - x^*\|. \tag{2.51}$$

Thus, $\{x^k\}$ is contained in a ball with radius $\|x^0 - x^*\|$ and centered in x^*, thus proving the second part of the theorem.
The third part of the theorem was already proved in the non convex case (Theorem 2.6.1).

□

Theorem 2.6.2 (Global Convergence) *The sequence $\{x^k\}$ generated by GPA with constant stepsize $0 < t < \frac{2}{L}$ converges to $x^* \in X^*$.*

Proof: From Lemma 2.6.3 it follows that $\{x^k\}$ is bounded and thus has an accumulation point x^∞. By Theorem 2.6.1, we have that x^∞ is a stationary point, that is, $T(x^\infty) = 0$.
All is left to prove is the uniqueness of the accumulation points. Suppose that x^∞, y^∞ are two accumulation points of $\{x^k\}$. The sequences $\{\|x^k - x^\infty\|\}, \{\|x^k - y^\infty\|\}$ are bounded and non-increasing (by Lemma 2.6.3) and thus have limits:

$$\begin{aligned} \lim_{k\to\infty} \|x^k - x^\infty\| &= l_1, \\ \lim_{k\to\infty} \|x^k - y^\infty\| &= l_2. \end{aligned}$$

Now,

$$\|x^k - x^\infty\|^2 - \|x^k - y^\infty\|^2 = -2\langle x^k, x^\infty - y^\infty\rangle + \|x^\infty\|^2 - \|y^\infty\|^2.$$

Take limits $x^{k_l} \overset{l\to\infty}{\longrightarrow} x^\infty$ and $x^{k_l} \overset{l\to\infty}{\longrightarrow} y^\infty$ and obtain:

$$\begin{aligned} l_1^2 - l_2^2 &= -\|x^\infty - y^\infty\|^2, \\ l_1^2 - l_2^2 &= \|x^\infty - y^\infty\|^2. \end{aligned}$$

Thus, $x^\infty = y^\infty$.

□

We can prove a sublinear rate of convergence of the function value sequence $\{f(x^k)\}$ without using the convergence of the sequence $\{x^k\}$.

Theorem 2.6.3 (Sublinear Rate of Convergence of the Function Values) *Let $\{x^k\}$ be the sequence generated by GPA with constant stepsize $0 < t < \frac{2}{L}$. Then for every $k \geq 1$ the following is satisfied:*

$$f(x^k) - f^* \leq \frac{C}{k},$$

for some positive constant C.

Proof: Let x^* be an arbitrary point in X^*. By the gradient inequality for convex functions we obtain

$$f(x^*) - f(x^k) \geq \langle \nabla f(x^k), x^* - x^k \rangle. \tag{2.52}$$

By Theorem A.1.3 we that:

$$\langle x^k - t\nabla f(x^k) - x^{k+1}, x^* - x^{k+1} \rangle \leq 0,$$

which is equivalent to the following inequality:

$$\langle x^k - x^{k+1}, x^* - x^{k+1} \rangle \leq t\langle \nabla f(x^k), x^* - x^{k+1} \rangle. \tag{2.53}$$

Moreover,

$$\begin{aligned}
\|\nabla f(x^k)\|^2 &= \|\nabla f(x^k) - \nabla f(x^0) + \nabla f(x^0)\|^2 \\
&\overset{(a+b)^2 \leq 2a^2+2b^2}{\leq} 2\|\nabla f(x^k) - \nabla f(x^0)\|^2 + 2\|\nabla f(x^0)\|^2 \\
&\leq 2L^2\|x^k - x^0\|^2 + 2\|\nabla f(x^0)\|^2 \\
&= 2L^2\|x^k - x^* + x^* - x^0\|^2 + 2\|\nabla f(x^0)\|^2 \\
&\overset{(a+b)^2 \leq 2a^2+2b^2}{\leq} 4L^2\|x^k - x^*\|^2 + 4L^2\|x^0 - x^*\|^2 + 2\|\nabla f(x^0)\|^2 \\
&\overset{(2.51)}{\leq} 8L^2\|x^* - x^0\|^2 + 2\|\nabla f(x^0)\|^2
\end{aligned} \tag{2.54}$$

As a result,

$$
\begin{aligned}
(f(x^k) - f^*)^2 &\overset{(2.52)}{\leq} \langle \nabla f(x^k), x^* - x^k \rangle^2 \\
&= \left(\langle \nabla f(x^k), x^* - x^{k+1} \rangle + \langle \nabla f(x^k), x^{k+1} - x^k \rangle \right)^2 \\
&\overset{(a+b)^2 \leq 2(a^2+b^2)}{\leq} 2\langle \nabla f(x^k), x^* - x^{k+1} \rangle^2 + 2\langle \nabla f(x^k), x^{k+1} - x^k \rangle^2 \\
&\overset{(2.53)}{\leq} 2\left(\frac{1}{t^2} \langle x^k - x^{k+1}, x^* - x^{k+1} \rangle^2 + \langle \nabla f(x^k), x^{k+1} - x^k \rangle^2 \right) \\
&\leq 2\left(\frac{1}{t^2} \|x^k - x^{k+1}\|^2 \|x^* - x^{k+1}\|^2 + \|\nabla f(x^k)\|^2 \|x^{k+1} - x^k\|^2 \right) \\
&\overset{(2.51)}{\leq} 2\left(\frac{1}{t^2} \|x^k - x^{k+1}\|^2 \|x^* - x^0\|^2 + \|\nabla f(x^k)\|^2 \|x^{k+1} - x^k\|^2 \right) \\
&= 2\|x^k - x^{k+1}\|^2 \left(\frac{1}{t^2} \|x^* - x^0\|^2 + \|\nabla f(x^k)\|^2 \right) \\
&\overset{(2.46)}{\leq} \frac{2}{\frac{1}{t} - \frac{L}{2}} \left(\frac{1}{t^2} \|x^* - x^0\|^2 + \|\nabla f(x^k)\|^2 \right) (f(x^k) - f(x^{k+1})) \\
&\overset{(2.54)}{\leq} \frac{2}{\frac{1}{t} - \frac{L}{2}} \left(\frac{1}{t^2} \|x^* - x^0\|^2 + 8L^2 \|x^* - x^0\|^2 + 2\|\nabla f(x^0)\|^2 \right) (f(x^k) - f(x^{k+1})) \\
&= \frac{2}{\frac{1}{t} - \frac{L}{2}} \left(\left(\frac{1}{t^2} + 8L^2 \right) \|x^* - x^0\|^2 + 2\|\nabla f(x^0)\|^2 \right) (f(x^k) - f(x^{k+1})) \\
&= A((f(x^k) - f^*) - (f(x^{k+1}) - f^*)),
\end{aligned}
$$

where $A = \frac{2}{\frac{1}{t} - \frac{L}{2}} \left(\left(\frac{1}{t^2} + 8L^2 \right) \|x^* - x^0\|^2 + 2\|\nabla f(x^0)\|^2 \right)$. The result then follows immediately from Lemma B.0.4. □

2.6.3 The Strongly Convex Case

In this subsection we assume assumptions 1,2,3,5. Under strong convexity it is well known that (P) has a unique minimum x^*. In particular, $X^* = \{x^*\}$. Also, in the strongly convex case, linear rate of convergence is proved:

Theorem 2.6.4 (Linear Rate of Convergence under Strong Convexity) *Let f be a strongly convex function on a closed convex set S with parameter $m > 0$. If $\{x^k\}$ is a* *sequence generated by GPA with a constant stepsize* $0 < t < \frac{2m}{L}$ *then it converges to the unique minimum* x^* *with a linear rate. In fact,*

$$\|x^{k+1} - x^*\| \leq \theta \|x^k - x^*\|,$$

where

$$\theta = \sqrt{1 - 2tm + t^2 L^2} < 1.$$

Proof: We follow the layout of the proof of Lemma 2.6.3. We have,

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|P_S(x^k - t\nabla f(x^k)) - P_S(x^* - t\nabla f(x^*))\|^2 \\
&\overset{\text{Theorem } A.1.4}{\leq} \|x^k - t\nabla f(x^k) - x^* + t\nabla f(x^*)\|^2 \\
&= \|x^k - x^*\|^2 - 2t\langle x^k - x^*, \nabla f(x^k) - \nabla f(x^*)\rangle \\
&\quad + t^2\|\nabla f(x^k) - \nabla f(x^*\|^2 \\
&\overset{\text{Theorem } B.0.6}{\leq} \|x^k - x^*\|^2 - 2tm\|x^k - x^*\|^2 + t^2\|\nabla f(x^k) - \nabla f(x^*)\|^2 \\
&\overset{\text{Assumption } 2}{\leq} \|x^k - x^*\|^2 - 2tm\|x^k - x^*\|^2 + L^2 t^2 \|x^k - x^*\|^2 \\
&= (1 - 2tm + t^2 L^2)\|x^k - x^*\|^2.
\end{aligned}$$

Since $0 < t < \frac{2m}{L}$, then $\theta = \sqrt{1 - 2tm + t^2 L^2} < 1$. □

2.6.4 The non convex case with the GREB assumption

Theorem 2.6.5 (Asymptotic Linear Convergence Rate of the Function Values) *Let* f *be a function with Lipschitz continuous gradient with Lipschitz constant* L*. Let* $\{x^k\}$ *be a sequence generated by GPA with constant stepsize* $0 < t < \frac{2}{L}$*. Suppose that* $\{x^k\}$ *is bounded and that GREB and assumption 7 is satisfied. Then,* $\{f(x^k)\}$ *converge to* f^* *where* $f^* = f(x^*)$ *for some stationary point* x^**. Furthermore, there is* $0 < \beta < 1$ *and* $K > 0$ *such that,*

$$\forall k > K \quad f(x^{k+1}) - f^* < \beta(f(x^k) - f^*),$$

where $f^* = f(x^*)$*.*

Proof: $\{x^k\}$ bounded thus contained in some ball $B = \{x : \|x\| \leq M\}$. By the GREB assumption, there exists a σ_B such that:

$$d(x^k, X^*) \leq \sigma_B T(x^k). \tag{2.55}$$

As a consequence of the second part of Theorem 2.6.1 we have,

$$d(x^k, X^*) \to 0,$$
$$\|x^k - x^{k+1}\| \to 0.$$

Thus there is a $K > 0$ such that $\forall k > K$ one has,

$$\begin{aligned} d(x^k, X^*) &\leq \frac{\epsilon}{4}, \\ \|x^k - x^{k+1}\| &\leq \frac{\epsilon}{4}. \end{aligned}$$

Now, for every $k > K$, $f(P_{X^*}(x^k))$ is constant and will be denoted by f^*. The reason for that is that if by contradiction there is a $k_0 > K$ such that $f(P_{X^*}(x^{k_0})) \neq f(P_{X^*}(x^{k_0+1}))$ then by assumption 7:

$$\|P_{X^*}(x^{k_0}) - P_{X^*}(x^{k_0+1})\| \geq \epsilon.$$

On the other hand

$$\begin{aligned} \|P_{X^*}(x^{k_0}) - P_{X^*}(x^{k_0+1})\| &= \|P_{X^*}(x^{k_0}) - x^{k_0} - P_{X^*}(x^{k_0+1}) + x^{k_0+1} + x^{k_0} - x^{k_0+1}\| \\ &\leq \|P_{X^*}(x^{k_0}) - x^{k_0}\| + \|P_{X^*}(x^{k_0+1}) - x^{k_0+1}\| + \|x^{k_0} - x^{k_0+1}\| \\ &= d(x^{k_0}, X^*) + d(x^{k_0+1}, X^*) + \|x^{k_0} - x^{k_0+1}\| \\ &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{3\epsilon}{4} < \epsilon, \end{aligned}$$

which is a contradiction to assumption 7. By (2.46),

$$f(x^k) - f(x^{k+1}) \geq \left(\frac{1}{t} - \frac{L}{2}\right) T^2(x^k),$$

which is equivalent to

$$(f(x^k) - f^*) - (f(x^{k+1}) - f^*) \geq \left(\frac{1}{t} - \frac{L}{2}\right) T^2(x^k). \tag{2.56}$$

By Theorem A.1.3 we have that:

$$\langle x^k - t\nabla f(x^k) - x^{k+1}, x^{k+1} - P_{X^*}(x^k)\rangle \geq 0. \tag{2.57}$$

By the mean value theorem we have that there is $\xi^k \in [P_{X^*}(x^k), x^{k+1}]$ such that:

$$f(x^{k+1}) - f^* = f(x^{k+1}) - f(P_{X^*}(x^k)) = \langle \nabla f(\xi^k), x^{k+1} - P_{X^*}(x^k)\rangle.$$

Thus,

$$\begin{aligned}
f(x^{k+1}) - f^* &= \langle \nabla f(\xi^k), x^{k+1} - P_{X^*}(x^k)\rangle \\
&= \langle \nabla f(\xi^k) - \nabla f(x^k), x^{k+1} - P_{X^*}(x^k)\rangle + \langle \nabla f(x^k), x^{k+1} - P_{X^*}(x^k)\rangle \\
&\overset{(2.57)}{\leq} \|\nabla f(\xi^k) - \nabla f(x^k)\| \cdot \|x^{k+1} - P_{X^*}(x^k)\| + \frac{1}{t}\langle x^k - x^{k+1}, x^{k+1} - P_{X^*}(x^k)\rangle \\
&\leq \left(L\|\xi^k - x^k\| + \frac{1}{t}\|x^k - x^{k+1}\|\right)\|x^{k+1} - P_{X^*}(x^k)\|.
\end{aligned}$$

Using the inequalities $\|\xi^k - x^k\| \leq \|x^{k+1} - x^k\| + \|P_{X^*}(x^k) - x^k\|$ and $\|x^{k+1} - P_{X^*}(x^k)\| \leq \|x^{k+1} - x^k\| + \|x^k - P_{X^*}(x^k)\|$ we deduce that there is a constant $c_1 > 0$ such that:

$$f(x^{k+1}) - f^* \leq c_1(\|P_{X^*}(x^k) - x^k\| + \|x^k - x^{k+1}\|)^2.$$

By the GREB assumption we have that there is a constant $c_2 > 0$ such that $\|P_{X^*}(x^k) - x^k\| \leq c_2\|x^k - x^{k+1}\|$ and thus there is a constant $c_3 > 0$ such that:

$$f(x^{k+1}) - f^* \leq c_3\|x^k - x^{k+1}\|^2. \tag{2.58}$$

Combining (2.57),(2.58) we obtain:

$$\begin{aligned}
f(x^{k+1}) - f^* &\overset{(2.58)}{\leq} c_3\|x^k - x^{k+1}\|^2 \\
&\overset{(2.56)}{\leq} \frac{c_3}{\frac{1}{t} - \frac{L}{2}}((f(x^k) - f^*) - f(x^{k+1} - f^*))
\end{aligned}$$

The result then follows with $\beta = \frac{\alpha}{\alpha+1}$ where $\alpha = \frac{c_3}{\frac{1}{t} - \frac{L}{2}}$.

□

Chapter 3

The Convex Feasibility Problem

3.1 Introduction

This chapter considers the Convex Feasibility Problem (CFP) which consists of finding a point in the intersection of closed convex sets in $\Re^n$.

The Convex feasibility problem : Given m closed convex sets $C_1, C_2, \ldots, C_m$ of $\Re^n$ such that $C \equiv \bigcap_{i=1}^m C_i \neq \emptyset$. Find a point $x \in C$.

The convex feasibility problem has many applications in diverse branches such as best approximation theory, image reconstruction (both discrete and continuous models) and subgradient methods (for a more detailed review of applications see [2] and references therein) . The algorithms discussed in this chapter are projection algorithms. For any set $S \subseteq \Re^n$, P_S denotes the projection operator. At each iteration of the algorithm the current point is a convex combination of the projections of the previous point on the convex sets:

Projection Algorithm

first step: Take an arbitrary $x^0 \in \Re^n$
general step: $x_{n+1} = \sum_{i=1}^m \alpha_i^n P_{C_i}(x_n)$

where $\alpha_i^n > 0 \ \ \forall i = 1, \ldots, m$ and $\sum_{i=1}^m \alpha_i = 1$. Simple instances of these type of methods are the cyclic projection algorithm (in short: CPA):

Cyclic Projection Algorithm (CPA)

first step: Take an arbitrary $x^0 \in \Re^n$
general step: $x_{n+1} = P_{C_{(n \bmod m)+1}}(x_n)$.

The maximum distance projection algorithm:

Maximum Distance Projection Algorithm (MDPA)

first step: Take an arbitrary $x^0 \in \Re^n$
general step: $x_{n+1} = P_{C_j}(x_n)$ where $j = \operatorname*{argmax}_{1 \le i \le m} d(x_n, C_i)$.

and the mean projection algorithm (in short MPA):

Mean Projection Algorithm (MPA)

first step: Take an arbitrary $x^0 \in \Re^n$
general step: $x_{n+1} = \sum_{i=1}^m \alpha_i P_{C_i}(x_n)$.

where $\alpha_1, \ldots, \alpha_m$ are positive constants. The cyclic projection algorithm goes back to von Neumann [40] who considered the case of two subspaces and the mean projection algorithm with equal weights (i.e. $\alpha_i = \frac{1}{m}$) was proposed by Cimmino [12] who considered the case where each C_i is a halfspace. In this chapter we investigate only these three schemes (MPA, MDPA, CPA). More general schemes can be found in the literature (see for example [2] and references therein).
The first natural question concerning these methods is the question of convergence and rate of convergence. Proofs of global convergence of these methods can be found in [21, 2]. Auslander [**?**] proved the convergence of the cyclic projection method for the general case where the sets are closed convex sets. Gubin, Polyak and Raik [21] proved that the sequence generated by CPA and by MDPA converges to a point in C and that if there exists a $1 \le j \le m$ such that $C_j \cap \operatorname{int}(\cap_{i \neq j} C_i) \neq \emptyset$ then the sequence converges to a point in C with a linear rate. Bauschke and Browein [2] introduced a property called "bounded linear regularity" that implies linear convergence of the method and in [3] proved that the standard Slater condition implies "bounded linear regularity".

The second main question concerning these methods is the question of error bounds. An error bound is a quantity that becomes zero whenever a point is in the solution set. In this chapter we investigate the error bound $T(x) = \max_{1 \le i \le m} d(x, C_i)$. Indeed, in this case $T(x) = 0$ if and only if $x \in C$. The error bound assumption states that the error bound is in some sense an upper bound on the distance of the point from optimality. There are two kind of error bounds for (CFP): Global Error Bound (GEB) and Local Error Bound (LEB)

Definition 3.1.1 (GEB) *m closed convex sets $C_1, \ldots, C_m$ are said to satisfy GEB if there exists $\theta > 0$ such that:*

$$\forall x \in \Re^n \quad d(x, C) \le \theta \max_{i=1,\ldots,m} \{d(x, C_i)\}.$$

Definition 3.1.2 (LEB) *m closed convex sets $C_1, \ldots, C_m$ are said to satisfy LEB if for every bounded set B of $\Re^n$ there exists $\theta_B > 0$ such that:*

$$\forall x \in B \quad d(x, C) \le \theta_B \max_{i=1,\ldots,m} \{d(x, C_i)\}.$$

Notice that in both definitions $d(x, C)$ is usually impossible to estimate (otherwise the original problem is trivial). However, the error bound $T(x)$ can in most cases be trivially calculated. Thus, GEB and LEB state that we can bound an unknown quantity by a computable quantity. GEB is satisfied only in rare cases. One of the cases that satisfies GEB is the case where all the C_i are polyhedral sets. This is the celebrated Hoffmann's Lemma [22]. For the non-polyhedral case GEB is usually not satisfied. However LEB is satisfied under some regularity conditions that will be discussed in this chapter. In chapter 2, we have seen that error bounds can also be defined for the convex optimization problem. Suppose we are given the convex optimization problem:

$$\min_{x \in S} f(x)$$

where S is a closed convex set and f is a differentiable convex function. Suppose that the optimal set $X^* = \{x^* : f(x^*) = \min_{x \in S} f(x)\}$ is nonempty. Here we use the error bound $R(x) = \|x - P_S(x - \alpha \nabla f(x))\| \ \forall \alpha > 0$. Obviously, $R(x) = 0$ if and only if $x \in X^*$ so that $R(x)$ qualifies as an error bound. The error bound assumption here is called GREB (see chapter 2) :

Assumption 10 (GREB) *Let $\alpha > 0$. For every closed bounded set B there exists $\sigma_B > 0$ such that:*

$$\forall x \in B \cap S \quad d(x, X^*) \leq \sigma_B \|x - P_S(x - \alpha \nabla f(x))\|,$$

where X^* is the optimal set.

The fact that LEB implies linear rate of convergence of MDPA and CPA was proven in [21],[2]. It is known that if LEB is satisfied then the rate of convergence of projection algorithms depends on the initial point, however if GEB is satisfied (e.g. in the case of polyhedral sets) then the rate of convergence is independent of the choice of the initial starting point.

One of the important questions that arise is: when does LEB holds?. Gubin, Polyak and Raik [21] proved that if there exists a $1 \leq j \leq m$ such that $C_j \cap \text{int}\,(\cap_{i \neq j} C_i) \neq \emptyset$ then LEB is satisfied and thus linear rate of convergence is proven. In [3] it was proven that the standard Slater condition implies LEB (LEB is called there "bounded linear regularity"). However, the argument leading to this result was rather long and tedious. The contribution of this chapter is summarized as follows:

1. We prove the basic convergence results about projection algorithms using elementary geometric facts. We give a new and simple proof based on elementary convexity arguments of the fact that Slater's condition implies LEB (and thus implies linear rate of convergence).

2. We show that projection methods can be very slow if the Slater condition is not satisfied. Moreover, without the Slater condition we can not bound the rate of convergence. For example, we give an example where the sequence $\{x_n\}$ generated by CPA satisfies $d(x_n, C) \geq \frac{1}{n^{\frac{1}{1000}}}$ which is a very slow rate of convergence. However, we show that we can bound the rate of convergence of the error bound. More precisely, that there exists a constant D such that $\max_{1 \leq i \leq m} d(x_n, C_i) \leq \frac{D}{\sqrt{n}}$.

3. A relation between projection algorithms for convex feasibility problems and the gradient projection method for optimization problems is established and a relation between the associated error bounds (i.e. GREB and LEB) is found.

4. We find that the best possible convex combination (in some sense) is the solution to a certain convex optimization problem.

The chapter is organized as follows. Section 3.2 recalls the basic results about projection algorithms that will be needed in the rest of the chapter and proves the global convergence of MPA , CPA and MDPA applied to the convex feasibility problem. In Section 3.3 we prove the linear rate of convergence of MPA, MDPA for a finite number of sets and the linear convergence of CPA for the case of two sets under a condition called LEB (Local Error Bound). An example demonstrates the fact that assumptions like LEB are to be made in order to insure the linear convergence. In section 3.4 we prove that the Slater condition implies LEB. In section 3.5 we demonstrate that not only linear rate of convergence is not guaranteed in projection algorithms but we can find examples of the method where the rate of convergence is as slow as we'd like. Section 3.6 recalls the gradient projection method. A connection between the gradient projection method and projection algorithms is established. Also, we prove a connection between error bounds for optimization problems and error bounds for convex feasibility problems. Section 3.7 suggests a rule for finding a good choice of convex combinations via the solution a related convex optimization problem. Section 3.8 presents an application of convex feasibility problems for conic optimization problems via projection algorithms.

3.2 Convergence of Projection Algorithms

Our objective is to prove that MPA and CPA converge to a point in C. The following technical lemma is the first step in proving the convergence result.

Lemma 3.2.1

$$\forall j = 1, \ldots, m \quad d^2(x_n, C_j) \leq d^2(x_n, C) - \|P_{C_j}(x_n) - P_C(x_n)\|^2. \tag{3.1}$$

Proof: Recall the firm non-expensiveness of the projection operator (Theorem A.1.4):

$$\|P_S x - P_S y\|^2 + \|(x - P_S x) - (y - P_S y)\|^2 \leq \|x - y\|^2. \tag{3.2}$$

Let $x = x_n, y = P_C(x_n), S = C_j$ then,

$$\begin{aligned} \|P_S x - P_S y\|^2 &= \|P_{C_j}(x_n) - P_{C_j} P_C(x_n)\|^2 \\ &= \|P_{C_j}(x_n) - P_C(x_n)\|^2. \end{aligned}$$

$$\begin{aligned}\|(x - P_S x) - (y - P_S y)\|^2 &= \|(x_n - P_{C_j}(x_n)) - (P_C(x_n) - P_{C_j}P_C(x_n))\|^2 \\ &= \|x_n - P_{C_j}(x_n)\|^2 \\ &= d^2(x_n, C_j).\end{aligned}$$

Moreover, one has,

$$\begin{aligned}\|x - y\|^2 &= \|x_n - P_C(x_n)\|^2 \\ &= d^2(x_n, C).\end{aligned}$$

Substitute these expressions in (3.2) we obtain:

$$\forall j = 1, \ldots, m \quad d^2(x_n, C_j) \le d^2(x_n, C) - \|P_{C_j}(x_n) - P_C(x_n)\|^2.$$

□

The following theorem is the key argument in proving the convergence of the sequence generated by MPA and CPA.

Theorem 3.2.1

$$\sum_{j=1}^{m} \alpha_j d^2(x_n, C_j) \le d^2(x_n, C) - d^2(x_{n+1}, C). \tag{3.3}$$

Proof:

$$\begin{aligned}d^2(x_{n+1}, C) &= \|x_{n+1} - P_C(x_{n+1})\|^2 \\ &\overset{d(x_{n+1},C) \le \|x_{n+1} - y\| \ \ \forall y \in C}{\le} \|x_{n+1} - P_C(x_n)\|^2 \\ &= \left\| \sum_{j=1}^{m} \alpha_j P_{C_j}(x_n) - P_C(x_n) \right\|^2 \\ &= \left\| \sum_{j=1}^{m} \alpha_j (P_{C_j}(x_n) - P_C(x_n)) \right\|^2 \\ &\overset{(*)}{\le} \sum_{j=1}^{m} \alpha_j \|P_{C_j}(x_n) - P_C(x_n)\|^2.\end{aligned}$$

$(*)$ is true by the convexity of $g(x) = \|x\|^2$. Thus,

$$\sum_{j=1}^{m} \alpha_j \|P_{C_j}(x_n) - P_C(x_n)\|^2 \geq d^2(x_{n+1}, C). \tag{3.4}$$

multiplying (3.1) by α_j we obtain:

$$\forall j = 1, \ldots, m \quad \alpha_j d^2(x_n, C_j) \leq \alpha_j d(x_n, C)^2 - \alpha_j \|P_{C_j}(x_n) - P_C(x_n)\|^2.$$

Adding these m inequalities one thus has:

$$\begin{aligned} \sum_{j=1}^{m} \alpha_j d^2(x_n, C_j) &\leq \left(\sum_{j=1}^{m} \alpha_j\right) d(x_n, C)^2 - \sum_{j=1}^{m} \alpha_j \|P_{C_j}(x_n) - P_C(x_n)\|^2 \\ &\overset{(3.4)}{\leq} d(x_n, C)^2 - d^2(x_{n+1}, C), \end{aligned}$$

and thus the result follows. □

It is also very useful to notice another property of the sequence $\{x_n\}$. This property is called *Fejér monotonicity.*

Theorem 3.2.2 (Fejér monotonicity of the sequence with respect to C)

$$\|x_{n+1} - y\| \leq \|x_n - y\| \quad \forall y \in C \tag{3.5}$$

Proof:

$$\begin{aligned} \|x_{n+1} - y\| &= \left\|\sum_{i=1}^{m} \alpha_i P_{C_i}(x_n) - y\right\| \\ &\overset{\sum_{i=1}^{m} \alpha_i = 1}{=} \left\|\sum_{i=1}^{m} \alpha_i (P_{C_i}(x_n) - y)\right\| \\ &\overset{\text{triangle inequality}}{\leq} \sum_{i=1}^{m} \alpha_i \|P_{C_i}(x_n) - y\| \\ &\overset{y \in C \subseteq C_i}{\leq} \sum_{i=1}^{m} \alpha_i \|P_{C_i}(x_n) - P_{C_i} y\| \\ &\overset{\text{Theorem}(A.1.4)}{\leq} \sum_{i=1}^{m} \alpha_i \|x_n - y\| \overset{\sum_{i=1}^{m} \alpha_i = 1}{=} \|x_n - y\|. \end{aligned}$$

□

We are now ready to prove the main convergence result of this section.

Theorem 3.2.3 (Global Convergence of MPA) *Let $\{x_n\}$ be a sequence generated by MPA. Then there is a point $c \in C$ such that:*

$$x_n \longrightarrow c.$$

Proof: First, from (3.5) we have that,

$$d(x_{n+1}, C) \leq d(x_n, C) \quad \forall n = 0, 1, 2, \ldots$$

Since $\{x_n\}$ is bounded (by (3.5)), it has at least one accumulation point x^*. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to x^*. By (3.3) one has :

$$\sum_{j=1}^{m} \alpha_j d^2(x_{n_k}, C_j) \leq d^2(x_{n_k}, C) - d^2(x_{n_k+1}, C) \leq d^2(x_{n_k}, C) - d^2(x_{n_{k+1}}, C). \tag{3.6}$$

Taking $k \to \infty$ we have by the continuity of the distance function that:

$$\sum_{j=1}^{m} \alpha_j d^2(x^*, C_j) \leq d^2(x^*, C) - d^2(x^*, C) = 0,$$

and thus $d(x^*, C_j) = 0$ for all $1 \leq j \leq m$ (because $\alpha_j > 0$). In other words, $x^* \in C$. All is left to prove is the uniqueness of the accumulation points. Suppose that x^∞, y^∞ are two accumulation points of $\{x^k\}$. The sequences $\{\|x^k - x^\infty\|\}, \{\|x^k - y^\infty\|\}$ are bounded and non-increasing so they have limits:

$$\begin{aligned}
\lim_{k\to\infty} \|x^k - x^\infty\| &= l_1, \\
\lim_{k\to\infty} \|x^k - y^\infty\| &= l_2.
\end{aligned}$$

Now,

$$\|x^k - x^\infty\|^2 - \|x^k - y^\infty\|^2 = -2\langle x^k, x^\infty - y^\infty\rangle + \|x^\infty\|^2 - \|y^\infty\|^2.$$

Taking limits $x^{k_l} \overset{l\to\infty}{\longrightarrow} x^\infty$,$x^{k_l} \overset{l\to\infty}{\longrightarrow} y^\infty$ we obtain:

$$l_1^2 - l_2^2 = -\|x^\infty - y^\infty\|^2, l_1^2 - l_2^2 = \|x^\infty - y^\infty\|^2.$$

Showing that $x^\infty = y^\infty$.

□

Remark: Besides clarity, there is no real reason to limit the discussion to algorithms which use the same convex combination of the projections of x_n to each of the m convex sets $C_1, C_2, \ldots, C_m$. We can define the general step by:

$$x_{n+1} = \sum_{i=1}^{m} \alpha_i^n P_{C_i}(x_n)$$

where $\sum_{i=1}^{m} \alpha_i^n = 1$ and $\alpha_1^n, \alpha_2^n, \ldots, \alpha_m^n > 0$. The only reservation is that Theorem (3.2.3) is true only if we add the assumption that there are numbers $\beta_1, \ldots, \beta_m > 0$ such that $\alpha_i^n \geq \beta_i \;\; \forall i = 1, \ldots, m, n = 0, 1, 2, \ldots$.

Theorem 3.2.4 (Global Convergence of MDPA) *Let $\{x_n\}$ be a sequence generated by MDPA. Then there is a point $c \in C$ such that:*

$$x_n \longrightarrow c.$$

Proof: The proof of the convergence of the sequence generated by MDPA is the same as the proof of the convergence of MPA except for inequality (3.6) that takes the following form:

$$\max_{j=1,\ldots,m} d(x_n, C_j) \leq d^2(x_{n_k}, C) - d^2(x_{n_k+1}, C) \leq d^2(x_{n_k}, C) - d^2(x_{n_{k+1}}, C).$$

□

The proof of the convergence of the sequence generated by CPA is slightly different.

Theorem 3.2.5 (Convergence of CPA) *Let $\{x_n\}$ be a sequence generated by CPA. Then there is a point $c \in C$ such that:*

$$x_n \longrightarrow c.$$

Proof: As noted in Theorem 3.2.3, $\{x_n\}$ is bounded and thus has at least one accumulation point x^*. We will prove that $x^* \in C$. Without loss of generality we assume that $\{x_n\}$ has infinite number of elements from the subsequence $\{x_{m(k-1)+1}\}_{k=1}^{\infty}$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to x^* such that $n_k \bmod m = 1$. By (3.3) one has:

$$d^2(x_{n_k}, C_1) \leq d^2(x_{n_k}, C) - d^2(x_{n_k+1}, C) \leq d^2(x_{n_k}, C) - d^2(x_{n_{k+1}}, C).$$

Passing to the limit as $k \to \infty$, we have by the continuity of the distance function that:

$$d^2(x^*, C_1) \leq d^2(x^*, C) - d^2(x^*, C) = 0,$$

and thus $x^* \in C_1$. By the definition of $\{x_{n_k}\}$ we have $x_{n_k} \in C_m$ and so $x^* \in C_1 \cap C_m$. Now, consider the sequence $\{x_{n_k+1}\} = \{P_{C_1}(x_{n_k})\}$. By the continuity of the projection operator we have that:

$$x_{n_k+1} = P_{C_1}(x_{n_k}) \longrightarrow P_{C_1}(x^*) \stackrel{x^* \in C_1}{=} x^*.$$

By the definition of x_{n_k+1} we have that $x_{n_k+2} = P_{C_2}(x_{n_k+1})$ and thus by (3.3):

$$d^2(x_{n_k+1}, C_2) \leq d^2(x_{n_k+1}, C) - d^2(x_{n_k+2}, C) \leq d^2(x_{n_k+1}, C) - d^2(x_{n_{k+1}+1}, C).$$

As $k \to \infty$ we have by the continuity of the distance function that:

$$d^2(x^*, C_2) \leq d^2(x^*, C) - d^2(x^*, C) = 0,$$

and thus $x^* \in C_2$. Continuing this process we obtain that $x^* \in \cap_{i=1}^m C_i = C$. The uniqueness of the accumulation point is proven as in Theorem 3.2.3. □

3.3 Linear Convergence of Projection Algorithms

3.3.1 Linear Convergence of MPA and MDPA

Convergence of the algorithm was proved under the mild assumption that $C \neq \emptyset$. In order to prove linear convergence, an additional assumption is necessary.

The assumption on the sets $C_1, C_2, \ldots, C_m$ that will be discussed is the *Local Error Bound* assumption (shortly written as LEB).

Definition 3.3.1 (LEB) *m closed convex sets $C_1, \ldots, C_m$ are said to satisfy LEB if for every bounded set B there exists a $\theta_B > 0$ such that:*

$$\forall x \in B \quad d(x, C) \le \theta_B \max_{i=1,\ldots,m} \{d(x, C_i)\}.$$

Theorem 3.3.1 (LEB implies linear rate of convergence of MPA) *If LEB is satisfied then MPA converges with a linear rate. More specifically,*

$$d(x_{n+1}, C) \le \gamma_B d(x_n, C), \tag{3.7}$$

where,

$$\begin{aligned} \gamma_B &= \sqrt{1 - \frac{\min_{j=1,\ldots,m} \{\alpha_j\}}{\theta_B^2}} \\ B &= \{x : \|x - y\| \le \|x_0 - y\|\}, \end{aligned} \tag{3.8}$$

and y is an arbitrary point in C.

Proof: First, we prove that $x_n \in B$ for every $n \ge 0$. By (3.5) we have that for all $y \in C$:

$$\|x_n - y\| \le \|x_{n-1} - y\| \le \ldots \le \|x_0 - y\|,$$

and thus $x_n \in B$. By LEB there is a $\theta_B > 0$ such that

$$d(x_n, C) \le \theta_B \max_{i=1,\ldots,m} \{d(x_n, C_i)\} \quad \forall n \ge 0.$$

Now, for all $n \ge 0$:

$$\begin{aligned} d^2(x_n, C) &\le \theta_B^2 \max_{i=1,\ldots,m} \{d^2(x_n, C_i)\} \\ &\le \frac{\theta_B^2}{\min_{j=1,\ldots,m} \{\alpha_j\}} \sum_{i=1}^{m} \alpha_i d^2(x_n, C_i) \\ &\overset{(3.3)}{\le} \frac{\theta_B^2}{\min_{j=1,\ldots,m} \{\alpha_j\}} (d^2(x_n, C) - d^2(x_{n+1}, C)). \end{aligned}$$

Define $A = \frac{\theta_B^2}{\min\limits_{j=1,\ldots,m}\{\alpha_j\}}$ and we obtain that:

$$d^2(x_{n+1}, C) \le \left(1 - \frac{1}{A}\right) d^2(x_n, C)$$

proving the desired result. $\square$

The linear convergence of MDPA under the LEB assumption is proved in a similar way:

Theorem 3.3.2 (LEB implies linear rate of convergence of MDPA) *If LEB is satisfied then MDPA converges with a linear rate. More specifically,*

$$d(x_{n+1}, C) \le \gamma_B d(x_n, C).$$

where,

$$\begin{aligned} \gamma_B &= \sqrt{1 - \frac{1}{\theta_B^2}} \\ B &= \{x : \|x - y\| \le \|x_0 - y\|\}. \end{aligned}$$

Proof: The proof is the same as the proof of linear convergence of MPA, but here we have the following:

$$\begin{aligned} d^2(x_n, C) &\le \theta_B^2 \max_{i=1,\ldots,m}\{d^2(x_n, C_i)\} \\ &\overset{(3.3)}{\le} \theta_B^2 (d^2(x_n, C) - d^2(x_{n+1}, C)). \end{aligned}$$

and the result follows. $\square$

Remark: The speed of the convergence that we have proved is dependent on the initial point x_0 (γ_B is dependent on x_0). This dependency can be removed if a stronger assumption, called GEB (Global Error Bound) is assumed:

Definition 3.3.2 (GEB) *m closed convex sets $C_1, \ldots, C_m$ satisfy GEB if there exists a $\theta > 0$ such that:*

$$d(x, C) \le \theta \max_{i=1,\ldots,m}\{d(x, C_i)\}.$$

3.3.2 Linear Convergence of CPA for two sets

For two sets, it is easy using our previous results to prove that the sequence generated by CPA converges to a point in C with a linear rate of convergence. However for the general problem of m sets we cannot prove a result like (3.7) for the simple reason that it is not true. Take for example the case where we have three closed convex sets C_1, C_2, C_3 such that $C_1 = C_2$ then obviously $x_{n+1} = x_n$ every three times thus equation (3.7) cannot hold for $\gamma_b < 1$.

Theorem 3.3.3 (LEB implies linear rate of convergence of CPA for two sets) *If LEB is satisfied then CPA converges with a linear rate. More specifically,*

$$d(x_{n+1}, C) \le \gamma_B d(x_n, C).$$

where,

$$\begin{aligned} \gamma_B &= \sqrt{1 - \frac{1}{\theta_B^2}} \\ B &= \{x : \|x - y\| \le \|x_0 - y\|\}. \end{aligned}$$

y is an arbitrary point in C.

Proof: CPA for two sets is the same as MDPA for two sets and thus the result follows. □

3.3.3 Linear Convergence of the Sequence

Until now, we have proved that under the assumption that $\bigcap_{i=1}^m C_i \neq \emptyset$ there is a $x^* \in C$ such that $x_n \to x^*$. We have also proved that under the LEB assumption $d(x_n, C) \to 0$ with a linear rate of convergence. Now we will prove that $x_n \to x^*$ with a linear rate of convergence.

Theorem 3.3.4 *Let $\{x_n\}$ be a sequence generated by MPA or by MDPA. Then, there is a $x^* \in C$ such that:*

$$\|x_n - x^*\| \le D\gamma_B^n,$$

where $D = \frac{d(x_0,C)}{1-\gamma_B} > 0$ and γ_B is defined by (3.8).

Proof: We have already proved that there is a $x^* \in C$ such that $x_n \to x^*$ and that $d(x_{n+1}, C) \leq \gamma_B d(x_n, C)$ for every $n \geq 0$. Now,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \left\|\sum_{i=1}^{m} \alpha_i P_{C_i}(x_n) - x_n\right\| \\
&= \left\|\sum_{i=1}^{m} \alpha_i (P_{C_i}(x_n) - x_n)\right\| \\
&\leq \sum_{i=1}^{m} \alpha_i \|P_{C_i}(x_n) - x_n\| \\
&= \sum_{i=1}^{m} \alpha_i d(x, C_i) \\
&\overset{C \subseteq C_i}{\leq} \sum_{i=1}^{m} \alpha_i d(x, C) \\
&= d(x_n, C) \\
&\leq t\gamma_B^n
\end{aligned}$$

where $t = d(x_0, C)$. Thus, for every $N > n$:

$$\|x_N - x_n\| \leq \sum_{j=n}^{N-1} \|x_{j+1} - x_j\| \leq \sum_{j=n}^{N-1} t\gamma_B^j = t\gamma_B^n \left(\frac{1 - \gamma_B^{N-n}}{1 - \gamma_B}\right).$$

Taking $N \to \infty$ we have:

$$\|x_n - x^*\| \leq \frac{t}{1 - \gamma_B} \gamma_B^n.$$

Substituting $D = \frac{t}{1-\gamma_B}$, we obtain the result. □

3.3.4 An Example

We consider two closed convex sets C_1, C_2 such that $C_1 \cap C_2 \neq \emptyset$. These sets do not satisfy LEB and we will prove that the sequence generated by CPA does not converge with a linear rate. Thus the we can not guarantee linear convergence in the absence of the LEB assumption.

Two Sets That Don't Satisfy LEB

Define the following closed convex sets in $\Re^2$:

$$\begin{aligned} C_1 &= \{(x,0) : x \in \Re\}, \\ C_2 &= \{(x,y) : y \geq x^2\}. \end{aligned}$$

Notice that $C_1 \cap C_2 = \{(0,0)\}$. Now, we prove that LEB is not satisfied. Otherwise, take the bounded sets $B = \{(x,0) : |x| \leq 1\}$ and by the LEB condition we have that there is a $\theta_B > 0$ such that:

$$\forall x \in B \quad d(x, C_1 \cap C_2) \leq \theta_B \max\{d(x,C_1), d(x,C_2)\}. \tag{3.9}$$

Set $x_k = \left(\frac{1}{k}, 0\right)$ for $k = 1, 2, \ldots$ and obtain:

$$d(x_k, C_1 \cap C_2) = \frac{1}{k}, \tag{3.10}$$

also $x_k \in C_1$ and thus $d(x_k, C_1) = 0$. All is left to calculate is $d(x_k, C_2)$. We know that:

$$\forall y \in C_2 \quad d(y, C_2) \leq \|x_k - y\|.$$

Take $y_k = \left(\frac{1}{k}, \frac{1}{k^2}\right) \in C_2$ and thus

$$d(x_k, C_2) \leq \|x_k - y_k\| = \frac{1}{k^2}. \tag{3.11}$$

Substitute (3.10) and (3.11) in (3.9) and obtain

$$\frac{1}{k} \leq \theta_B \frac{1}{k^2} \quad \forall k = 1, 2, \ldots.$$

which is equivalent to:

$$\theta_B \geq k \quad \forall k = 1, 2, \ldots$$

and is clearly impossible.

CPA Does Not Converge with a linear rate

Convergence of the sequence generated by CPA to $(0,0)$ is guaranteed by Theorem 3.3.3. We will now show that the CPA algorithm does not converge to (0,0) with a linear rate. The projection on C_1 has an explicit and simple expression:

$$P_{C_1}(x,y) = (x,0).$$

The projection on C_2 does not have a simple expression. The following lemma finds an implicit expression for the projection on C_2:

Lemma 3.3.1 *Let $x_0 > 0$. Then,*

$$P_{C_2}(x_0,0) = \left(\frac{x_0}{1+\lambda}, \frac{\lambda}{2}\right),$$

where $\lambda > 0$ satisfies the following condition:

$$\frac{x_0^2}{(1+\lambda)^2} = \frac{\lambda}{2}.$$

Proof: The projection of $(x_0,0)$ on C_2 is the solution to the following minimization problem:

$$\begin{array}{ll} \text{minimize} & (x-x_0)^2 + y^2 \\ \text{s.t.} & x^2 - y \le 0 \end{array}$$

By the KKT conditions there is a $\lambda \ge 0$ such that:

$$\begin{cases} 2(y-y_0) - \lambda = 0 \\ 2(x-x_0) + 2\lambda x = 0. \end{cases}$$

Thus,

$$x = \frac{x_0}{1+\lambda} \;\;,\;\; y = \frac{\lambda}{2}.$$

$(x_0, y) \notin \text{int}(C_2)$ and thus $(x,y) = P_{C_2}(x_0,0) \in bd(C_2)$ which yields $x^2 = y$. □
Define the sequence generated by CPA (we only compute the first component because the second component is zero):

$$\begin{aligned} x_0 &> 0 \text{ arbitrary}, \\ x_{n+1} &= \frac{x_n}{1+\lambda_n}. \end{aligned}$$

where $\lambda_n > 0$ satisfies:

$$\frac{x_n^2}{(1+\lambda_n)^2} = \frac{\lambda_n}{2}.$$

Lemma 3.3.2 *$x_n \to 0$ but not with a linear rate of convergence.*

Proof: By Theorem 3.2.3 $x_n \to 0$. Assume that $\{x_n\}$ does converge with a linear rate to 0 then, there is a $0 < \alpha < 1$ and a natural number N such that $x_{n+1} \leq \alpha x_n$ for every $n \geq N$. Recall that $x_{n+1} = \frac{x_n}{1+\lambda_n}$ where $(1+\lambda_n)^2\lambda_n = 2x_n^2$. As a result:

$$\lambda_n^3 < (1+\lambda_n)^2\lambda_n = 2x_n^2$$

Thus,

$$\lambda_n < \sqrt[3]{2x_n^2},$$

and so for any $n \geq N$:

$$\alpha x_n \geq x_{n+1} = \frac{x_n}{1+\lambda_n} > \frac{x_n}{1+\sqrt[3]{2x_n^2}}$$

which implies:

$$\alpha x_n > \frac{x_n}{1+\sqrt[3]{2x_n^2}}.$$

Dividing the later inequality by x_n we get:

$$\alpha > \frac{1}{1+\sqrt[3]{2x_n^2}}$$

and as $x_n \to 0$ we thus have that $\alpha \geq 1$ which contradicts the assumption $\alpha \in (0,1)$. □

3.4 The Slater Condition Implies LEB

Recall the Slater condition for the collection of closed convex sets $\{C_i\}_{i=1}^m$.
The Slater Condition: Let $C_1, \ldots, C_m$ be m closed convex sets. Suppose that $C_1, \ldots, C_k$ ($k \leq m$) are polyhedral sets. Then, $C_1, \ldots, C_m$ are said to satisfy the *Slater condition* if:

$$\left(\bigcap_{i=1}^{k} C_i\right) \bigcap \left(\bigcap_{i=k+1}^{m} \mathrm{ri}(C_i)\right) \neq \emptyset$$

The aim of this section is to prove that the Slater condition implies LEB using elementary convexity arguments and Hoffmann's Lemma. This result was recently derived in [3] through a quite long and rather complex machinery which thus appear to be unnecessary. We begin with the following result which is adopted from Gubin-Polyak [21].

Lemma 3.4.1 *Let $C_1, C_2 \subseteq \Re^n$ be two closed convex sets such that $C_1 \bigcap int(C_2) \neq \emptyset$. Then LEB is satisfied i.e. for every bounded set B there is a $\theta_B > 0$ such that*

$$\forall x \in B \quad d(x, C_1 \cap C_2) \leq \theta_B \max\{d(x, C_1), d(x, C_2)\}.$$

Proof: Let $x \in \Re^n$, denote $\eta = 2\max\{d(x, C_1), d(x, C_2)\}$. For every $z \in C_1 \cap C_2$ we have:

$$\begin{aligned}
d(x, C_1 \cap C_2) &\leq \|x - z\| \\
&\leq \|x - P_{C_1}(x)\| + \|P_{C_1}(x) - z\| \\
&= d(x, C_1) + \|P_{C_1}(x) - z\| \\
&\leq \frac{\eta}{2} + \|P_{C_1}(x) - z\|. \qquad (3.12)
\end{aligned}$$

$d(\bullet, C_2)$ is Lipschitz with constant 1 and thus,

$$d(y, C_2) \leq \|y - x\| + d(x, C_2) \quad \forall x, y \in \Re^n.$$

Set $y = P_{C_1}(x)$ and obtain:

$$\begin{aligned}
d(P_{C_1}(x), C_2) &\leq \|P_{C_1}(x) - x\| + d(x, C_2) \\
&= d(x, C_1) + d(x, C_2) \\
&\leq \eta \qquad (3.13)
\end{aligned}$$

Let $u \in C_1 \cap \operatorname{int}(C_2)$. $u \in \operatorname{int}(C_2)$ and thus there is a $\epsilon > 0$ such that:

$$\|u - v\| \leq \epsilon \Rightarrow v \in C_2.$$

Let $v = u + \mu(P_{C_1}(x) - P_{C_2}(P_{C_1}(x)))$. Then,

$$\|v - u\| = \mu\|P_{C_1}(x) - P_{C_2}(P_{C_1}(x))\| = \mu d(P_{C_1}(x), C_2) \overset{(3.13)}{\leq} \mu\eta.$$

So pick $\mu = \frac{\epsilon}{\eta}$ and therefore $\|v - u\| \leq \epsilon \Rightarrow v \in C_2$. Now, construct a specific z in $C_1 \cap C_2$:

$$z = \frac{1}{\mu+1}\underbrace{v}_{\in C_2} + \frac{\mu}{\mu+1}\underbrace{P_{C_2}(P_{C_1}(x))}_{\in C_2} \overset{C_2 \text{ is convex}}{\Longrightarrow} z \in C_2.$$

Using the definition of v we also have:

$$\begin{aligned} z &= \frac{1}{\mu+1}v + \frac{\mu}{\mu+1}P_{C_2}(P_{C_1}(x)) \\ &= \frac{1}{\mu+1}(u + \mu(P_{C_1}(x) - P_{C_2}(P_{C_1}(x)))) + \frac{\mu}{\mu+1}P_{C_2}(P_{C_1}(x)) \\ &= \frac{1}{\mu+1}\underbrace{u}_{\in C_1} + \frac{\mu}{\mu+1}\underbrace{P_{C_1}(x)}_{\in C_1} \end{aligned}$$

Thus $z \in C_1$ (and as a conclusion $z \in C_1 \cap C_2$). So now we have:

$$\begin{aligned} \|z - P_{C_1}(x)\| &= \left\| \overbrace{\frac{1}{\mu+1}u + \frac{\mu}{\mu+1}P_{C_1}(x)}^{z} - P_{C_1}(x) \right\| = \frac{1}{\mu+1}\|u - P_{C_1}(x)\| \\ &\leq \frac{1}{\mu}\|u - P_{C_1}(x)\| \\ &= \frac{1}{\mu}\|P_{C_1}u - P_{C_1}(x)\| \\ &\leq \frac{1}{\mu}\|u - x\| \\ &= \frac{\eta}{\epsilon}\|u - x\|. \end{aligned}$$

Using (3.12) we have:

$$d(x, C_1 \cap C_2) \leq \frac{\eta}{2} + \frac{\eta}{\epsilon}\|u - x\| \quad \forall x \in \Re^n, u \in C_1 \cap C_2.$$

Assuming $x \in B$ we have from the boundedness of B that there is a $M > 0$ such that $\|x\| \leq M$ and thus:

$$\begin{aligned} d(x, C_1 \cap C_2) &\leq \frac{\eta}{2} + \frac{\eta}{\epsilon}\|u - x\| \leq \frac{\eta}{2} + \frac{\eta}{\epsilon}(\|u\| + \|x\|) \leq \frac{\eta}{2} + \frac{\eta}{\epsilon} \overbrace{M'}^{M+\|u\|} \\ &= 2\left(\frac{1}{2} + \frac{M'}{\epsilon}\right) \max\{d(x, C_1), d(x, C_2)\} \\ &= \theta \max\{d(x, C_1), d(x, C_2)\}. \end{aligned}$$

where $\theta = 1 + \frac{2M'}{\epsilon} > 0$. □

Corollary 3.4.1 *Let $D_1, \ldots, D_m \subseteq \Re^n$ be m closed convex sets. If $\bigcap_{i=1}^m int(D_i) \neq \emptyset$ then LEB is satisfied i.e., for every bounded set B there is a $\theta_B > 0$ such that:*

$$\forall x \in B \quad d\left(x, \bigcap_{i=1}^m D_i\right) \leq \theta_B \max_{i=1,\ldots,m}\{d(x, D_i)\}.$$

Proof: Define:

$$\begin{aligned} C_1 &= \left\{\underbrace{(x, x, \ldots, x)}_{m \text{ times}} : x \in \Re^n\right\} \\ C_2 &= D_1 \times D_2 \times \ldots \times D_m \end{aligned}$$

Now,

$$C_1 \cap \text{int}(C_2) = \left\{(x, x, \ldots, x) : x \in \bigcap_{i=1}^m \text{int}(D_i)\right\}.$$

By the assumption $\bigcap_{i=1}^m \text{int}(D_i) \neq \emptyset$ we have that $C_1 \cap \text{int}(C_2) \neq \emptyset$. Thus, by Lemma 3.4.1 there is a $\theta_B > 0$ such that:

$$\forall y \in B^m \cap (\Re^n)^m \quad d(y, C_1 \cap C_2) \leq \theta_B \max\{d(y, C_1), d(y, C_2)\}, \tag{3.14}$$

where $B^m = \underbrace{B \times \ldots \times B}_{m \text{ times}}$. Let $y = (x, x, \ldots, x)$ then:

$$\begin{aligned} d(y, C_1 \cap C_2) &= m \cdot d\left(x, \bigcap_{i=1}^{m} D_i\right), \\ d(y, C_1) &= 0, \\ d(y, C_2) &= \sqrt{\sum_{j=1}^{m} d^2(x, D_j)} \leq \sqrt{m} \max_{j=1,\ldots,m} \{d(x, D_j)\}. \end{aligned}$$

Substituting these equations in (3.14) we obtain:

$$\forall x \in B \quad d\left(x, \bigcap_{i=1}^{m} D_i\right) \leq \frac{\theta_B}{\sqrt{m}} \max_{i=1,\ldots,m} \{d(x, D_i)\}.$$

□

The next simple result on convex sets will allow us to pass from interiors to relative interiors.

Lemma 3.4.2 *Let C be a closed convex set in $\Re^n$. Then there exists a closed convex sets $\tilde{C}$ such that $\tilde{C} \subseteq C$ and the following is satisfied:*

$$\begin{aligned} \mathit{aff}(C) \cap \mathit{int}(\tilde{C}) &= \mathit{ri}(C), \\ \mathit{aff}(C) \cap \tilde{C} &= C. \end{aligned}$$

Proof: Take $\tilde{C}$ to be:

$$\tilde{C} = C + M,$$

where M is the orthogonal complement to the linear subspace parallel to aff(C). First, we prove that aff$(C) \cap \tilde{C} = C$. It is obvious that $C \subseteq$ aff(C), $C \subseteq \tilde{C}$ and thus $C \subseteq$ aff$(C) \cap \tilde{C}$. Now, we'll prove the second direction: aff$(C) \cap \tilde{C} \subseteq C$. Let $x \in$ aff$(C) \cap \tilde{C}$. $x \in \tilde{C}$ so there are y, z such that:

$$x = y + z \qquad y \in C, z \in M.$$

Thus,

$$x - y = z \in M.$$

Since $x, y \in \text{aff}(C)$ then $x - y \in M^\perp$ which yields that $z = 0$. As a conclusion $x = y \in C$ which proves that $\text{aff}(C) \cap \tilde{C} = C$. Now,

$$\text{ri}(C) = \text{ri}(\text{aff}(C) \cap \tilde{C}) = \text{ri}(\text{aff}(C)) \cap \text{ri}(\tilde{C}) = \text{aff}(C) \cap \text{ri}(\tilde{C})$$

All that is left to verify is that $\text{aff}(\tilde{C}) = \Re^n$, and indeed one has:

$$\text{aff}(\tilde{C}) = \text{aff}(C + M) = \text{aff}(C) + M = \Re^n$$

□

Theorem 3.4.1 (Slater Implies LEB) *Let $C_1, \ldots, C_k$ be polyhedral sets and let $D_1, \ldots, D_m$ be closed convex sets. If the Slater condition is satisfied i.e.,*

$$\left(\bigcap_{i=1}^{k} C_i\right) \cap \left(\bigcap_{i=1}^{m} ri(D_i)\right) \neq \emptyset, \tag{3.15}$$

then LEB is satisfied, i.e., for every bounded set B there is a $\theta_B > 0$ such that,

$$\forall x \in B \quad d\left(x, \left(\bigcap_{i=1}^{k} C_i\right) \cap \left(\bigcap_{i=1}^{m} D_i\right)\right) \leq \theta_B \max_{i=1,\ldots,m, j=1,\ldots,k} \{d(x, C_j), d(x, D_i)\}.$$

proof: Let $\tilde{D}_1, \ldots, \tilde{D}_m$ be defined as in Lemma 3.4.2 i.e.,

$$\begin{aligned} \text{ri}(D_j) &= \text{aff}(D_j) \cap \text{int}(\tilde{D}_j) \\ D_j &= \text{aff}(D_j) \cap \tilde{D}_j. \end{aligned}$$

Then (3.15) is equivalent to:

$$\underbrace{\left(\bigcap_{i=1}^{k} C_i\right) \cap \left(\bigcap_{i=1}^{m} \text{aff}(D_i)\right)}_{E} \cap \text{int}\left(\underbrace{\bigcap_{i=1}^{m} \tilde{D}_j}_{F}\right) \neq \emptyset.$$

Since $E \cap \text{int}(F) \neq \emptyset$ we have from Lemma 3.4.1 that there is a $\theta_B > 0$ such that:

$$\forall x \in B \quad d(x, E \cap F) \leq \theta_B \max\{d(x,E), d(x,F)\}.$$

By Corollary 3.4.1 there is a $\gamma_B > 0$ such that:

$$\forall x \in B \quad d(x, F) \leq \gamma_B \max_{j=1,\ldots,m}\{d(x, \tilde{D}_j)\}.$$

On the other hand, by Hoffmann's Lemma there is a $\delta_B > 0$ such that:

$$\begin{aligned} \forall x \in B \quad d(x, E) &= d\left(x, \left(\bigcap_{i=1}^{k} C_i\right) \cap \left(\bigcap_{i=1}^{m} \operatorname{aff}(D_i)\right)\right) \\ &\leq \delta_B \max_{i=1,\ldots,m, j=1,\ldots,k}\{d(x, C_j), d(x, \operatorname{aff}(D_i))\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \forall x \in B \quad d(x, E \cap F) &\leq \theta_B \gamma_B \delta_B \max_{i=1,\ldots,m, j=1,\ldots,k}\{d(x, \tilde{D}_i), d(x, C_j), d(x, \operatorname{aff}(D_i))\} \\ &\leq \theta_B \gamma_B \delta_B \max_{i=1,\ldots,m, j=1,\ldots,k}\{d(x, C_j), d(x, D_i)\}. \end{aligned}$$

The last inequality is true because $D_i \subseteq \operatorname{aff}(D_i), D_i \subseteq \tilde{D}_i$ and thus,

$$d(x, \tilde{D}_i), d(x, \operatorname{aff}(D_i)) \leq d(x, D_i).$$

□

3.5 The Rate of Convergence of Projection Algorithms

We have already proven that the sequence $\{x_n\}$ generated by CPA converges to a point $x^* \in C_1 \cap C_2$. We will now see that if the Slater condition is not valid then for every even p we can find an example of closed convex sets C_1, C_2 such that:

$$\frac{1}{(An+B)^{\frac{1}{2p-2}}} \leq \|x - x^*\| \leq \frac{1}{(Cn+D)^{\frac{1}{2p-2}}}.$$

That is, the rate of convergence can be very slow. The two closed convex sets that we will consider are:

$$
\begin{aligned}
C_1 &= \{(x,0) : x \in \Re\}, \\
C_2 &= \{(x,y) : x^p - y \leq 0\}.
\end{aligned}
$$

Notice that $C_1 \cap C_2 = \{(0,0)\}$ and the Slater condition is not satisfied. The sequence generated by CPA is:

$$
\begin{aligned}
(x_0, y_0) &= (1,0), &(3.16)\\
(x_{n+1}, 0) &= P_{C_1}(P_{C_2}(x_n, 0)). &(3.17)
\end{aligned}
$$

The following lemma states that the sequence $\{x_n\}$ satisfies a recursive relation:

Lemma 3.5.1 *Let p be an even integer. The sequence generated by CPA $\{x_n\}$ satisfies the following relation.*

$$
\begin{aligned}
x_0 &= 1 \\
x_n &= x_{n+1} + p x_{n+1}^{2p-1}\ ,\ n = 0, 1, \ldots &(3.18)
\end{aligned}
$$

Proof: Denote $P_{C_2}(x_n, 0) = (x_{n+1}, y_{n+1})$. Then, (x_{n+1}, y_{n+1}) is the solution of the following optimization problem:

$$
\begin{aligned}
\text{minimize} \quad & (x - x_n)^2 + y^2 \\
\text{s.t.} \quad & x^p - y \leq 0
\end{aligned}
$$

By the KKT condition we have that:

$$
\begin{cases}
2y_{n+1} - \lambda = 0 \\
2(x_{n+1} - x_n) + p\lambda x_{n+1}^{p-1} = 0
\end{cases}
$$

Also, $(x_{n+1}, y_{n+1}) \in bd(C_2)$ and thus $y_{n+1} = x_{n+1}^p$. To conclude, we have:

$$
2(x_{n+1} - x_n) \overset{(3.5)}{=} -p\lambda x_{n+1}^{p-1} = -2p y_{n+1} x_{n+1}^{p-1} = -2p x_{n+1}^{2p-1}.
$$

□

In the next result we bound the value of x_{n+1} with respect to the value of x_n. These bounds will play a crucial role in the investigation of the convergence rate of the sequence.

Lemma 3.5.2 *There exists $0 < \gamma < 1$ such that for every n:*

$$\gamma x_n < x_{n+1} < x_n.$$

Proof: $x_{n+1} < x_n$ by the definition of the sequence. Also,

$$x_{n+1} = x_n - x_{n+1}^{2p-1} \overset{x_{n+1}<x_n}{>} x_n - x_n^{2p-1}.$$

On the other hand, by the convergence of the sequence generated by CPA we have that $x_n \to 0$ and thus there exists a natural N such that for every $n > N$ we have $x_n < \frac{1}{2}$. Thus, for every $n > N$,

$$x_{n+1} = x_n - x_{n+1}^{2p-1} > x_n - \left(\frac{1}{2}\right)^{2p-2} x_n = x_n \left(1 - \left(\frac{1}{2}\right)^{2p-2}\right).$$

Define γ to be greater than $\max\left\{1 - \left(\frac{1}{2}\right)^{2p-2}, \frac{x_1}{x_0}, \frac{x_2}{x_1}, \ldots, \frac{x_{N+1}}{x_N}\right\}$ and the lemma is proved. □

We are now ready to prove the main result of this subsection: "CPA can converge as slow as we wish".

Theorem 3.5.1 *Let $\{x_n\}$ be the sequence generated by CPA as described by (3.16),(3.17). Then there exists numbers A, B, C, D such that:*

$$\frac{1}{(An+B)^{\frac{1}{2p-2}}} \le x_n \le \frac{1}{(Cn+D)^{\frac{1}{2p-2}}}.$$

Proof: Notice that:

$$\begin{aligned}
\frac{1}{x_{n+1}^{2p-2}} - \frac{1}{x_n^{2p-2}} &= \frac{x_n^{2p-2} - x_{n+1}^{2p-2}}{x_{n+1}^{2p-2} x_n^{2p-2}} \\
&= \frac{(x_n - x_{n+1})\left(\sum_{k=0}^{2p-3} x_n^k x_{n+1}^{2p-3-k}\right)}{x_{n+1}^{2p-2} x_n^{2p-2}}
\end{aligned}$$

$$\stackrel{(3.5)}{=} \frac{p x_{n+1}^{2p-1}\left(\sum_{k=0}^{2p-3} x_n^k x_{n+1}^{2p-3-k}\right)}{x_{n+1}^{2p-2} x_n^{2p-2}}$$
$$= \frac{p x_{n+1}\left(\sum_{k=0}^{2p-3} x_n^k x_{n+1}^{2p-3-k}\right)}{x_n^{2p-2}}$$

We can thus bound the expression $\frac{1}{x_{n+1}^{2p-2}} - \frac{1}{x_n^{2p-2}}$ from above:

$$\frac{1}{x_{n+1}^{2p-2}} - \frac{1}{x_n^{2p-2}} = \frac{p x_{n+1}\left(\sum_{k=0}^{2p-3} x_n^k x_{n+1}^{2p-3-k}\right)}{x_n^{2p-2}} \stackrel{x_{n+1}<x_n}{<} p(2p-2),$$

and from below:

$$\frac{1}{x_{n+1}^{2p-2}} - \frac{1}{x_n^{2p-2}} = \frac{p x_{n+1}\left(\sum_{k=0}^{2p-3} x_n^k x_{n+1}^{2p-3-k}\right)}{x_n^{2p-2}} \stackrel{x_{n+1}>\gamma x_n}{>} p(2p-2)\gamma^{2p-2},$$

Summing this inequalities we obtain:

$$\frac{1}{x_n^{2p-2}} - 1 = \sum_{k=0}^{n-1}\left(\frac{1}{x_{k+1}^{2p-2}} - \frac{1}{x_k^{2p-2}}\right) < p(2p-2)n,$$
$$\frac{1}{x_n^{2p-2}} - 1 = \sum_{k=0}^{n-1}\left(\frac{1}{x_{k+1}^{2p-2}} - \frac{1}{x_k^{2p-2}}\right) > \gamma^{2p-2}p(2p-2)n.$$

Thus,

$$\frac{1}{(p(2p-2)n+1)^{\frac{1}{2p-2}}} < x_n < \frac{1}{(\gamma^{2p-2}p(2p-2)+1)^{\frac{1}{2p-2}}}.$$

Define $A = p(2p-2), B = 1, C = \gamma^{2p-2}p(2p-2), D = 1$ and the theorem is proved. □

3.6 Two Points Projection Algorithms

3.6.1 Definition of TPA

In this section we consider only convex feasibility problems with two sets. We begin by recalling the gradient projection algorithm (see chapter 2) to solve the following optimization problem:

$$(OP)\min_{x\in S} f(x),$$

Here we assume that S is a closed convex set and f is a differentiable function with Lipschitz gradient with Lipschitz constant L, i.e.:

$$\|\nabla f(x)-\nabla f(y)\|\leq L\|x-y\| \;\; \forall x,y\in S.$$

We assume that the optimal set of (OP) is nonempty and denote the optimal value by f^*. The Gradient Projection Algorithm (in short, GPA) is defined as follows:

Gradient Projection Algorithm (GPA)

first step: Take an arbitrary $x^0\in\Re^n$
general step: $x_{n+1}=P_S(x_n-\alpha\nabla f(x_n))$.

where α is a step size. One of the main results about the gradient projection algorithm is the sublinear rate of convergence of the function values:

Theorem 3.6.1 (Sublinear Rate of Convergence of the Function Values) *Let $\{x^k\}$ be the sequence generated by GPA with constant step size $0<\alpha<\frac{2}{L}$. Then for every $k\geq 1$ the following is satisfied:*

$$f(x^k)-f^*\leq\frac{C}{k},$$

for some constant C.

Proof: See [25].

There is a simple connection between the projection algorithms previously defined (CPA,MPA) and the gradient projection algorithm. Consider the convex feasibility problem with two closed convex sets C_1, C_2, then the feasibility problem is equivalent to the solution of the following optimization problem:

$$(P)\quad \min\left\{\frac{1}{2}\|x-y\|^2 : x\in C_1, y\in C_2\right\}.$$

The optimal set of (P) is $(C_1\cap C_2)\times(C_1\cap C_2)$ and the optimal value $f^*=0$.

The gradient projection algorithm applied to (P) considers two points in each iteration (one for each set). Thus the algorithm will be called TPA (Two Points projection Algorithm) and has the following form:

Two Point Projection Algorithm (TPA)

initial step: Take an arbitrary $x_0 \in C_1, y_0 \in C_2$
general step: $x_{n+1} = P_{C_1}((1-\alpha)x_n + \alpha y_n), y_{n+1} = P_{C_2}((1-\alpha)y_n + \alpha x_n)$

when α is a step size and thus by Theorem 3.6.1 the convergence is guaranteed only when $\alpha < \frac{2}{L}$ where L is the Lipschitz constant of the gradient of the objective function. It is easy to see that here $L = 2$ and thus the algorithm will converge for every $\alpha < 1$. Few important remarks are considered below:
Remarks:

1. In the case $\alpha = \frac{1}{2}$ TPA becomes MPA with equal weights ($x_{n+1} = \frac{P_{C_1}(x_n)+P_{C_2}(x_n)}{2}$) the convergence is already guaranteed by results already derived in this chapter but also from the results known about the gradient projection algorithm.

2. In the case $\alpha = 1$, convergence is not guaranteed by the theorems known for the gradient projection algorithm, but when $\alpha = 1$ TPA becomes CPA and so convergence is guaranteed by the theorems considered in this chapter.

3.6.2 The Rate of Convergence of TPA

In the previous subsection we noted that TPA is just MPA with equal weights when $\alpha = \frac{1}{2}$. Thus, TPA is also a kind of projection algorithm as already discussed in section 3.5 and can converge as slowly as we'd like (or don't like, as a matter of fact). However, we can bound the rate of convergence of the error bound: $\max\{d(x_n, C_2), d(y_n, C_1)\}$ (Recall that by the definition of TPA $x_n \in C_1, y_n \in C_2$).

Theorem 3.6.2 *Let C_1, C_2 be two closed convex sets with nonempty intersection and let $0 < \alpha < 1$. Then the sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ generated by TPA with constant step size α converges to a point in $(C_1 \cap C_2) \times (C_1 \cap C_2)$ and the following is satisfied: there exists a constant A such that:*

$$\max\{d(x_n, C_2), d(y_n, C_1)\} \le \frac{A}{\sqrt{n}}.$$

Proof: Denote $f(x,y) = \frac{1}{2}\|x-y\|^2$. Then by Theorem 3.6.1 we have that there exists C such that:

$$\|x_n - y_n\|^2 \le \frac{C}{n},$$

or equivalently,

$$\|x_n - y_n\| \le \frac{\sqrt{C}}{\sqrt{n}}.$$

Since $x_n \in C_1, y_n \in C_2$, this implies the following two inequalities:

$$d(x_n, C_2) \le \|x_n - y_n\| \le \frac{\sqrt{C}}{\sqrt{n}}, d(y_n, C_1) \le \|x_n - y_n\| \le \frac{\sqrt{C}}{\sqrt{n}}.$$

Define $A = \sqrt{C}$ and the theorem is proved. □

3.6.3 A Relation between GREB and LEB

Recall that for optimization problems one has the following error bound :
Assumption (GREB) For every closed bounded set B there exists a $\sigma_B > 0$ such that:

$$\forall x \in B \cap S \quad d(x, X^*) \le \sigma_B \|x - P_S(x - \alpha \nabla f(x))\|,$$

where X^* is the optimal set.

It is known that if GREB is satisfied then the sequence generated by GPA converges to an optimal point with a linear rate. By writing GREB for the optimization problem (P) (induced by the feasibility problem) we obtain a new error bound for the feasibility problem:

For every bounded set there exists $\sigma_B > 0$ such that:

$$d((x,y), (C_1 \cap C_2) \times (C_1 \cap C_2)) \le \sigma_B \|(x - P_{C_1}((1-\alpha)x + \alpha y), y - P_{C_2}((1-\alpha)y + \alpha))\|.$$

We can use the equivalence of norms in finite dimensional spaces and obtain the following error bound that will be called TPEB (Two Points Error Bound)

Definition 3.6.1 (TPEB) *Two sets C_1, C_2 are said to satisfy TPEB with constant α if for every bounded set B there exists $\sigma_B > 0$ such that for every $x \in B \cap C_1, y \in B \cap C_2$:*

$$\max\{d(x, C_1 \cap C_2), d(y, C_1 \cap C_2)\} \leq \sigma_B \max\{\|x - P_{C_1}((1-\alpha)x + \alpha y)\|, \|y - P_{C_2}((1-\alpha)y + \alpha x)\|\},$$

By using the results derived for the GPA algorithm we obtain that TPEB implies linear rate of convergence of the sequence generated by TPA.

Theorem 3.6.3 (Linear Rate of Convergence of the sequence) *Let C_1, C_2 be two closed convex sets with nonempty intersection. Suppose that TPEB is satisfied. Let $\{(x_n, y_n)\}$ be a sequence generated by TPA with $\alpha \in (0,1)$. Then there is $0 < \eta < 1$ such that,*

$$\begin{aligned} d(x_{n+1}, C_1 \cap C_2) &\leq \eta d(x_n, C_1 \cap C_2), \\ d(y_{n+1}, C_1 \cap C_2) &\leq \eta d(y_n, C_1 \cap C_2) \end{aligned}$$

Proof: Follows immediately from Theorem 2.3.2. □

Both conditions: LEB, TPEB imply linear convergence of the sequence generated by their associated algorithms (MPA,TPA respectively). As already noted, TPA with $\alpha = \frac{1}{2}$ is in fact MPA with equal weights. Thus, both conditions imply the linear convergence of MPA with equal weights. The question that naturally arises is: *what is the weaker condition?* . The next theorem answers this question, and it turns out that LEB implies TPEB.

Theorem 3.6.4 (LEB → TPEB) *Let C_1, C_2 be two closed convex sets with nonempty intersection that satisfy LEB. Then, for every $0 < \alpha < 1$ the TPEB condition with constant α is satisfied.*

Proof: Let B be a be a bounded set. Then, by LEB we have:

$$\forall x \in B \quad d(x, C_1 \cap C_2) \leq \theta_B \max\{d(x, C_1), d(x, C_2)\}.$$

Thus, for every $x \in C_1 \cap B, y \in C_2 \cap B$:

$$\begin{aligned} d(x, C_1 \cap C_2) &\leq \theta_B \max\{d(x, C_1), d(x, C_2)\} \\ &= \theta_B d(x, C_2) \\ &\leq \theta_B d(x, P_{C_2}((1-\alpha)x + \alpha y). \end{aligned}$$

By taking y instead of x we obtain the inequality:

$$d(y, C_1 \cap C_2) \leq \theta_B d(y, P_{C_1}((1-\alpha)y + \alpha x).$$

Combining these two inequalities we obtain TPEB:

$$\max\{d(x, C_1 \cap C_2), d(y, C_1 \cap C_2)\} \leq \theta_B \max\{\|x - P_{C_1}((1-\alpha)x + \alpha y)\|, \|y - P_{C_2}((1-\alpha)y + \alpha x)\|\},$$

□

3.7 Finding The Optimal Convex Combination in Projection Algorithms

Suppose that we are given m closed convex sets $C_1, C_2, \ldots, C_m$ such that $\bigcap_{i=1}^m C_i \neq \emptyset$. Consider the Projection algorithm given by:

$$x_{n+1} = \sum_{i=1}^m \alpha_i P_{C_i}(x_n),$$

where $\sum_{i=1}^m \alpha_i = 1$ and $\alpha_i \geq 0$ for all i.

The convex combination is not necessarily the same in every step of the algorithm. The question is :*what is the best convex combination?* We have:

$$\begin{aligned}
d(x_{n+1}, C)^2 &= \left\| \sum_{j=1}^m \alpha_j P_{C_j}(x_n) - P_C(x_{n+1}) \right\|^2 \\
&\leq \left\| \sum_{j=1}^m \alpha_j P_{C_j}(x_n) - P_C(x_n) \right\|^2 \\
&= \left\| \sum_{j=1}^m \alpha_j (P_{C_j}(x_n) - P_C(x_n)) \right\|^2 \\
&= \sum_{j=1}^m \|\alpha_j (P_{C_j}(x_n) - P_C(x_n))\|^2 + \sum_{i \neq j} \alpha_i \alpha_j \langle P_{C_i}(x_n) - P_C(x_n), P_{C_j}(x_n) - P_C(x_n) \rangle \\
&= \sum_{j=1}^m \alpha_j^2 \|P_{C_j}(x_n) - P_C(x_n)\|^2
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{i \neq j} \alpha_i \alpha_j \left(\frac{\|P_{C_i}(x_n) - P_C(x_n)\|^2 + \|P_{C_j}(x_n) - P_C(x_n)\|^2 - \|P_{C_i}(x_n) - P_{C_j}(x_n)\|^2}{2} \right) \\
\leq \quad & \sum_{j=1}^{m} \alpha_j^2 d(x_n, C)^2 + \sum_{i \neq j} \alpha_i \alpha_j d(x_n, C)^2 - \sum_{i \neq j} \alpha_i \alpha_j \frac{\|P_{C_i}(x_n) - P_{C_j}(x_n)\|^2}{2} \\
= \quad & \sum_{i,j} \alpha_i \alpha_j d(x_n, C)^2 - \sum_{i \neq j} \alpha_i \alpha_j \frac{\|P_{C_i}(x_n) - P_{C_j}(x_n)\|^2}{2} \\
= \quad & \left(\sum_{i=1}^{m} \alpha_i \right)^2 d(x_n, C)^2 - \frac{1}{2} \sum_{i \neq j} \alpha_i \alpha_j \|P_{C_i}(x_n) - P_{C_j}(x_n)\|^2 \\
= \quad & d(x_n, C)^2 - \frac{1}{2} \sum_{i \neq j} \alpha_i \alpha_j \|P_{C_i}(x_n) - P_{C_j}(x_n)\|^2.
\end{aligned}
$$

That is, the difference $d(x_n, C)^2 - d(x_{n+1}, C)^2$ is greater than:

$$
f(\alpha) = \sum_{i \neq j} \alpha_i \alpha_j \|P_{C_i}(x_n) - P_{C_j}(x_n)\|^2.
$$

This suggests that we should take the convex combination which is the solution of the maximization program:

$$
\begin{aligned}
\text{maximize} \quad & f(\alpha) \\
\text{s.t.} \quad & \sum_{j=1}^{m} \alpha_j = 1 \\
& \alpha_j \geq 0
\end{aligned}
$$

On first sight, this does not seem to be a convex programming problem because $f(\alpha)$ is a quadratic function with an indefinite matrix. But, using the condition $\sum_{i=1}^{m} \alpha_i = 1$ we can find a convex representation of the problem as a convex programming problem. Indeed, denote $v_i = P_{C_i}(x_n)$. Then, :

$$
\begin{aligned}
f(\alpha) \quad & = \quad \sum_{i \neq j} \alpha_i \alpha_j \|v_i - v_j\|^2 \\
& \overset{\|v_i - v_i\| = 0}{=} \quad \sum_{i,j} \alpha_i \alpha_j \|v_i - v_j\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i,j} \alpha_i \alpha_j (\|v_i\|^2 - 2v_i^T v_j + \|v_j\|^2) \\
&= \sum_{i,j} \alpha_i \alpha_j \|v_i\|^2 - 2\sum_{i,j} \alpha_i \alpha_j v_i^T v_j + \sum_{i,j} \alpha_i \alpha_j \|v_j\|^2 \\
&= \sum_{j=1}^{m} \alpha_j \sum_{i=1}^{m} \alpha_i \|v_i\|^2 - 2\left\|\sum_{i=1}^{m} \alpha_i v_i\right\|^2 + \sum_{i=1}^{m} \alpha_i \sum_{j=1}^{m} \alpha_j \|v_j\|^2 \\
&= 2\sum_{i=1}^{m} \alpha_i \|v_i\|^2 - 2\left\|\sum_{i=1}^{m} \alpha_i v_i\right\|^2
\end{aligned}
$$

The objective function is now a concave quadratic function because the matrix is seminegative-definite (for every $\alpha_1, \ldots, \alpha_m : -\|\sum_{i=1}^{m} \alpha_i v_i\|^2 \leq 0$).

Remark: If the set $\{P_{C_j}(x_n)\}$ is linearly independent then the target function is strictly concave.

3.8 Conic Programming

We introduce here potential algorithms that can be devised to solve conic optimization problems via CFP type algorithms and give the explicit formulas for the projections (details of computations are in appendix A).

3.8.1 The General Framework

The Conic Feasibility Problem

Let $C \subseteq \Re^n$ be a closed convex cone, and let A be a $m \times n$ real matrix and $b \in \Re^m$ then the following feasibility problem is called *The Conic Feasibility Problem*

$$
(CFP) \begin{array}{c} Ax = b \\ x \in C \end{array} .
$$

This problem can be solved by MPA when considering the closed convex sets $C_1 = \{x : Ax = b\}$, $C_2 = C$. Note that P_{C_1} is just an affine transformation. The form of P_{C_2} depends on the cone C.

Conic Programming

Let $C \subseteq \Re^n$ be a closed convex cone, and let A be a $m \times n$ real matrix and $b \in \Re^m, a \in \Re^n$ then the following optimization problem is called *Conic Programming*

$$
\text{(CP)} \quad \begin{array}{ll} \inf & a^T x \\ \text{s.t.} & Ax = b \\ & x \in C \end{array}
$$

The dual problem is of the form: $(\lambda \in \Re^m)$

$$
\text{(DCP)} \quad \begin{array}{ll} \sup & -b^T \lambda \\ \text{s.t.} & A^T \lambda + a = 0 \\ & \lambda \in C^* \end{array}
$$

where C^* is the polar cone defined by:

$$
C^* = \{x : x^T y \geq 0 \;\; \forall y \in C\}.
$$

If strong duality is satisfied then (x, λ) are optimal primal-dual solutions iff they satisfy the following system :

$$
\begin{array}{l} Ax = b \\ a^T x + b^T \lambda = 0 \\ x \in C \\ A^T \lambda + a \in C^* \end{array}
$$

We introduce the slack variables $s \in \Re^n$ for the purpose of obtaining a standard conic feasible problem:

$$
(CFP) \quad \begin{array}{l} Ax = b \\ a^T x + b^T \lambda = 0 \\ s = A^T \lambda + a \\ x \in C \\ s \in C^* \end{array} \tag{3.19}
$$

This is just a standard (CFP). We can see that by defining the cone $\tilde{C}$, the matrix $\tilde{A}$ and the vector $\tilde{b}$ as follows:

$$\begin{aligned}
\tilde{C} &= C \times \Re^m \times C^* \\
\tilde{A} &= \begin{pmatrix} A & 0 & 0 \\ a^T & b^T & 0 \\ 0 & -A^T & I_n \end{pmatrix} \\
\tilde{b} &= \begin{pmatrix} b \\ 0 \\ -a \end{pmatrix}
\end{aligned}$$

Thus, the conic programming problem is equivalent to the following convex feasibility problem:

$$(\widetilde{CFP}) \quad \begin{array}{l} \tilde{A}\tilde{x} = \tilde{b} \\ \tilde{x} \in \tilde{C} \end{array}$$

where $\tilde{x} = \begin{pmatrix} x \\ \lambda \\ s \end{pmatrix}$. The Projection on $\tilde{C}$ is given in terms of the projection on C, C^*:

$$P_{\tilde{C}} \begin{pmatrix} x \\ \lambda \\ s \end{pmatrix} = \begin{pmatrix} P_C(x) \\ \lambda \\ P_{C^*}(s) \end{pmatrix}$$

Now, we can apply MPA on the closed convex sets

$$\begin{aligned}
C_1 &= \{\tilde{x} : \tilde{A}\tilde{x} = \tilde{b}\}, \\
C_2 &= \tilde{C}.
\end{aligned}$$

Theorem (3.2.3) implies that MPA converges to a feasible point of $(\widetilde{CFP})$. If LEB is satisfied then convergence is guaranteed with a linear rate. This happens in the case of linear programming where LEB is satisfied by Hoffmann's Lemma. In the case of other conic programming problems like second order cone problems and semidefinite programming a Slater type condition must be assumed in order to ensure linear convergence. For this reason we consider a slightly different system, instead of (CFP) we consider its ϵ perturbation (CFP_ϵ):

$$\begin{aligned}
&Ax = b \\
&a^T x + b^T \lambda \geq \epsilon \\
&x \in C \\
&A^T \lambda + a \in C^*
\end{aligned}$$

Introducing slack variables we obtain the following cone feasibility problem:

$$(CFP_\epsilon) \quad \begin{aligned}
&Ax = b \\
&s = A^T \lambda + a \\
&t = a^T x + b^T \lambda - \epsilon \\
&x \in C \\
&s \in C^* \\
&t \geq 0
\end{aligned}$$

A solution to this system is not necessarily optimal but has an ϵ duality gap. This system is of course also a standard conic feasible problem. In order to ensure linear convergence we need to assume the Slater condition:

Assumption 11 (The Slater Condition For Conic Optimization Problems) *There exists a x, λ such that:*

$$\begin{aligned}
Ax &= b \\
x &\in ri(C) \\
A^T \lambda + a &\in ri(C^*) \\
a^T x + b^T \lambda &> \epsilon
\end{aligned}$$

3.8.2 Linear Programming

Linear Programming is the problem of minimizing a linear functional under linear constraints and non-negativity constraints:

$$\text{(LP)} \quad \begin{array}{ll} \inf & a^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

where $a \in \Re^n, A \in \Re^{m \times n}, b \in \Re^m$. Here the cone is the non-negative orthant: $C = \Re^n_+$. C is a self dual, i.e. $C^* = C = \Re^n_+$. Thus, by (CFP), the feasibility problem associated to with (LP) is

$$\begin{aligned} & Ax = b \\ & a^T x + b^T \lambda = 0 \\ & s = A^T \lambda + a \\ & x, s \geq 0 \end{aligned}$$

Now, we follow the general framework and define C_1, C_2 as in (3.20). By Hoffmann's lemma LEB is satisfied for C_1, C_2 because both are defined by system of linear equalities and inequalities. Thus, we can define a projection algorithm (for example: MPA) that will converge with a linear rate. The projection operator on each of the sets C_1, C_2 is trivial. The projection on C_1 is just an affine transformation and the projection on C_2 is just:

$$P_{C_2} \begin{pmatrix} x \\ \lambda \\ s \end{pmatrix} = \begin{pmatrix} [x]_+ \\ \lambda \\ [s]_+ \end{pmatrix}$$

3.8.3 Second Order Cone Programming

The standard representation of the second order cone programming optimization problem (SOCP) is the following:

$$\text{(SOCP)} \quad \begin{array}{ll} \inf & a^T x \\ \text{s.t.} & \|A_i x + b_i\| \leq c_i^T x + d_i \;\; i = 1, \ldots, m \end{array}$$

where $a, c_i \in \Re^n, A_i \in \Re^{k_i \times n}, b_i \in \Re^{k_i}$. This is equivalent to the following problem:

$$
\begin{array}{lll}
 & \inf & a^T x \\
\text{(SOCP')} & \text{s.t.} & \|z_i\| \leq w_i \;\; i = 1, \ldots, m \\
 & & z_i = A_i x + b_i, w_i = c_i^T x + d_i \quad i = 1, \ldots, m
\end{array}
$$

where $z_i \in \Re^{k_i}, w_i \in \Re$. Here we have a standard conic optimization problem where the associated cone is $C = \Re^n \times \mathcal{L}_{k_1} \times \ldots \times \mathcal{L}_{k_m}$. $\mathcal{L}_k$ is the *Lorentz Cone* also called *the ice cream cone*, and defined by:

$$
\mathcal{L}_k = \{(z, w) : \|z\| \leq w, z \in \Re^k, w \in \Re\}.
$$

The projection on the Ice Cream Cone can be expressed explicitly:

$$
P_{\mathcal{L}_k}(x, r) = \begin{cases} (x, r), & \text{if } \|x\| \leq r \\ (0, 0), & \text{if } \|x\| \leq -r \\ \frac{\|x\|+r}{2}\left(\frac{x}{\|x\|}, 1\right) & \text{else} \end{cases}
$$

3.8.4 Semidefinite Programming

The semidefinite Programming problem is the problem of minimizing a linear functional under linear constraints and positive semidefinite constraints.

$$
\begin{array}{lll}
 & \inf & a^T x \\
\text{(SDP)} & \text{s.t.} & AX = b \\
 & & X \succeq 0
\end{array}
$$

Here the associated cone is S_+^n which is the cone of all positive semidefinite matrices. The projection of a matrix to this cone involves finding the spectral decomposition of the matrix.

Chapter 4

Mirror Descent and Nonlinear Projected Subgradient Methods for Convex Optimization

4.1 Introduction

In this chapter we study projection based algorithms for nonsmooth constrained convex minimization problems. Consider the following nonsmooth convex minimization problem,

$$(P) \qquad \text{minimize} \;\; f(x) \;\; s.t. \;\; x \in X \subset \Re^n.$$

Throughout the chapter we make the following assumptions on problem (P):
Assumption A

- X is a closed convex subset in $\Re^n$ with nonempty interior.
- The objective function $f : X \to \Re$ is a convex Lipschitz continuous function with Lipschitz constant L_f with respect to a fixed given norm $\|\cdot\|$, i.e.,

$$|f(x) - f(y)| \leq L_f \|x - y\|, \; \forall x, y \in X.$$

- The optimal set of (P) denoted X^* is nonempty.
- A subgradient of f at $x \in X$ is computable. We denote by $f'(x)$ an element of the subdifferential $\partial f(x)$.

We are interested in finding an approximate solution to problem (P), within $\varepsilon > 0$, i.e., to find $x \in X$ such that

$$f(x) - f^* := f(x) - \min_{x \in X} f(x) \leq \varepsilon.$$

A standard method to solve (P) is the subgradient projection algorithm, (see e.g. [6] and references therein), which generates iteratively the sequence $\{x^k\}$ via

$$x_{k+1} = \pi_X(x^k - t_k f'(x^k)), \tag{4.1}$$

where $t_k > 0$ are some positive stepsizes and

$$\pi_X(x) = \operatorname{argmin}\{||x - y|| \mid y \in X\}.$$

is the Euclidean projection onto X.

The key advantage of the subgradient algorithm is its simplicity, provided that projections can be easily computed, which is the case when the constraints set X is described by simple sets, e.g., hyperplanes, balls, bound constraints, etc... Its main drawback is that it has a very slow rate of convergence. Indeed, consider the convex problem (P) with f convex and Lipschitz continuous on X, with Lipschitz constant L_f. Then, by the subgradient inequality for the convex function f one has,

$$\|f'(x)\|_* \leq L_f \quad \forall x \in X \tag{4.2}$$

where $\|\cdot\|_*$ is the dual norm. Suppose that X is a convex compact subset of $\Re^n$, and denote by $\operatorname{Diam}(X)$ the diameter of X, i.e., $\operatorname{Diam}(X) := \max_{x,y \in X} ||x - y|| < \infty$. Then, with the stepsizes chosen as

$$t_k = \operatorname{Diam}(X) k^{-1/2}, \; k = 1, \ldots \tag{4.3}$$

the optimal efficiency estimate for the subgradient method is (see [32]):

$$f(x^k) - \min_{x \in X} f(x) \leq O(1) L_f \operatorname{Diam}(X) k^{-1/2}, \tag{4.4}$$

where $O(1)$ stands for a positive absolute constant. Thus, like all gradient based methods, one can obtain in a very small number of iterations a *low accuracy* optimal value, (say one or two digits) but then within further iterations no more progress in accuracy can be achieved and the method is essentially jamming. However, a key feature of gradient methods is also the fact that while their rate of convergence is very slow, the rate is *almost* *independent* of the dimension of the problem. In contrast to this, more efficient sophisticated algorithms, such as for example interior point based methods, which require for example at each iteration Newton type computations, i.e., the solution of a linear system, are often defeated even for problems with a few thousands of variables, and a fortiori for very large scale nonsmooth problems. Therefore, for constrained problems where low accurate solutions is sufficient and the dimension is huge, gradient type methods appear as natural candidates for developing potential practical algorithms. The recent paper [4] on computerized tomography demonstrates very well this situation through an algorithm based on the *Mirror Descent Algorithm* (MDA for short) introduced by [32]. It is shown there that it is possible to solve efficiently a convex minimization problem over the unit simplex, with millions of variables.

Motivated by the recent work of [4], in this chapter we concentrate on the analysis of the basic steps of the (MDA) which is recalled in Section 2. We show in Section 3, that the (MDA) can be viewed as a simple *nonlinear subgradient projection* method, where the usual Euclidean projection operator is replaced by a nonlinear/non-orthogonal type projection operator based on a Bregman-like distance function, (see e.g., [8],[11],[39]) and references therein. With this new interpretation of the (MDA), we derive in a simple and systematic way convergence proofs and efficiency estimates, see Section 4. In Section 5 we concentrate on optimization problems over the unit simplex and propose a new algorithm called the Entropic Mirror Descent Algorithm (EMDA). The EMDA is proven to exhibit an efficiency estimate which is almost independent in the dimension n of the problem and in fact shares the same properties of an algorithm proposed in [4] for the same class of problems, but is given explicitly by a simple formula. Finally, in the last section we outline some potential applications and extensions for further work.

4.2 The Mirror Descent Algorithm (MDA)

The idea of the algorithm is based on dealing with the structure of the Euclidean norm rather than with local behavior of the objective function in problem (P). Roughly speaking, the method originated from functional analytic arguments arising within the infinite dimensional setting, between primal and dual spaces. The mathematical objects associated with f and x are not vectors from a vector space E, but elements of the dual vector space to E, which consists of linear forms on E. The Euclidean structure is not the only way to identify the primal-dual spaces, and it is possible to identify the primal and dual spaces within a wider family which includes as particular case, the classical Euclidean structures. This idea and approach was introduced by Nemirovsky and Yudin [32], and the reader is refereed to their book for a more detailed motivation and explanations. We will show below, that there is a much simpler and easy way to motivate, explain, and construct the MDA. For now, let us consider the basic steps involve in the original MDA.

Consider the problem (P) satisfying Assumption A. The (MDA) further assumes the following objects, which can be freely chosen as long as they satisfy the following hypothesis:

- Fix any norm $||\cdot||$ in $\Re^n$ (which will play a role in the choice of the stepsize).
- Let $\psi : X \to \Re$ be a continuously differentiable and strongly convex function on X with strong convexity parameter $\sigma > 0$.
- The conjugate of ψ, defined by

  $$\psi^*(y) = \max_{x \in X}\{\langle x, y\rangle - \psi(x)\}$$

  is assumed to be easily computable.

The basic steps of the Mirror Descent Algorithm can be described as follows, see, [32], and [4] (for comparison the set Y there is set to be equal to X in [4, p.84]).

The Mirror Descent Algorithm-MDA. Start with $y^1 \in \operatorname{dom} \nabla\psi^*$ and generate the sequence $\{x^k\} \in X$ via the iterations

$$x^k = \nabla\psi^*(y^k) \tag{4.5}$$

$$y^{k+1} = \nabla\psi(x^k) - t_k f'(x^k) \tag{4.6}$$

$$x_{k+1} = \nabla\psi^*(y^{k+1}) = \nabla\psi^*(\nabla\psi(x^k) - t_k f'(x^k)), \tag{4.7}$$

where $t_k > 0$ are appropriate step sizes.
The method looks at this stage somewhat hard to understand or even to motivate (besides the very rough explanation given above). In the next section we will give a very simple interpretation which will explain and reveal the structure of this algorithm. In the mean time, let us consider a basic example which clearly indicates that the MDA appears to be as a natural generalization of the subgradient algorithm.

Example 1. Let $||\cdot||$ be the usual l_2 norm in $\Re^n$ and let $\psi(x) := \frac{1}{2}||x||^2$ for $x \in X$ and $+\infty$ for $x \notin X$. The function ψ is clearly proper, lsc and strongly convex with parameter $\sigma = 1$, and continuously differentiable on X. A straightforward computation shows that the conjugate of ψ is given by $\psi^* : \Re^n \to \Re$

$$\psi^*(z) = \frac{1}{2}(||z||^2 - ||z - \pi_X(z)||^2)$$

with $\nabla\psi^*(z) = \pi_X(z)$. Indeed, since $\partial\psi(x) = (I + N_X)(x)$, where N_X denotes the normal cone of the closed convex set X, using the well known relations $(I + N_X)^{-1} = \pi_X$ and $(\partial\psi)^{-1} = \partial\psi^*$, (see, [36]), one thus has

$$z \in \partial\psi(x) \iff x = (I + N_X)^{-1}(z) = \pi_X(z) = \nabla\psi^*(z).$$

Therefore, the (MDA) yields

$$x^k = \pi_X(y^k) \tag{4.8}$$

$$y^{k+1} = x^k - t_k f'(x^k) \tag{4.9}$$

$$x^{k+1} = \pi_X(x^k - t_k f'(x^k)), \tag{4.10}$$

i.e., we have recovered the subgradient projection algorithm.

4.3 Nonlinear Projection Methods

It is well known (see e.g., [6]) that the subgradient algorithm can be viewed as *linearization* of the so-called proximal algorithm [37], (or as an explicit scheme of the corresponding subdifferential inclusion). Indeed, it is immediate to verify that the projected subgradient iteration (4.1) can be rewritten equivalently as

$$x^{k+1} \in \underset{x \in X}{\operatorname{argmin}} \left\{ \langle x, f'(x^k) \rangle + \frac{1}{2t_k}||x - x^k||^2 \right\}.$$

In [39], it has been shown that more general proximal maps can be considered by replacing the usual Euclidean quadratic norms with some sort of more general distance-like functions. As explained there, the principal motivation for such kind of distances is to be able to use one which reflects the geometry of the given constraints set X, so that in particular with such an appropriate choice, the constraints can often be *automatically eliminated*. In a similar way, we can thus construct nonlinear projection subgradient methods, by considering iteration schemes of the form

$$x^{k+1} \in \underset{x \in X}{\operatorname{argmin}} \left\{ \langle x, f'(x^k) \rangle + \frac{1}{t_k} D(x, x^k) \right\}, \tag{4.11}$$

where $D(u,v)$ replaces $2^{-1}\|u-v\|^2$, and should verify the property $D(u,v)$ is nonnegative, and $D(u,v) = 0$ if and only if $u = v$. We prove below, that the (MDA) is nothing else, but the nonlinear subgradient projection method (4.11), with a particular choice of D based on a Bregman-like distance generated by a function ψ. Note, that the hypothesis on D will be somewhat different from the usual Bregman based distances assumed in the literature, see e.g., [24], [39], and references therein.

Let $\psi : X \to \Re$ be strongly convex and continuously differentiable on $\operatorname{int} X$. The distance-like function is defined by $B_\psi : X \times \operatorname{int}(X) \to \Re$ given by

$$\mathrm{B}_\psi(x, y) = \psi(x) - \psi(y) - \langle x - y, \nabla\psi(y) \rangle. \tag{4.12}$$

The basic subgradient algorithm based on B_ψ is as follows.

Subgradient Algorithm with Nonlinear Projections (SANP) Given B_ψ as defined in (4.12) with ψ as above, start with $x_1 \in \operatorname{int} X$, and generate the sequence $\{x^k\}$ via the iteration

$$x^{k+1} = \underset{x \in X}{\operatorname{argmin}} \left\{ \langle x, f'(x^k) \rangle + \frac{1}{t_k} \mathrm{B}_\psi(x, x^k) \right\}, \quad t_k > 0. \tag{4.13}$$

When $\nabla\psi$ can be continuously extended on X, (e.g., as in Example 1), then we can consider the function B_ψ defined on $X \times X$. Note that in this case one needs not to start with $x^1 \in \operatorname{int} X$ and (SANP) can start with any arbitrary point $x^1 \in \Re^n$. With $X = \Re^n$ and $\psi(x) = \frac{1}{2}\|x\|^2$ one obtains $B_\psi(x,y) = \frac{1}{2}\|x-y\|^2$ thus recovering the classical squared Euclidean distance and (SANP) is just the classical subgradient algorithm.

We now turn to the question of having (SANP) a well defined algorithm. When ψ is continuously differentiable on X, then the strong convexity assumption immediately implies that the algorithm which starts with $x^1 \in \Re^n$ is well defined and produces a sequence $x^k \in X$, $\forall k$. When ψ is only assumed to be differentiable on $\operatorname{int} X$, we clearly need to guarantee that the next iterate stays in the interior of X, so that B_ψ can be defined on $X \times \operatorname{int} X$. For that, it suffices to make the following assumption:

$$\|\nabla\psi(x_t)\| \to +\infty \;\text{ as } t \to \infty,\; \forall\, \{x_t\} \in \operatorname{int} X \;\text{ with } x_t \to x \in \partial X, \tag{4.14}$$

where ∂X denotes the boundary of X. Note that (4.14) is just to say that ψ is essentially smooth, (see [36]). With this additional assumption on ψ together with the strong convexity, it follows that the sequence $\{x^k\}$ is well defined i.e., $x^k \in \text{int}\, X,\ \forall k$. An interesting choice for ψ satisfying (4.14) will be considered in Section 5.

It is interesting to note the differences between the two classes of algorithms which then emerged from (SANP). The first class with ψ continuously differentiable on X leads to *non-interior* methods with iterates $x^k \in X$. This is exactly the setting of the (MDA). Typical examples of ψ in that case will involve power of norms on X, see Example 1 and [4]. On the other hand, the second class, with ψ satisfying (4.14) will be an *interior* type subgradient algorithm producing sequences $x^k \in \text{int}\, X$. Note that the analysis we develop in the rest of this chapter will hold for both classes of algorithms with the additional assumption (3.14) on ψ when needed.

We first recall some useful facts regarding strongly convex functions, and their relations with conjugates and subdifferentials. These results can be found in [38, Section 12H].

Proposition 4.3.1 *Let $\varphi : \Re^n \to \Re \cup \{+\infty\}$ be a proper convex and lsc function and let $\sigma > 0$. Consider the following statements:*
(a) φ is strongly convex with parameter σ;
(b) $\langle u-v, x-y\rangle \geq \sigma||x-y||^2$, whenever $u \in \partial\varphi(x), v \in \partial\varphi(y)$; i.e., the map $\partial\varphi$ is strongly monotone.
(c) The inverse map $(\partial\varphi)^{-1}$ is everywhere single valued and Lipschitz continuous with constant σ^{-1}.
(d) φ^* is finite everywhere and differentiable.
Then, $(a) \iff (b) \implies (c) \iff (d)$.

As written above in (4.13), the resemblance between (MDA) and (SANP) is still not obvious. However, we first note that the main step of (SANP) can be written in a more explicit way. Writing down formally the optimality conditions for (4.13), we obtain the following equivalent forms for (SANP):

$$\begin{aligned} 0 &\in t_k f'(x^k) + \nabla\psi(x^{k+1}) - \nabla\psi(x^k) + N_X(x^{k+1}) \\ (\nabla\psi + N_X)(x^{k+1}) &\in \nabla\psi(x^k) - t_k f'(x^k) \\ x^{k+1} &\in (\nabla\psi + N_X)^{-1}(\nabla\psi(x^k) - t_k f'(x^k)). \end{aligned} \tag{4.15}$$

Proposition 4.3.2 *The sequence $\{x^k\} \subseteq X$ generated by (MDA) corresponds exactly to the sequence generated by (SANP).*

Proof. By definition of the conjugate function, one has $\psi^*(z) = \max_{x \in X}\{\langle x, z\rangle - \psi(x)\}$. Writing the optimality conditions for the later we obtain $0 \in z - \nabla\psi(x) - N_X(x)$, which is the same as $x \in (\nabla\psi + N_X)^{-1}(z)$. But, since ψ is strongly convex on X, then using Proposition 4.3.1, ψ^* is finite everywhere and differentiable and one has: $\nabla\psi^* = (\partial\psi)^{-1}$. Thus, the later inclusion is just the equation

$$x = (\nabla\psi + N_X)^{-1}(z) = \nabla\psi^*(z) = (\partial\psi)^{-1}.$$

Using these relations, (SANP) can be written as follows. Let $y^{k+1} := \nabla\psi(x^k) - t_k f'(x^k)$ and set $x^k = \nabla\psi^*(y^k)$. Then, (SANP) given by (4.15) reduces to $x^{k+1} = \nabla\psi^*(y^{k+1})$, which are exactly the iterations generated by (MDA). $\square$

Note that when ψ satisfies (4.14), then (SANP) reduces to: $x^{k+1} = (\nabla\psi)^{-1}(\nabla\psi(x^k) - t_k f'(x^k))$.

4.4 Convergence Analysis

With this interpretation of the (MDA), viewed as SANP, its convergence analysis can be derived in a simple way. The key of the analysis, relies essentially on the following simple identity which appears to be a natural generalization of the quadratic identity valid for the Euclidean norm.

Lemma 4.4.1 *([13]). Let $S \subset \Re^n$ be an open set with closure $\bar{S}$ and let $\psi : \bar{S} \to \Re$ be continuously differentiable on S. Then for any three points $a, b \in S$ and $c \in \bar{S}$ the following identity holds true*

$$\mathrm{B}_\psi(c, a) + \mathrm{B}_\psi(a, b) - \mathrm{B}_\psi(c, b) = \langle \nabla\psi(b) - \nabla\psi(a), c - a\rangle. \tag{4.16}$$

We will need some further notations. Let

$$||z||_* = \max\{\langle x, z\rangle \mid x \in \Re^n,\ \|x\| \le 1\}$$

be the (dual) conjugate norm. The convergence results for the SANP (and hence MDA) are given in the following theorem. We assume that the sequence x^k produced by (SANP) is well defined (see Section 3 for the appropriate condition on ψ).

Theorem 4.4.1 *Suppose that assumption A is satisfied for the convex optimization problem (P). Let $\{x^k\}$ be the sequence generated by (SANP) with starting point $x^1 \in int(X)$. Then, for every $k \geq 1$ one has*

$$(a) \quad \min_{1 \leq s \leq k} f(x^s) - \min_{x \in X} f(x) \leq \frac{B_\psi(x^*, x^1) + 2\sigma^{-1} \sum_{s=1}^k t_s^2 ||f'(x^s)||_*^2}{\sum_{s=1}^k t_s}. \tag{4.17}$$

(b) In particular, the method converges, i.e., $\min_{1 \leq s \leq k} f(x^s) - \min_{x \in X} f(x) \to 0$ *provided that*

$$\sum_s t_s = \infty, \; t_k \to 0, \; k \to \infty.$$

Proof. Let x^* be an optimal solution of (P). Optimality for (4.13) implies:

$$\langle x - x^{k+1}, t_k f'(x^k) + \nabla \psi(x^{k+1}) - \nabla \psi(x^k) \rangle \geq 0, \quad \forall x \in X,$$

and thus in particular for $x = x^*$ we obtain

$$\langle x^* - x^{k+1}, \nabla \psi(x^k) - \nabla \psi(x^{k+1}) - t_k f'(x^k) \rangle \geq 0. \tag{4.18}$$

Using the subgradient inequality for the convex function f one obtains

$$\begin{aligned}
0 \leq t_k(f(x^k) - f(x^*)) &\leq t_k \langle x^k - x^*, f'(x^k) \rangle \\
&= \langle x^* - x^{k+1}, \nabla \psi(x^k) - \nabla \psi(x^{k+1}) - t_k f'(x^k) \rangle \qquad (4.19) \\
&+ \langle x^* - x^{k+1}, \nabla \psi(x^{k+1}) - \nabla \psi(x^k) \rangle \qquad (4.20) \\
&+ \langle x^k - x^{k+1}, t_k f'(x^k) \rangle. \qquad (4.21) \\
&:= s_1 + s_2 + s_3, \qquad (4.22)
\end{aligned}$$

where s_1, s_2, s_3 denotes the three righthand side terms (4.19)-(4.21). Now, we have

$$\begin{aligned}
s_1 &\leq 0, \quad \text{[by (4.18)]}, \\
s_2 &= \mathrm{B}_\psi(x^*, x^k) - \mathrm{B}_\psi(x^*, x^{k+1}) - \mathrm{B}_\psi(x^{k+1}, x^k), \quad \text{[by Lemma 4.4.1]} \\
s_3 &\leq (2\sigma)^{-1} t_k^2 ||f'(x^k)||_*^2 + 2^{-1}\sigma ||x^k - x^{k+1}||^2,
\end{aligned}$$

the later inequality following from $\langle a, b\rangle \leq (2\sigma)^{-1}||a||^2 + 2^{-1}\sigma||b||_*^2, \quad \forall a, b \in \Re^n$. Therefore, recalling that $\mathrm{B}_\psi(\cdot,\cdot)$ is σ-strongly convex, i.e., $-\mathrm{B}_\psi(x^{k+1}, x^k) + 2^{-1}\sigma||x^k - x^{k+1}||^2 \leq 0$, it follows that

$$t_k(f(x^k) - f(x^*)) = s_1 + s_2 + s_3 \leq \mathrm{B}_\psi(x^*, x^k) - \mathrm{B}_\psi(x^*, x^{k+1}) + (2\sigma)^{-1}t_k^2||f'(x^k)||_*^2. \quad (4.23)$$

Summing (4.23) over $k = 1, \ldots, s$ one obtains,

$$\sum_{k=1}^{s} t_k(f(x^k) - f(x^*)) \leq \mathrm{B}_\psi(x^*, x^1) - \mathrm{B}_\psi(x^*, x^{s+1}) + (2\sigma)^{-1}\sum_{k=1}^{s} t_k^2||f'(x^k)||_*^2.$$

Since $\mathrm{B}_\psi(\cdot,\cdot) \geq 0$, it follows from the last inequality that

$$\min_{1\leq s\leq k} f(x^s) - \min_{x\in X} f(x) \leq \frac{B_\psi(x^*, x^1) + (2\sigma)^{-1}\sum_{s=1}^{k} t_s^2||f'(x^s)||_*^2}{\sum_{s=1}^{k} t_s}, \quad (4.24)$$

proving (a). Assuming that $t_k \to 0$ and $\sum t_k = \infty$ as $k \to \infty$, it thus follows from (4.24) that $\min_{1\leq s\leq k} f(x^s) - \min_{x\in X} f(x) \to 0$ as $k \to \infty$, proving (b). □

The above convergence result allows for deriving the best efficiency estimate of the method, by choosing an appropriate step size. The best stepsize is obviously obtained by minimizing the right-hand side of the inequality (4.24), with respect to $t \in \Re^k_{++}$. We need the following technical result.

Proposition 4.4.1 *Given $c > 0$, $b \in \Re^d_{++}$ and D a symmetric positive definite matrix one has*

$$\inf_{z\in\Re^d_{++}} \frac{c + (2\sigma)^{-1}z^T Dz}{b^T z} = \sqrt{\frac{2c}{\sigma b^T D^{-1} b}},$$

with optimal solution $z^* = \sqrt{\frac{2c\sigma}{b^T D^{-1} b}} D^{-1} b$.

Proof. Writing the KKT optimality conditions for the (equivalent) convex problem

$$\inf_{z,u>0} \left\{ \frac{c + (2\sigma)^{-1}z^T Dz}{u} : \; b^T z = u \right\},$$

yields the desired result. □

We can now derive the efficiency estimate for (SANP).

Theorem 4.4.2 *Suppose that assumption A is satisfied for the convex optimization problem (P). Let $\{x^k\}$ be the sequence generated by (SANP) with starting point $x^1 \in int X$. Then, with the stepsizes chosen as*

$$t_k := \frac{\sqrt{2\sigma B_\psi(x^*, x^1)}}{L_f} \frac{1}{\sqrt{k}}, \tag{4.25}$$

one has the following efficiency estimate,

$$\min_{1 \le s \le k} f(x^s) - \min_{x \in X} f(x) \le L_f \sqrt{\frac{2B_\psi(x^*, x^1)}{\sigma}} \frac{1}{\sqrt{k}} \tag{4.26}$$

Proof. The right hand side of (4.24) is upper bounded by

$$\frac{B_\psi(x^*, x^1) + (2\sigma^{-1})L_f^2 \sum_{s=1}^k t_s^2}{\sum_{s=1}^k t_s}, \tag{4.27}$$

Minimizing (4.27) with respect to $t_1, t_2, \ldots, t_k > 0$, and invoking Proposition 4.4.1 with $c := B_\psi(x^*, x^1), b := e = (1, 1, \ldots, 1)^T$ and $D = L_f^2 \cdot I$ where I is the $k \times k$ identity matrix, one gets the desired step size and efficiency estimate. □

Clearly, in order to make this result practical, one has to be able to upper bound the quantity $B_\psi(x^*, x^1)$, which depends on the (obviously unknown) optimal solution x^*, so that the step size and the efficiency estimate can be computed. This can be done by defining for any $y \in \text{int}\, X$ $\gamma[\psi, y] := \max_{x \in X} B_\psi(x, y)$. Thus, one can replace in the estimate (4.17), the quantity $B(x^*, x^1)$ by $\gamma[\psi, x^1]$, provided the later quantity is finite. This is particularly true whenever X is assumed compact. At this point it is informative to compare our results and assumptions with the ones derived in [4] for (MDA). In the later work, the following assumptions were used:

(a) $X \subset \Re^n$ is a compact convex set.

(b) f is a convex Lipschitz continuous function on X, with Lipschitz constant L_f with respect to a given norm $\|\cdot\|$.

(c) $||f'(x)||_* \le L_f$, $\forall x \in X$, where $\|\cdot\|_*$ is the dual norm.

(d) $\psi : X \to \Re$ is strongly convex and continuously differentiable on X

(e) The quantity $\Gamma[\psi] := \max_{x,y \in X} B_\psi(x, y) < \infty$

Note that (c) used in [4] appears to be a redundant assumption since $(b) \Rightarrow (c)$, (cf. 4.2)). When (a) and (d) holds, clearly (e) holds and the results of [4] are recovered through Theorem

4.2. Furthermore, when in (d) we replace the differentiability assumption for ψ on $\text{int}\, X$ and (4.14) holds, then one obtains an interior subgradient algorithm to minimize f over X, where in (4.25) and (4.26) the quantity $B(x^*, x^1)$ is replaced by $\gamma[\psi, x^1] < \infty$ for any $x^1 \in \text{int}\, X$.

4.5 Application: minimization over the unit simplex

As explained before, the key elements needed to implement the MDA and analyze its efficiency are

- To be able to compute the conjugate function ψ^* of ψ efficiently
- To evaluate the strong convexity constant of ψ.
- To upperbound the quantity $B_\psi(x^*, x^1)$.

In this section, we begin by recalling briefly the results of [4], where the authors analyze the problem (P) of minimizing a convex function f over the unit simplex given by

$$\Delta := \{x \in \Re^n : \sum_{j=1}^{n} x_j = 1,\ x \geq 0\},$$

and we introduce a new method for this class of problems.

The MDA_1

Let $\psi_1(x) := 2^{-1}||x||_p^2$ with

$$p := 1 + \frac{1}{\ln n}.$$

The following results were derived in [4]. The number $O(1)$ stands for some positive absolute constant.

- The conjugate of ψ_1 over Δ can be computed, but not explicitly, as it requires the solution of a one dimensional equation. The authors [4] have then considered, the minimization of $f(x)$ over $Y := \{x \in \Re^n : ||x||_p \leq 1\} \supset \Delta$, and in that case ψ_1^* is explicitly computable, see the details there.
- The strong convexity parameter $\sigma = O(1)(\ln n)^{-1}$.
- $\Gamma(\psi_1) \leq O(1)$

Using these results, the following efficiency estimate for MDA_1 holds:

$$\min_{1\leq s\leq k} f(x^s) - \min_{x\in X} f(x) \leq 0(1)\frac{(\ln n)^{1/2}\max_{1\leq s\leq k}||f'(x^s)||_\infty}{\sqrt{k}}. \tag{4.28}$$

In comparison, using $\psi(x) = 2^{-1}||x||_2^2$ on $X = \Delta$, the best efficiency estimate is

$$\min_{1\leq s\leq k} f(x^s) - \min_{x\in X} f(x) \leq O(1)\frac{\max_{1\leq s\leq k}||f'(x^s)||_2}{\sqrt{k}}. \tag{4.29}$$

The ratio of these efficiency estimates is

$$R = 0(1)\frac{L_2}{L_1\sqrt{n}},$$

where $L_1 := \max_{1\leq s\leq k}||f'(x^s)||_\infty$, $L_2 := \max_{1\leq s\leq k}||f'(x^s)||_2$. Using the well known inequality

$$||x||_p \leq ||x||_1 \leq n^{(p-1)/p}||x||_p, \ \ \forall x\in \Re^n$$

it follows that $L_1 \leq L_2 \leq n^{(p-1)/p}L_1$. Therefore, the ratio R is always greater or equal than 1 and can be as large as $0(1)\sqrt{n}$. Thus, the MDA_1 with ψ_1 can outperformed the usual gradient method (obtained with ψ_2 on Δ) by a factor of $(n/\ln n)^{1/2}$, which for large n, can make a huge difference. This method was considered as a "nearly optimal" algorithm for the class of problems under consideration. Further details on these results are developed in [4].

We now propose a different choice for ψ to solve the minimization problem (P) over the unit simplex Δ, which shares the same efficiency estimate. The function appears to be quite "natural" due to the simplex constraints, and is the so-called entropy function defined by:

$$\psi_e(x) = \sum_{j=1}^n x_j \ln x_j \ \text{ if } \ x\in\Delta, \ \ +\infty \ \text{ otherwise}, \tag{4.30}$$

where we adopt the convention $0\ln 0 \equiv 0$.

The entropy function defined on Δ possesses some remarkable properties collected below.

Proposition 4.5.1 *Let $\psi_e : \Delta \to \Re$ be the entropy function defined in (4.30). Then,*
(a) ψ_e is 1-strongly convex over $int\,\Delta$ with respect to the $||\cdot||_1$ norm, i.e.,

$$\langle \nabla\psi_e(x) - \nabla\psi_e(y), x-y\rangle = \sum_{j=1}^n (x_j - y_j)\ln\frac{x_j}{y_j} \geq ||x-y||_1^2, \ \ \forall x,y\in int\,\Delta.$$

(b) The conjugate of ψ_e is the function $\psi_e^* : \Re^n \to \Re$ with $\psi_e^* \in C^\infty(\Re^n)$ and is given explicitly by the formula

$$\psi_e^*(z) = \ln \sum_{j=1}^n e^{z_j},$$

and $\|\nabla \psi_e(x)\| \to \infty$ as $x \to \bar{x} \in \Delta$.
(c) For the choice $x^1 = n^{-1}e$, and $\psi = \psi_e$ one has $B_\psi(x^*, x^1) \leq \ln n, \ \ \forall x^* \in \Delta$

Proof.(a) The strong convexity of ψ_e follows from a fundamental inequality in information Theory. For completeness we give here a different and simple proof. Let $\varphi : \Re_{++} \to \Re$ be defined by

$$\varphi(t) = (t-1)\ln t - 2\frac{(t-1)^2}{t+1}, \ \ \forall t > 0.$$

It is easy to verify that $\varphi(1) = \varphi'(1) = 0$ and that $\varphi''(t) > 0 \ \forall t > 0$. Therefore φ is convex on $(0, \infty)$ and it follows that $\varphi(t) \geq 0, \ \ \forall t > 0$. Therefore, with $t := \frac{x_j}{y_j}$ one has:

$$(x_j - y_j)\ln\frac{x_j}{y_j} \geq 2\frac{(x_j-y_j)^2}{x_j+y_j} \ \ \forall x_j, y_j > 0. \tag{4.31}$$

Using (4.31) one then obtains $\forall x, y \in \operatorname{int}\Delta$:

$$\begin{aligned}\sum_{j=1}^n (x_j - y_j)\ln\frac{x_j}{y_j} \quad &\geq \quad \sum_{j=1}^n \frac{1}{\frac{x_j+y_j}{2}}(x_j-y_j)^2 = \sum_{j=1}^n \frac{x_j+y_j}{2}\frac{(x_j-y_j)^2}{(\frac{x_j+y_j}{2})^2} \\ &\overset{(*)}{\geq} \quad \left(\sum_{j=1}^n \frac{x_j+y_j}{2}\frac{|x_j-y_j|}{\frac{x_j+y_j}{2}}\right)^2 = \left(\sum_{j=1}^n |x_j-y_j|\right)^2 = \|x-y\|_1^2\end{aligned}$$

The inequality (*) is true because $\frac{x+y}{2} \in \Delta$, and from the convexity of the function $h(t) = t^2$ it follows that for every $\alpha \in \Delta$ we have $\left(\sum_{j=1}^n \alpha_j x_j\right)^2 \leq \sum_{j=1}^n \alpha_j x_j^2$.
(b) Using the definition of the conjugate and simple calculus gives the desired results, see also [36].
(c) Substituting $\psi = \psi_e(x) = \sum_{j=1}^n x_j \ln x_j$ in the definition of B_ψ we obtain,

$$B_\psi(x^*, x^1) = \sum_{j=1}^n x_j^* \ln\left(\frac{x_j^*}{x_j^1}\right).$$

Substituting $x_j^1 = n^{-1}$, $\forall j$ we get for all $x^* \in \Delta$:

$$B_\psi(x^*, x^1) = \sum_{j=1}^n x_j^* \ln x_j^* + \ln n \leq \ln n,$$

the last inequality being true since the entropy function is always nonpositive on Δ. $\square$

Remark 5.1. If for some j one has $y_j = 0, x_j > 0$, the left hand side of the strong convexity inequality in (a) is $+\infty$ and there is nothing to prove. Likewise, when we reverse x with y. Thus, recalling that $0 \ln 0 \equiv 0$, it follows that the strong convexity inequality given in (a) remains true for all $x, y \in \Delta$.

Using the entropy function ψ_e in (3.12), we thus obtain a very simple algorithm for minimizing the convex function f over Δ, which is given explicitly by (as opposed to the case when using ψ_1 in (MDA_1).

The Entropic Descent Algorithm (EDA)

Start with $x^1 \in \operatorname{int} \Delta$ and generate for $k = 1, \ldots,$ the sequence $\{x_k\}$ via:

$$x_j^{k+1} = \frac{x_j^k e^{-t_k f_j'(x^k)}}{\sum_{j=1}^n x_j^k e^{-t_k f_j'(x^k)}}, \quad t_k = \frac{\sqrt{2 \ln n}}{L_f} \cdot \frac{1}{\sqrt{k}}.$$

where $f'(x) = (f_1(x)', \ldots, f_n'(x))^T \in \partial f(x)$.

Applying Theorem 4.4.2 and Proposition 4.5.1 we immediately obtain the following efficiency estimate for the EMDA.

Theorem 4.5.1 *Let $\{x^k\}$ be the sequence generated by EMDA with starting point $x^1 = n^{-1}e$. Then, for all $k \geq 1$ one has*

$$\min_{1 \leq s \leq k} f(x^s) - \min_{x \in X} f(x) \leq \sqrt{2 \ln n} \frac{\|f'(x^s)\|_\infty}{\sqrt{k}} \tag{4.32}$$

Thus, the EMDA appears as another useful candidate algorithm for solving large scale convex minimization problems over the unit simplex. Indeed, EMDA shares the same efficiency estimate than the (MDA_1) obtained with ψ_1, but has the advantage of being completely explicit, as opposed to the (MDA_1) which still requires the solution of one dimensional nonlinear equation at each step of the algorithm.

4.6 Concluding remarks and Further Applications

We have presented a new derivation and analysis of mirror descent type algorithms. In its current state, the proposed approach has given rise to new insights on the properties of Mirror descent methods, bringing it in line of subgradient projection algorithms based on Bregman based distance-like functions. This has led us to provide simple proofs for its convergence analysis and to introduce the new algorithm (EMDA) for solving convex problems over the unit simplex, with efficiency estimate mildly dependent on the problem's dimension. Many issues for potential extensions and further analysis include:

- Extension to the cases where $f(x) = \sum_{l=1}^{m} f_l(x)$ which can be derived along the analysis of incremental subgradients techniques [31], [4] and numerical implementations for the corresponding EMDA.

- The choice of other functions ψ can be considered in SANP, (see for example [39], [24]) to produce other interior subgradient (gradient) methods.

- Extension to semidefinite programs, in particular for problems with constraints of the type
$$Z \in S_n, \mathrm{tr}(Z) = 1, Z \succeq 0,$$
and which often arise in relaxations of combinatorial optimization problems. This can be analyzed within the use of a corresponding entropic function defined over the space of positive semidefinite symmetric matrices, see for example [15] and references therein.

Chapter 5

On the Efficiency of the Conditional Gradient Method for solving Convex Linear Systems

5.1 Introduction

This chapter continues our investigation and analysis of gradient based algorithms which can lead to simple algorithms with good convergence rates. In a recent work of Epelman and Freund [19] where the authors proposed and analyzed the complexity of an algorithm for solving a convex feasibility problem in conic linear form:

$$(CF) \begin{cases} Mx = g \\ x \in C \end{cases}$$

where $M : \Re^n \to \Re^m$ is a linear map, $g \in \Re^m$ is a given point and C is a closed convex cone. The main objectives of [19] were to develop and analyze an elementary algorithm which produces reliable solutions of well posed instances of (CF), and to estimate the complexity of this algorithm in terms of a suitable condition number depending on the problem's data (M, g). The heart of the proposed algorithm relies on what the authors of [19] have called a Generalized Von Neumann Algorithm (GVNA, for short), which is devised to solve conic linear systems in compact form, that is, a system of the form (CF) but with the added constraint $u^t x = 1$ where $u \in \Re^n$ is a fixed and given point for which $C \cap \{x : u^T x = 1\}$ is bounded. The resulting algorithm to solve the more general problem (CF) consists essentially of calling several times the algorithm GVNA to a sequence of data instances of very similar forms. This algorithm was justly called *elementary*, since it requires only few relatively simple computations at each iteration (e.g. matrix-vector multiplications) and the solution of one conic section optimization problem per iteration of GVNA. The complexity of the resulting algorithm is expressed in terms of a condition number $\mathcal{C}(M, g)$ developed by Renegar see e.g., [35] and has iteration complexity exponential in $\ln \mathcal{C}(M, g)$. Thus, such an algorithm is not competitive with polynomial-like interior point methods and we can legitimately ask why one could be interested in such methods? There are in fact a number of good reasons for studying alternative and simpler algorithms. Firstly, as pointed out by Epelman and Freund [19], in contrast to the sophisticated interior algorithms which usually requires very heavy computational steps, such an algorithm has the advantage of requiring only simple operations. For very large scale problems, this can be crucial, since the problem structure can be fully exploited; this is not the case in general within the use interior point methods. Secondly, for large scale problems arising from practical models, it is often preferable to obtain quickly a low accuracy solution (as often produced for example by gradient-like methods) rather than an accurate solution requiring very slow and heavy computational time. Thirdly, while polynomial algorithms have theoretical complexity bounds depending on the dimension of the problem, the iteration complexity bounds of more primitive algorithms are often independent or mildly dependent of the problem's dimension, which is a clear advantage for very large scale problems. The recent work of [4] which uses also an elementary algorithm called *Mirror Descent*, demonstrates the later fact and shows that very large scale problems in nuclear medicine can be solved successfully, while interior efficient algorithms are defeated due to the size of these problems attaining millions of variables. Finally, in most practical applications the input data is anyway known only very roughly and thus it is unclear to what extent an accurate solution might be meaningful.

In this chapter we concentrate on the theoretical efficiency analysis of GVNA and ask if a new algorithmic framework has been found? We answer to this and related questions on the iteration complexity bounds of this algorithm.

We consider the convex feasibility problem

$$(I) \begin{cases} Mx = g \\ x \in S \end{cases}$$

where $S \subset \Re^n$ is closed and bounded (not necessarily a cone) and its associated equivalent optimization formulation

$$(OP) \quad \min\{||Mx - g||^2 : \; x \in S\}.$$

First, we show that the basic algorithm GVNA proposed by Epelman and Freund is in fact a special case of the *Conditional Gradient Method* (CGM) when applied to (OP). The conditional gradient method is a feasible direction method and is applicable only when the feasible set S is compact. At each iteration of the algorithm, the *best* feasible direction (with respect to the linear approximation of the function) is chosen and then a line search is performed along that direction. The conditional gradient algorithm has been studied by several researchers, see for example, Bertsekas [6], Dunn [16], Levitin and Polyak [25] and references therein. The convergence of CGM can be established under relatively mild assumptions on the problem's data and is in fact an extension of the Frank and Wolfe algorithm [20] originally devised to minimize a quadratic function over a polyhedron. The advantage of the CGM is its simplicity, in particular when applied to problem of the form (OP), yet the efficiency of CGM is far less attractive. Sublinear rate of convergence of the function values was established by [25]. However, the improvement toward the derivation of a linear rate of convergence of the function values has been established only under very restrictive assumptions. Indeed, in [25] linear rate of convergence is proven under the assumptions that the feasible set is strongly (uniformly) convex and that $\|\nabla f(x)\|$ is bounded below by a positive number, which are severe and rarely met assumptions in most optimization models of interest. Later, in [16], the conditions for deriving linear convergence are given in terms of a function

$$a(\sigma) = \inf_{y \in S, \|y-\xi\| \geq \sigma} \langle \nabla f(\xi), y - \xi \rangle$$

where ξ is an optimal point of the minimization problem $\min_{x \in S} f(x)$. If $a(\sigma) \geq A\sigma^2$ for some $A > 0$ then linear rate of convergence of the function values is proven in [16]. Unfortunately, none of these general results are even applicable to the simple problem of minimizing the convex quadratic function over a the compact convex set S described in (OP), which is the problem we intend to study.

The second and main contribution of this chapter is to establish the linear convergence of CGM when applied to (OP). We prove that under the mild and standard Slater's constraint qualification on the system (I), the CGM converges to a solution of (I) at a linear rate. The rate of convergence depends on the matrix M, the vector g and on the radius of the largest ball contained in the feasible set of (I). This result is then compared with the linear rate of convergence result proven in [19] for the system (CF), i.e., whenever the set S is the particular compact set described by the intersection of a cone C and the constraint $\{x : u^T x = 1\}$. The crucial quantity measuring the rate of convergence in [19] is defined by

$$r(M, g) = \inf\{||g - h|| \ : \ h \in \partial H\}$$

where $H := \{Mx : x \in S\}$. The system (I) is defined to be well-posed whenever $r(M, g) > 0$, while for $r(M, g) = 0$, the problem (I) is feasible but arbitrary small changes in the data (M, g) can yield problems (I) with no feasible solution, and hence can naturally be called "ill-posed". We prove that the assumption $r(M, g) > 0$ made in [19] is in fact equivalent to Slater's condition for the system (I).

The chapter is organized as follows. In Section 2 we recall the conditional gradient method (CGM) and some other preliminary results. In Section 3, we show that the GVNA presented by Epelman and Freund may be viewed as a special case of (CGM) when applied to the optimization problem (OP). We then introduce the new quantity used to measure the rate of convergence of (CGM) that depends on the Slater's point and the matrix M and we prove the announced linear rate of convergence result. We then compare this result with the assumptions and results derived in [19]. In the last section, we apply the results to conic linear systems. Our notations are mostly standard. The Euclidean space is denoted by $\Re^n$ with inner product $\langle \cdot, \cdot \rangle$ and associated induced l_2 norm $|| \cdot ||$. For any matrix A, the norm of A is defined by $||A|| = \max\{||Ax|| : \ ||x|| \leq 1\}$. For any set $S \subset \Re^n$ we denote by $\text{int}(S), \text{cl}(S)$ respectively the interior and closure of S and by $\partial S = \text{cl}(S) \setminus \text{int}(S)$ the boundary of S. For a cone $K \subseteq \Re^n$ the polar cone is $K^* = \{x^* : \langle x, x^* \rangle \leq 0 \ \ \forall x \in K\}$.

5.2 The Conditional Gradient Method and Preliminary Results

In this section we recall the basic steps and convergence results on the conditional gradient method, see e.g. [6] for details and references, as well as some other technical results that will be needed in the rest of this chapter.

Consider the convex optimization problem:

$$(P) \quad \min_{x \in S} f(x)$$

Unless otherwise specified, throughout this section we assume that f is a convex continuously differentiable function on the closed and bounded convex set $S \subset \Re^n$, with Lipschitz gradient ∇f on S, i.e.

$$\exists L > 0 \text{ such that } \|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \ \forall x, y \in S.$$

Conditional Gradient Method-CGM: Start with $x^0 \in S$. Generate the sequence $\{x^k\}$, $\forall k = 1, 2, \ldots$ via the following steps:

1. Compute $p^{k-1} = \text{argmin}\{\langle p - x^{k-1}, \nabla f(x^{k-1})\rangle : p \in S\}$.

2. Stopping Criteria: Let $S(x) := \min_{p \in S}\langle p - x, \nabla f(x)\rangle$. If $S(x^{k-1}) = \langle p^{k-1} - x^{k-1}, \nabla f(x^{k-1})\rangle = 0$ STOP. Else, goto step 3.

3. Line search: Compute $\lambda^{k-1} = \underset{\lambda \in [0,1]}{\text{argmin}}\, f(x^{k-1} + \lambda(p^{k-1} - x^{k-1}))$.
   Update $x^k = x^{k-1} + \lambda^{k-1}(p^{k-1} - x^{k-1})$.

4. Set $k \leftarrow k + 1$. Goto step 1.

Note on stopping Criteria: Let $\min_{x \in S} f(x) = f^*$. By the definition of f^* we have that $f(x^{k-1}) \geq f^*$ for every k. On the other hand,

$$\begin{aligned}
f^* - f(x^{k-1}) &= \min_{u \in S}(f(u) - f(x^{k-1})) \\
&\overset{\text{f is convex}}{\geq} \min_{u \in S}\langle u - x^{k-1}, \nabla f(x^{k-1})\rangle \\
&= \langle p^{k-1} - x^{k-1}, \nabla f(x^{k-1})\rangle \\
&= S(x^{k-1}).
\end{aligned}$$

Therefore, we conclude that:

$$f(x^{k-1}) \geq f^* \geq f(x^{k-1}) + S(x^{k-1}). \tag{5.1}$$

Thus one always has $S(x^{k-1}) \leq 0$ and $S(x^{k-1}) = 0$ if and only if x^{k-1} is an optimal solution of problem (P).

The bulk of computation in the CGM are in Step 1 and Step 3. The latter requires to find a step size λ^{k-1} by solving the following one dimensional problem. Given x, p in S find λ^* solution of

$$\min_{\lambda\in[0,1]} f(x+\lambda(p-x)).$$

This step can in fact be computed analytically by using an appropriate quadratic approximation of the function f. Such approximation exists since we assumed here that ∇f is Lipschitz continuous, (see Appendix). Thus the only remaining computational step in CGM is step 1 which in many applications might be very easy to solve. For example, whenever S is a simplex, in which case the solution is immediate or whenever the constraints set in a polyhedron, namely we have to solve a linear programming problem. Thus, CGM is an attractive simple algorithm whenever step 1 can be performed efficiently. As in [19] we suppose that the solution of step 1 can be achieved with low cost complexity, for otherwise, the algorithm looses much of its potential efficiency and practical usefulness. The main results on the conditional gradient method without any more assumptions (except for the ones we have already assumed) are summarized in the following proposition. Since many of these results have been scattered in several references in the literature (see e.g., [6],[16],[25]), for convenience and the interested reader on general results for CGM, we have given in an appendix compact proofs.

Proposition 5.2.1 *Let $f \in C^1(\Re^n)$ be a convex function with Lipschitz continuous gradient and Lipschitz constant $L > 0$. Let $\{x^k\}$ be a sequence generated by the conditional gradient method. Then,*

(i) $x^k \in S$, *the sequence* $\{f(x^k)\}$ *is monotone decreasing and every limit point of the sequence* $\{x^k\}$ *solves* $\min_{x\in S} f(x)$.

(ii) $\lim_{n\to\infty} f(x^n) = f^* = \min_{x\in S} f(x)$.

(iii) *There exists a positive constant c, which depends on L and the diameter* $\delta_S := \sup_{x,y\in S} ||x-y||$ *such that* $f(x^n) - f^* \leq \frac{c}{n}$.

Note that convexity is not needed to derive the first statement of the proposition. In that case of course, the statement on the sequence $\{x^k\}$ is that every limit point is a stationary point, i.e., it satisfies the necessary local optimality conditions for problem (P).

The sublinear rate of convergence for function values cannot be improved, see e.g., Cannon and Cullum [10], unless, as we already mentioned in the introduction, we make some further stronger assumptions on the constraints set S, and this even if we assume that the objective function is convex quadratic which is our problem of interest in this chapter. The next section develops the required analysis to achieve a linear rate of convergence of CGM for such class of problems. We end this section with two elementary and well known results on the rate of convergence of nonnegative sequences of real numbers and which will be useful to us.

Lemma 5.2.1 *Let $\{a_k\}_{k=0}^m$ be a nonnegative sequence of real numbers.*
(i) Sublinear rate: If $\{a_k\}$ is such that $a_{k-1} - a_k \geq \gamma a_{k-1}^2$ for some $\gamma > 0$ and for any $k = 1, \ldots, m$, then

$$a_m \leq \frac{a_0}{1 + m\beta a_0} < (\gamma m)^{-1}.$$

(ii) Linear Rate: If $\{a_k\}$ is such that $a_{k-1} - a_k \geq \gamma_k a_{k-1}$ for some $\gamma_k \geq 0$, $\forall\ k = 1, \ldots, m$, then

$$a_m \leq a_0 e^{-\sum_{k=1}^m \gamma_k}.$$

5.3 Linear rate of convergence analysis of CGM

We consider the problem of finding a point satisfying:

$$(I) \begin{cases} Mx = g \\ x \in S \end{cases}$$

where S is a closed convex and bounded set. For ease of comparison with the results derived by Epelman and Freund, [19], we tried to keep as close as possible the notations used in that paper.

To solve this problem we consider the equivalent optimization problem:

$$(OP) \quad v^* := \min_{x \in S} \frac{1}{2} \|Mx - g\|^2$$

Clearly, if (I) is feasible the optimal function value of (OP) is $v^* = 0$, otherwise one has $v^* > 0$.

We will apply the conditional gradient method CGM to (OP). The line search applied to the case of a convex quadratic objective is simple as it has an analytic expression (as it obviously does not require the use of a quadratic approximation of the objective). Indeed, with $f(x) = \frac{1}{2}\|Mx - g\|^2$ one has $\nabla f(x) = M^T(Mx - g)$ and we immediately obtain the following identity: for any $x, p \in \Re^n$ and any $\lambda \in \Re$:

$$g(\lambda) := f(x + \lambda(x - p)) = f(x) + \lambda\langle p - x, \nabla f(x)\rangle + \frac{1}{2}\lambda^2\|M(x - p)\|^2. \tag{5.2}$$

In order to simplify the expressions we use the following notations:

$$\begin{aligned} v_{k-1} &= g - Mx^{k-1}, \\ w_{k-1} &= g - Mp^{k-1}. \end{aligned}$$

Using the identity (5.2) at the points $x = x^{k-1}, p = p^{k-1}$ and denoting by $g_k(\lambda)$ the resulting function, the step size computation in the line search of Step 3 of CGM, consists of finding $\lambda^* = \underset{\lambda\in[0,1]}{\operatorname{argmin}}\, g_k(\lambda)$. This is a simple one dimensional convex quadratic minimization problem over the interval $[0, 1]$. One has $g_k'(\lambda) = 0$ if and only if:

$$\begin{aligned} \lambda &= -\frac{\langle p^{k-1} - x^{k-1}, \nabla f(x^{k-1})\rangle}{\|M(p^{k-1} - x^{k-1})\|^2} \\ &= -\frac{\langle p^{k-1} - x^{k-1}, M^T M x^{k-1} - M^T g\rangle}{\|M(p^{k-1} - x^{k-1})\|^2} \\ &= \frac{\langle M(p^{k-1} - x^{k-1}), g - Mx^{k-1}\rangle}{\|M(p^{k-1} - x^{k-1})\|^2} \\ &= \frac{\langle (g - Mx^{k-1}) - (g - Mp^{k-1}), g - Mx^{k-1}\rangle}{\|(g - Mx^{k-1}) - (g - Mp^{k-1})\|^2} \\ &= \frac{\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle}{\|v_{k-1} - w_{k-1}\|^2}. \end{aligned}$$

Thus,

$$\lambda^* = \underset{\lambda\in[0,1]}{\operatorname{argmin}}\, f(x^{k-1} + \lambda(p^{k-1} - x^{k-1})) \tag{5.3}$$

$$= \begin{cases} \frac{\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle}{\|v_{k-1} - w_{k-1}\|^2} & \text{if } \frac{\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle}{\|v_{k-1} - w_{k-1}\|^2} < 1 \\ 1 & \text{if } \frac{\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle}{\|v_{k-1} - w_{k-1}\|^2} \geq 1 \end{cases} \tag{5.4}$$

Now, the main computational step of the conditional gradient method given in CGM-Step 1 is $p^{k-1} = \underset{p\in S}{\operatorname{argmin}}\{\langle p - x^{k-1}, \nabla f(x^{k-1})\rangle\}$. Substituting the expression of the gradient of f: $\nabla f(x) = M^T(Mx - g)$ we have:

$$\begin{aligned}
p^{k-1} &= \underset{p\in S}{\operatorname{argmin}}\langle p - x^{k-1}, M^T(Mx^{k-1} - g)\rangle \\
&= \underset{p\in S}{\operatorname{argmin}}\langle v_{k-1}, M(x^{k-1} - p)\rangle \\
&= \underset{p\in S}{\operatorname{argmin}}\langle v_{k-1}, g - Mp\rangle + \underbrace{\langle v_{k-1}, Mx^{k-1} - g\rangle}_{-\|v_{k-1}\|^2} \\
&= \underset{p\in S}{\operatorname{argmin}}\langle v_{k-1}, g - Mp\rangle. \qquad (5.5)
\end{aligned}$$

To summarize, the basic steps of the conditional gradient method for the quadratic problem (OP) has the following form:

<u>The conditional gradient method applied to (OP): CGM-OP</u>

Initialization step: Start with an arbitrary $x^0 \in S$

General step: Solve: $p^{k-1} = \underset{p\in S}{\operatorname{argmin}}\langle v_{k-1}, g - Mp\rangle \quad k = 1, 2, \ldots$

and compute: $\lambda^{k-1} = \begin{cases} \frac{\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle}{\|v_{k-1} - w_{k-1}\|^2} & \text{if } \frac{\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle}{\|v_{k-1} - w_{k-1}\|^2} < 1 \\ 1 & \text{if } \frac{\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle}{\|v_{k-1} - w_{k-1}\|^2} \geq 1 \end{cases}$.

Update: $x^k = x^{k-1} + \lambda^{k-1}(p^{k-1} - x^{k-1})$

The stopping function $S(\cdot)$ defined in step 2 of CGM can be expressed as follows:

$$\begin{aligned}
S(x^{k-1}) &= \langle p^{k-1} - x^{k-1}, \nabla f(x^{k-1})\rangle \\
&= \langle g - Mx^{k-1}, Mx^{k-1} - g + g - Mp^{k-1}\rangle = \langle v_{k-1}, w_{k-1}\rangle - \|v_{k-1}\|^2.
\end{aligned}$$

Furthermore, using the right hand side of the inequality (5.1), since here $f^* = v^* = 0$ and $f(x^{k-1}) = \frac{1}{2}\|v_{k-1}\|^2$ we obtain

$$\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle \geq \frac{1}{2}\|v_{k-1}\|^2. \qquad (5.6)$$

The algorithm CGM-OP will produce an optimal solution at iteration k whenever $\langle v_{k-1}, w_{k-1}\rangle = \|v_{k-1}\|^2$. If $v_{k-1} \neq 0$, i.e., $\langle v_{k-1}, w_{k-1}\rangle > 0$, then CGM-OP will stop with an infeasible solution of (OP) (which means that the original problem (I) is infeasible).

Applying CGM-OP to the special case when $S = \{x \in C : u^T x = 1\}$, where C is closed convex cone and S is bounded, the above development shows that we have precisely recovered the algorithm GVNA proposed in [19, p.461-462].

As a byproduct of this equivalence between CGM and GVNA we can thus derive as an immediate consequence of Proposition 5.2.1 that $\{\|v_k\|^2\}$ converges to 0 at a sublinear rate. i.e, $\exists \eta > 0 : \|v_k\| \leq \frac{\eta}{\sqrt{k}}$. Note that in the quadratic case we don't need to use the quadratic approximation in the line search (see Appendix) and thus we can write η explicitly in terms of (M, g, u). This provides an alternative simple proof to the results derived in Epelman and Freund [19, Lemma 1 and 3].

We are now going to prove our main result concerning the efficiency of CGM-OP. Our approach is patterned after the key proposition derived in [19, Proposition 6] for establishing linear convergence of GVNA, but introduces a new idea that leads to a different way of measuring the rate.

We denote the distance from a point $b \in \Re^n$ to the boundary ∂S of a closed convex set of $\Re^n$ by

$$d(b, \partial S) := \inf\{||z - b|| : \ z \in \partial S\}.$$

One thus have

$$d(b, \partial S) = \begin{cases} \min\{||z - b|| : \ z \in S\} & \text{if } b \notin S \\ \max\{r : B(b, r) \subset S\} & \text{if } b \in S \end{cases}$$

where $B(b, r)$ is the ball centered at b with radius r.
We assume that the row vectors of the matrix M are linearly independent. This implies that the Gram matrix MM^T is positive definite and thus has an inverse. Note that this assumption is without loss of generality since it simply means that there are no redundant equations in the system $Mx = g$.

Proposition 5.3.1 *Let $\{x^k\}$ be the sequence generated by CGM-OP, let p^k be the direction computed in the general step at iteration $k+1$ and let $v_k = g - Mx^k$. Suppose that the Slater condition for the convex linear system (I) is satisfied, i.e.,*

$$\exists \ \hat{x} \in int(S) \ \text{ such that } \ M\hat{x} = g.$$

Then,

$$\langle v_k, g - Mp^k \rangle + R_S(\hat{x}, M)\|v_k\| \leq 0, \tag{5.7}$$

where

$$R_S(\hat{x}, M) = \frac{d(\hat{x}, \partial S)}{\sqrt{\|(MM^T)^{-1}\|}}.$$

Proof: First, note that one has:

$$v_k = g - Mx^k = M\hat{x} - Mx^k = M(\hat{x} - x^k) := Md.$$

Thus, the system $Md = v_k$ has at least one solution. Among all possible solutions, we pick the one with minimum norm, that is we are interested in finding d^*, which solves the following optimization problem:

$$d^* = \min_{Md=v_k} \|d\|^2.$$

It is easy to see that the optimum of this minimization problem is attained at $d^* = M^T(MM^T)^{-1}v_k$ and $\|d^*\|^2 = v_k^T(MM^T)^{-1}v_k$. As a consequence,

$$\begin{aligned} \|d^*\| &= \sqrt{v_k^T(MM^T)^{-1}v_k} \\ &\leq \sqrt{\|(MM^T)^{-1}\| \cdot \|v_k\|^2} \\ &= \sqrt{\|(MM^T)^{-1}\|} \cdot \|v_k\|. \end{aligned} \tag{5.8}$$

Define $s := d(\hat{x}, \partial S)$. Since we assumed $\hat{x} \in \text{int}(S)$ one has $s > 0$. From the definition of s it follows that: $x = \hat{x} + s\frac{d^*}{\|d^*\|} \in S$, and hence,

$$\begin{aligned} Mx &= M\left(\hat{x} + s\frac{d^*}{\|d^*\|}\right) = M\hat{x} + s\frac{Md^*}{\|d^*\|} \\ &= g + s\frac{v_k}{\|d^*\|}. \end{aligned}$$

Thus, one has $g - Mx = -s\frac{v_k}{\|d^*\|}$ and therefore using (5.8) it follows that,

$$\langle v_k, g - Mp^k\rangle \overset{(5.5)}{\leq} \langle v_k, g - Mx\rangle = -\frac{s\|v_k\|^2}{\|d^*\|} \leq -\frac{s}{\sqrt{\|(MM^T)^{-1}\|}}\|v_k\|$$

proving the desired result. □

Remark 3.1 Proposition 5.3.1 can be easily extended to sets of the form $S = T \cap \{x : Ax = b\}$ where T is a closed convex set. Under the slater condition (i.e., there exists

$\hat{x} \in \text{int}(T)$ such that $M\hat{x} = g$ and $A\hat{x} = b$) (5.7) is satisfied but here $R_S(\hat{x}, M)$ is defined by $R_S(\hat{x}, M) = \frac{d(\hat{x}, \partial S)}{\sqrt{\|(\tilde{M}\tilde{M}^T)^{-1}\|}}$ where $\tilde{M} = \begin{pmatrix} M \\ A \end{pmatrix}$.

Proposition 5.3.1 can now be used to prove the following linear convergence rate for the conditional gradient method.

Proposition 5.3.2 *Suppose that the slater condition is satisfied at the point $\hat{x}$ for the system (I) and let ρ_S be the radius of a ball containing the compact set S. Then, the conditional gradient method has a linear rate of convergence:*

$$\|v_k\| \leq (1 - q^2)^{\frac{1}{2}} \|v_{k-1}\| \quad \forall k = 1, 2, \ldots.$$

where $q = \frac{R_S(\hat{x}, M)}{\|g\| + \rho_S \|M\|}$. Equivalently, this means that

$$||v_k|| \leq ||v_0|| e^{-\frac{kq}{2}}, \;\; \forall k = 1, \ldots$$

Proof: First recall that from the CGM-OP one has $w_{k-1} = g - Mp^{k-1}, v_{k-1} = g - Mx^{k-1}$ and $x^k = x^{k-1} + \lambda^*(p^{k-1} - x^{k-1})$ where $\lambda^* = \min\left\{\frac{\langle v_{k-1}, v_{k-1} - w_{k-1}\rangle}{\|v_{k-1} - w_{k-1}\|^2}, 1\right\}$. Then,

$$\begin{aligned} \|v_k\|^2 &= \|g - Mx^k\|^2 \\ &= \|g - M(x^{k-1} + \lambda^*(p^{k-1} - x^{k-1}))\|^2 \\ &= \|(1 - \lambda^*)(g - Mx^{k-1}) + \lambda^*(g - Mp^{k-1})\|^2 \\ &= \|(1 - \lambda^*)v_k + \lambda^* w_k\|^2 \\ &= \|v_k + \lambda^*(w_k - v_k)\|^2 \\ &= (\lambda^*)^2 \|v_k - w_k\|^2 + 2\lambda^* \langle v_k, w_k - v_k \rangle + \|v_k\|^2 \end{aligned} \tag{5.9}$$

By (5.7) we have that $\langle v_{k-1}, w_{k-1} \rangle \leq 0$. Therefore,

$$\begin{aligned} \langle v_{k-1}, v_{k-1} - w_{k-1} \rangle &= \|v_{k-1}\|^2 - \langle v_{k-1}, w_{k-1} \rangle \\ &\overset{\langle v_{k-1}, w_{k-1} \rangle \leq 0}{\leq} \|v_{k-1}\|^2 - \langle v_{k-1}, w_{k-1} \rangle + (\|w_{k-1}\|^2 - \langle v_{k-1}, w_{k-1} \rangle) \\ &= \|v_{k-1} - w_{k-1}\|^2, \end{aligned}$$

and hence $\frac{\langle v_{k-1}, v_{k-1} - w_{k-1} \rangle}{\|v_{k-1} - w_{k-1}\|^2} \leq 1$ which implies that $\lambda^* = \frac{\langle v_{k-1}, v_{k-1} - w_{k-1} \rangle}{\|v_{k-1} - w_{k-1}\|^2}$. Substituting this value of λ^* in (5.9) yields:

$$\|v_k\|^2 = \frac{\|v_{k-1}\|^2 \|w_{k-1}\|^2 - \langle v_{k-1}, w_{k-1} \rangle^2}{\|v_{k-1} - w_{k-1}\|^2}. \tag{5.10}$$

Now, since S is a bounded set, it is contained in some ball $B(0, \rho_S)$ and thus one has $\|w_{k-1}\| = \|g - Mp^{k-1}\| \leq \|g\| + \|M\|\rho_S$. Moreover, note that $\|v_{k-1} - w_{k-1}\|^2 = \|v_{k-1}\|^2 - 2\langle v_{k-1}, w_{k-1}\rangle + \|w_{k-1}\|^2 \geq \|w_{k-1}\|^2$. Therefore we obtain from (5.10) (we set here $R := R_S(\hat{x}, M)$):

$$\begin{aligned}
\|v_k\|^2 &= \frac{\|v_{k-1}\|^2\|w_{k-1}\|^2 - \langle v_{k-1}, w_{k-1}\rangle^2}{\|v_{k-1} - w_{k-1}\|^2} \\
&\overset{(5.7)}{\leq} \frac{\|v_{k-1}\|^2(\|w_{k-1}\|^2 - R^2)}{\|v_{k-1} - w_{k-1}\|^2} \\
&\overset{\|v_{k-1}-w_{k-1}\|^2 \geq \|w_{k-1}\|^2}{\leq} \frac{\|v_{k-1}\|^2(\|w_{k-1}\|^2 - R^2)}{\|w_{k-1}\|^2} \\
&= \left(1 - \frac{R^2}{\|w_{k-1}\|^2}\right)\|v_{k-1}\|^2 \\
&\leq \left(1 - \left(\frac{R}{\|g\| + \rho_S\|M\|}\right)^2\right)\|v_{k-1}\|^2,
\end{aligned}$$

proving the first statement of the Proposition. From the last inequality it follows that $\|v_{k-1}\|^2 - \|v_k\|^2 \geq \frac{R}{||g||+\rho_S||M||}\|v_{k-1}\|^2$. Invoking Lemma B.0.4(ii) to the nonnegative sequence $a_k := ||v_k||$ the equivalent part of the Proposition is obtained.

□

We can apply the above result to find an approximate solution of (I) with fixed accuracy. Given $\varepsilon > 0$, an ε-solution of (I), namely a point $x \in S$ such that $||Mx - g|| \leq \varepsilon$, is obtained in no more that

$$k = \left\lceil 2\frac{\|g\| + \rho_S\|M\|}{R_S(\hat{x}, M)} \ln\left(\frac{\|g - Mx^0\|}{\epsilon}\right)\right\rceil$$

iterations of CGM.

It is interesting to compare the key quantity $R_S(\hat{x}, M)$ involved in the linear rate of convergence result derived in Proposition 3.1 with the one derived in [19]. The linear convergence rate derived there was obtained in terms of the quantity $r(M, g)$ defined by:

$$r(M, g) = \inf\{\|g - h\| : h \in \partial H\}, \tag{5.11}$$

where,

$$H = \{Mx : x \in S\} = M(S).$$

Recall, that the authors [19] have studied the algorithm GVNA only for the case $S = C \cap \{x : u^T x = 1\}$ where C is a closed convex cone with nonempty interior, and is pointed, i.e., contains no lines.

The sufficient condition to derive linear convergence for a feasible problem (I) in terms of the analysis derived in [19] is:

$$r(M, g) > 0. \tag{5.12}$$

It is not an easy task to compare the *values* of the quantities $r(M, g)$ and $R_S(\hat{x}, M)$. The former depends on the data (M, g) and the set S, and requires to find the solution of the optimization problem (5.11). The latter depends on the choice of a Slater point $\hat{x}$, the matrix M and the radius of the largest ball contained in S. Yet, and interestingly enough, it turns out that the condition (5.12) imposed in the analysis of [19] is in fact *equivalent* to the Slater condition.

Proposition 5.3.3 *Suppose that the convex feasibility problem (I) is feasible. Then* $r(M, g) > 0$ *if and only if there exists* $\bar{x} \in int(S)$ *such that* $M\bar{x} = g$.

Proof: Under the given feasibility assumption, problem (I) has a solution and thus we have that $g \in H$. Now, $r(M, g) > 0$ is equivalent to $g \notin \partial H$ and thus $g \in \operatorname{int} H$. Using relative interior calculus ([36, Proposition 6.6, p.48]) and the fact that the relative interior and the interior are the same in this case one has $\operatorname{int}(H) = \operatorname{int} M(S) = M(\operatorname{int}(S))$. Therefore, $g \in \operatorname{int} H$ translates to: there exists $\bar{x} \in \operatorname{int}(S)$ such that $g = M\bar{x}$.□

5.4 Application to a Class of Conic Problems

In this section we consider the conic feasibility problem: find x such that

$$(CFP) \qquad \begin{array}{l} Ax = b \\ x \in C \end{array}$$

where C is a closed convex cone. We assume that C is pointed (i.e. has no lines) and that $T = \{x : Ax = b, x \in C\}$ is bounded. Since here C is obviously not compact (unless $C = \{0\}$), a direct application of CGM-OP is not possible.

We show below that (CFP) can be rewritten in such a way that our results can be applied. For another line of analysis with more detailed and precise results to resolve a general conic linear system, see Epelman-Freund [19, Sections 4–5]).

First, notice that the fact that T is bounded implies that the following system

$$\begin{aligned} &Ax = 0 \\ &x \in C \end{aligned} \tag{5.13}$$

does not have a nonzero solution. We add a variable θ and consider the following conic feasibility problem:

$$(\widetilde{CFP}) \quad \begin{aligned} &Ax - \theta b = 0 \\ &x \in C \\ &\theta \geq 0 \end{aligned}$$

Clearly, any non zero solution of $(\widetilde{CFP})$ satisfies $\theta > 0$ because (5.13) does not have a nonzero solution. Since C is a cone, any solution $\begin{pmatrix} y \\ \theta \end{pmatrix}$ of $(\widetilde{CFP})$ induces a solution to (CFP) via the substitution $\theta^{-1}y$.

Now since C is a closed convex pointed cone, its polar C^* has a nonempty interior, i.e., $\text{int}(C^*) \neq \emptyset$. Take $u \in -\text{int}(C^*)$ and consider the following feasibility problem:

$$(\widehat{CFP}) \quad \begin{aligned} &Ax - \theta b = 0 \\ &x \in C \\ &\theta \geq 0 \\ &u^T x + \theta = 1. \end{aligned}$$

Denoting:

$$\tilde{C} = C \times \Re_+, \quad \tilde{u} = \begin{pmatrix} u \\ 1 \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} x \\ \theta \end{pmatrix}, \quad M = (A, -b),$$

we then obtain the following formulation for $(\widehat{CFP})$:

$$(\widehat{CFP}) \quad \begin{aligned} &M\tilde{x} = 0 \\ &\tilde{x} \in \tilde{C} \\ &\tilde{u}^T \tilde{x} = 1. \end{aligned}$$

Notice that $(\widehat{CFP})$ is a nonempty set if and only if $(\widetilde{CFP})$ is a nonempty set. $\tilde{C}$ is a closed convex pointed cone and $\tilde{u} \in \text{int}(\tilde{C}^*)$. The next lemma states that $S = \{\tilde{x} : \tilde{x} \in \tilde{C}, \tilde{u}^T\tilde{x} = 1\}$ is bounded.

Lemma 5.4.1 *Let C be a closed convex pointed cone and let $u \in -\,int(C^*)$. Then the set $S = \{\tilde{x} : \tilde{x} \in \tilde{C}, \tilde{u}^T\tilde{x} = 1\}$ is closed and bounded.*

Proof: In order to prove the boundedness of S it is necessary and sufficient to prove that $\{d : d \in C, u^T d = 0\} = \{0\}$ (i.e., the recession cone of S is the Singleton $\{0\}$). Suppose on the contrary that there exists $d \neq 0$ such that $d \in C, u^T d = 0$. Since $u \in -\,\text{int}(C^*)$ then there exists $\epsilon > 0$ such that $u - \epsilon d \in -C^*$. But then $(u - \epsilon d)^T d = u^T d - \|d\|^2 = -\|d\|^2 < 0$ and as a consequence $d \notin C$ which is a contradiction. $\square$

As a result of Lemma 4.1, problem (CFP) can be solve by applying the conditional gradient method CGM-OP for the optimization problem:

$$\min_{\tilde{x} \in S} \|M\tilde{x}\|^2.$$

5.5 Appendix: Proofs of Basic Convergence Results for CGM

We outline here the results needed to prove Proposition 5.2.1 when minimizing a convex continuously differentiable function with Lipschitz gradient over a compact convex set S. In what follows $x^k \in S$ is the sequence produced by CGM as outlined in Section 2.

Lemma 5.5.1

$$f(x^{k-1}) - f(x^k) \geq \frac{1}{2}\frac{S^2(x^{k-1})}{\|x^{k-1} - p^{k-1}\|^2} \cdot \min\left\{\frac{1}{L}, \frac{\|x^{k-1} - p^{k-1}\|}{\|\nabla f(x^{k-1})\|}\right\}.$$

Proof. The proof follows by applying the descent Lemma [6, Proposition A.24] which gives $\forall \lambda \in [0,1]$,

$$f(x^{k-1}) - f(x^{k-1} + \lambda(p^{k-1} - x^{k-1})) \geq \lambda\langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1})\rangle - \frac{L}{2}\lambda^2\|x^{k-1} - p^{k-1}\|^2. \quad (5.14)$$

The later inequality is in particular true for

$$\lambda^* = \underset{0\le\lambda\le 1}{\operatorname{argmax}} \left\{ \lambda\alpha - \frac{1}{2}\beta\lambda^2 \right\} = \begin{cases} 1 & \text{if } 1 \le \frac{\alpha}{\beta} \\ \frac{\alpha}{\beta} & 1 > \frac{\alpha}{\beta} \end{cases} ,$$

where,

$$\alpha = \langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle \ \ , \ \ \beta = \|x^{k-1} - p^{k-1}\|^2.$$

Thus, the step size in Step 3 of CGM can be taken as $\lambda^* = \min\left\{1, \frac{\langle x^{k-1}-p^{k-1}, \nabla f(x^{k-1})\rangle}{L\|x^{k-1}-p^{k-1}\|^2}\right\}$. If $\lambda^* = 1$,then in this case,

$$1 \le \frac{\langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle}{L\|x^{k-1} - p^{k-1}\|^2}. \tag{5.15}$$

and therefore one obtains:

$$\begin{aligned} f(x^{k-1}) - f(x^k) &\ge \langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle - \frac{L}{2}\|x^{k-1} - p^{k-1}\|^2 \\ &\overset{(5.15)}{\ge} \frac{1}{2}\langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle \\ &\overset{a^T b \ge \frac{(a^T b)^2}{\|a\|\cdot\|b\|}}{\ge} \frac{1}{2}\frac{\langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle^2}{\|x^{k-1} - p^{k-1}\| \cdot \|\nabla f(x^{k-1})\|} \\ &= \frac{1}{2}\frac{S^2(x^{k-1})}{\|x^{k-1} - p^{k-1}\| \cdot \|\nabla f(x^{k-1})\|}. \end{aligned} \tag{5.16}$$

In a similar way, in the other case $\lambda^* = \frac{\langle x^{k-1}-p^{k-1}, \nabla f(x^{k-1})\rangle}{L\|x^{k-1}-p^{k-1}\|^2.}$ one has,

$$1 > \frac{\langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle}{L\|x^{k-1} - p^{k-1}\|^2}. \tag{5.17}$$

and we obtain

$$\begin{aligned} f(x^{k-1}) - f(x^k) &\ge \frac{\langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle^2}{L\|x^{k-1} - p^{k-1}\|^2} - \frac{L}{2}\frac{\langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle^2}{L^2\|x^{k-1} - p^{k-1}\|^2} \\ &= \frac{1}{2}\frac{\langle x^{k-1} - p^{k-1}, \nabla f(x^{k-1}) \rangle^2}{L\|x^{k-1} - p^{k-1}\|^2} \\ &= \frac{1}{2}\frac{S^2(x^{k-1})}{L\|x^{k-1} - p^{k-1}\|^2}. \end{aligned} \tag{5.18}$$

□

Lemma 5.5.2 *For any $f : \Re^n \to \Re$ which is continuously differentiable with Lipschitzian gradient and Lipschitz constant L over $S \subseteq \Re^n$ closed, bounded and convex one has:*
(i) $\sup_{x \in S} \|\nabla f(x)\| \leq c_2$ *for some constant c_2.*
(ii) $\forall x \in S \quad \|p(x) - x\| \leq c_1$ *for some $c_1 > 0$ where*

$$p(x) := \underset{p \in S}{argmin} \langle p - x, \nabla f(x) \rangle.$$

Proof.(i) Since the gradient of f is Lipschitz, we obtain for any $x, y \in S : \quad ||\nabla f(x)\| = \|\nabla f(x) - \nabla f(y) + \nabla f(y)\| \leq L\|x - y\| + \|\nabla f(y)\| \leq L\delta_S + \|\nabla f(y)\|$, where, $\delta_S = \sup_{x,y \in S} \|x - y\|$ and (i) is proved with $c_2 := L\delta + \|\nabla f(y)\|$.
(ii) Since S is compact and for all $x \in S$ we have that $p(x) \in S$ thus there is a constant $c_1 > 0$ such that $\|p(x) - x\| \leq c_1 \quad \forall x \in S$.

□

Applying the results of the previous lemma to lemma 5.5.1 we obtain:

Proposition 5.5.1

$$f(x^{k-1}) - f(x^k) \geq CS^2(x^{k-1}), \quad \forall k = 1, \ldots \tag{5.19}$$

with, $C = \min\left\{\frac{1}{2c_1c_2}, \frac{1}{2Lc_1^2}\right\} > 0$.

Proof: By lemma 5.5.1 we have that:

$$\begin{aligned} f(x^{k-1}) - f(x^k) &\geq \frac{1}{2}\frac{S^2(x^{k-1})}{\|x^{k-1} - p^{k-1}\|^2} \cdot \min\left\{\frac{1}{L}, \frac{\|x^{k-1} - p^{k-1}\|}{\|\nabla f(x^{k-1})\|}\right\} \\ &\overset{\text{lemma 5.5.2}}{\geq} \left\{\frac{1}{2c_1c_2}, \frac{1}{2Lc_1^2}\right\} S^2(x^{k-1}) \end{aligned}$$

□

Before proving Proposition 5.2.1, the next result shows that every limit point of CGM is a stationary point of $\min_{x \in S} f(x)$. No convexity assumption is needed.

Proposition 5.5.2 *Let $f : \Re^n \to \Re$ be a continuously differentiable function with Lipschitz gradient and Lipschitz constant $L > 0$ over the compact set S. Let $\{x^k\}$ be the sequence generated by CGM then,*
(i) $x^k \in S$ and $\{f(x^k)\}$ is a monotone decreasing sequence.
(ii)$S(x^k) \to 0$ as $k \to \infty$.
(iii) Every limit point x^* of $\{x^k\}$ is a stationary point. i.e., it satisfies the necessary conditions for local minimum:

$$\langle \nabla f(x^*), x - x^* \rangle \geq .0$$

Proof. (i) follows immediately from (5.19), while (ii) is a consequence of

$$\sum_{k=1}^{n} S^2(x^{k-1}) \overset{(5.19)}{\leq} C^{-1}(f(x^0) - f(x^n)) \leq C^{-1}(f(x^0) - f(x^*)) < \infty.$$

which implies that $S(x^k) \to 0$ as $k \to \infty$. To show (iii), suppose first that there is a k such that $S(x^{k-1}) = 0$. Then $\langle \nabla f(x^{k-1}), p - x^{k-1} \rangle \geq 0 \; \forall p \in S$ thus x^{k-1} is a stationary point by definition and the proposition is proved. Otherwise, one has $f(x^k) < f(x^{k-1}) \;\; \forall k = 1, , \ldots$. Let x^* be a limit point of $\{x^k\}$. Then there exists a subsequence $\{x^{n_k}\}$ that converges to x^* and we have:

$$\begin{cases} \langle p^{n_k}, \nabla f(x^{n_k}) \rangle \leq \langle p, \nabla f(x^{n_k}) \rangle \quad \forall p \in S \\ \langle x^{n_k} - p^{n_k}, \nabla f(x^{n_k}) \rangle \to 0 \end{cases}$$

$\{p^{n_k}\} \subseteq S$ and thus it is a bounded sequence and consequently has a limit point $\bar{p}$. Also, ∇f is continuous and we have:

$$\begin{cases} \langle \bar{p}, \nabla f(x^*) \rangle \leq \langle p, \nabla f(x^*) \rangle \quad \forall p \in S \\ \langle x^* - \bar{p}, \nabla f(x^*) \rangle = 0 \end{cases}$$

Therefore $\langle x^*, \nabla f(x^*) \rangle \leq \langle p, \nabla f(x^*) \rangle \;\; \forall p \in S$, which proves that x^* is a stationary point.

□

Proof of Proposition 5.2.1 For convex functions the optimal points are exactly the stationary points and thus (i) has already been proven. By (5.1) we have that:

$$S(x^{k-1}) \leq f^* - f(x^{k-1}) \leq 0 \quad \forall k = 1, 2, \ldots, \tag{5.20}$$

and since $S(x^{k-1}) \to 0$ we obtain that $\lim_{n\to\infty} f(x^n) = f^*$ which proves (ii). It remains to prove the sublinear rate in function values (iii). From (5.19) we have

$$(f(x^{k-1}) - f^*) - (f(x^k) - f^*) \geq CS^2(x^{k-1}),$$

but from (5.20) we have $S^2(x^{k-1}) \geq (f(x^{k-1}) - f^*)^2$. Defining $a_k = f(x^{k-1}) - f^*, \;\; \gamma := C$, the result follows from Lemma B.0.4(i). $\square$.

Appendix A

Projection on Convex Sets

A.1 Definition and Properties

In this section we will define the concept of *Projection* on a closed convex set. The definition can be made possible due to the following theorem:

Theorem A.1.1 (Existence and Uniqueness of the Projection) *Let $C \subseteq \Re^n$ be a closed convex set and let $\|\cdot\|$ be the Euclidean norm. For every $x \in \Re^n$ there exists a unique vector $z^* \in C$ that minimizes $\|z - x\|$ over all $z \in C$. i.e.,*

$$z^* = \underset{z \in C}{argmin} \|z - x\|.$$

Proof: Let w be some element in C then,

$$\min_{z \in C} \|x - z\| = \min_{z \in C, \|z - x\| \leq \|w - x\|} \|z - x\|.$$

The closed set $C \cap \{z : \|x - z\| \leq \|w - x\|\}$ is compact and thus by the Weiestrass' theorem we have that a minimizer exists. The uniqueness of the minimizer follows from the strict convexity of the function $g(z) = \|x - z\|^2$. $\square$

The previous theorem enable us to define the concept of projection of a point on a convex set:

Definition A.1.1 *Let $C \subseteq \Re^n$ be a closed convex set and let $x \in \Re^n$. The* Projection of x on C *is denoted by $P_C(x)$ and defined to be:*

$$P_C(x) \equiv \underset{z \in C}{argmin} \|z - x\|. \tag{A.1}$$

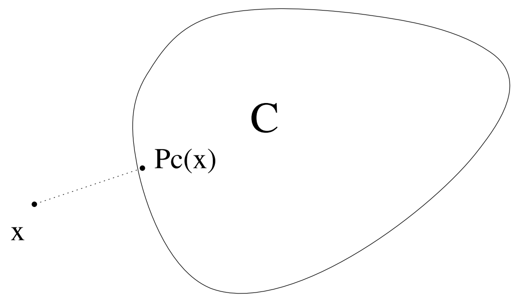

Figure A.1: The Projection of a point on a convex set

We first notice the following geometrical properties which are quite obvious from figure A.1:

Theorem A.1.2 *Let $C \subseteq \Re^n$ be a closed convex set. Then,*

1. $x \in C$ *iff* $P_C(x) = x$.
2. *if* $x \notin C$ *then* $P_C(x) \in bd(C)$.

Proof:

1. Follows directly from the definition of projection.
2. Assume otherwise that $P_C(x) \notin bd(C)$. $P_C(x)$ belongs to C and thus $P_C(x)$ must be in $int(C)$. Thus, there is $\epsilon > 0$ such that every point with distance less then ϵ from $P_C(x)$ also belongs to C. The point $y = P_C(x) + \frac{\epsilon}{2} \frac{x - P_C(x)}{\|x - P_C(x)\|}$ satisfies the following:

$$\begin{aligned} \|P_C(x) - y\| &< \epsilon \\ \|x - y\| &< \|x - P_C(x)\| \end{aligned}$$

from the first inequality we obtain that $y \in C$ and so y is a point in C closer to x than $P_C(x)$ which is a contradiction to the definition of projection (A.1). Thus, $P_C(x) \in bd(C)$. $\square$

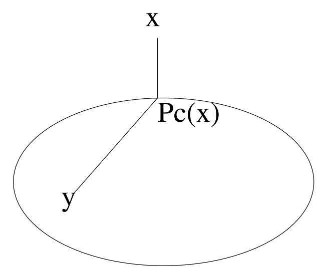

Figure A.2: $\langle x - P_C(x), y - P_C(x)\rangle \leq 0$

The definition of projection on a closed convex set induces the following definition of the distance between a point to a closed convex set.

Definition A.1.2 *Let $C \subseteq \Re^n$ be a closed convex set and let $x \in \Re^n$ then* the distance of x from C *is denoted by $d(x, C)$ and defined by:*

$$d(x, C) = \|x - P_C(x)\|.$$

One of the most important properties of the projection operator is the following:

Theorem A.1.3 *Let $C \subseteq \Re^n$ be a closed convex set and let $x \in \Re^n$. a vector z^* is equal to $P_C(x)$ iff:*

$$\langle y - z^*, x - z^*\rangle \leq 0 \quad \forall y \in C. \tag{A.2}$$

Proof: Recall that $P_C(x) = \operatorname*{argmin}_{z \in C} \|x - z\|$. As a result $P_C(x)$ is equal to $\operatorname*{argmin}_{z \in C} \|x - z\|^2$. $g(z) = \|x - z\|^2$ is a differentiable convex function and thus $z^* = \operatorname*{argmin}_{z \in C} g(z)$ iff:

$$\langle \nabla g(z^*), y - z^*\rangle \geq 0 \quad \forall y \in C.$$

Substituting $\nabla g(z^*) = 2(z^* - x)$ we have that:

$$\langle x - z^*, y - z^*\rangle \leq 0 \quad \forall y \in C,$$

and the result follows. $\square$

Theorem A.1.4 (Non-expensiveness of the projection) *Let $C \subseteq \Re^n$ be a closed convex set. Then,*

1. **Firm non-expensiveness:**

$$\|P_C(x) - P_C(y)\|^2 + \|(x - P_C(x)) - (y - P_C(y))\|^2 \leq \|x - y\|^2 \quad \forall x, y \in \Re^n. \tag{A.3}$$

2. **Non-expensiveness:**

$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \quad \forall x, y \in \Re^n$$

Proof: 1. First, recall that by (A.2) we have that for every $x, y \in \Re^n$:

$$\langle P_C(x) - x, P_C(x) - P_C(y)\rangle \leq 0 \tag{A.4}$$

$$\langle y - P_C(y), P_C(x) - P_C(y)\rangle \leq 0 \tag{A.5}$$

Now, for every $x, y \in \Re^n$ we have:

$$\begin{aligned}
\|x - y\|^2 &= \|(x - P_C(x)) - (y - P_C(y)) + P_C(x) - P_C(y)\|^2 \\
&= \|(x - P_C(x)) - (y - P_C(y))\|^2 + \|P_C(x) - P_C(y)\|^2 \\
&\quad + 2\langle (x - P_C(x)) - (y - P_C(y)), P_C(x) - P_C(y)\rangle \\
&= \|(x - P_C(x)) - (y - P_C(y))\|^2 + \|P_C(x) - P_C(y)\|^2 \\
&\quad + 2\langle x - P_C(x), P_C(x) - P_C(y)\rangle - 2\langle y - P_C(y), P_C(x) - P_C(y)\rangle \\
&\overset{(A.4)-(A.5)}{\geq} \|(x - P_C(x)) - (y - P_C(y))\|^2 + \|P_C(x) - P_C(y)\|^2.
\end{aligned}$$

2. A direct result from firm non-expensiveness. □

The firm non expensiveness of the projection mapping applies the nonexpensivness of the distance function:

Theorem A.1.5 (Non Expensiveness of the distance function) *Let $C \subseteq \Re^n$ be a closed convex set. Then,*

$$|d(x, C) - d(y, C)| \leq \|x - y\| \quad \forall x, y \in \Re^n.$$

Proof: By (A.3) we have that:

$$\|(x - P_C(x)) - (y - P_C(y))\| \leq \|x - y\| \quad \forall x, y \in \Re^n, \tag{A.6}$$

and thus for every $x, y \in \Re^n$ we have:

$$\begin{aligned} d(x, C) &= \|x - P_C(x)\| \\ &= \|(x - P_C(x)) - (y - P_C(y)) + (y - P_C(y))\| \\ &\leq \|(x - P_C(x)) - (y - P_C(y))\| + \|y - P_C(y)\| \\ &\overset{(A.6)}{\leq} \|x - y\| + \|y - P_C(y)\| \\ &= \|x - y\| + d(y, C). \end{aligned}$$

The following property demonstrates that the projection operator has monotonicity features:

Theorem A.1.6 *Let $C \subseteq \Re^n$ be a closed convex set. Then for every $x, y \in \Re^n$:*

$$\langle P_C(x) - P_C(y), x - y \rangle \geq \|P_C(x) - P_C(y)\|^2.$$

Proof: Adding equations (A.4) and (A.5) we have that:

$$\langle P_C(x) - P_C(y) - x + y, P_C(x) - P_C(y) \rangle \leq 0.$$

Thus,

$$\|P_C(x) - P_C(y)\|^2 - \langle x - y, P_C(x) - P_C(y) \rangle \leq 0,$$

and the result follows. □
The following theorem has a nice geometrical illustration demonstrated by figure A.1.

Theorem A.1.7 *Let $C \subseteq \Re^n$ be a closed convex set. Then,*

$$P_C^{-1}(x) = x + N_C(x) \quad \forall x \in \Re^n.$$

Proof: $y \in x + N_C(x)$ iff $y - x \in N_C(x)$. By the definition of $N_C(x)$, this is equivalent to the following:

$$\langle y - x, z - x \rangle \leq 0 \quad \forall z \in C. \tag{A.7}$$

By theorem A.1.3 we conclude that (A.7) is equivalent to $P_C(y) = x$, in other words, $y \in P_C^{-1}(x)$. □

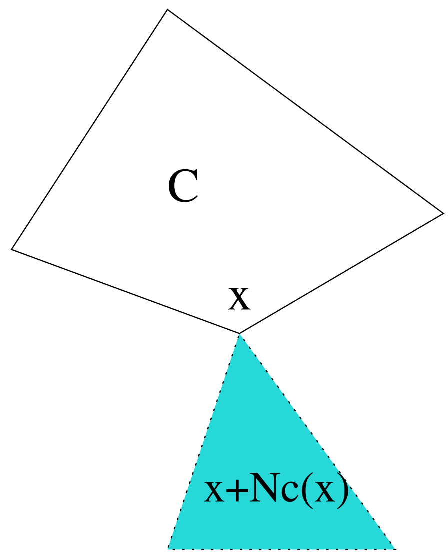

Figure A.3: $P_C^{-1}(x) = x + N_C(x)$

Finding an explicit expression for $P_C(\cdot)$ is not an easy task and there are only few examples of convex sets for which we find an analytical expression for the projection (see the next section). Thus, the importance of the following theorem which enable us to calculate the projection for complicated sets.

Theorem A.1.8 *Let $C_1 \subseteq \Re^n, C_2 \subseteq \Re^m$ be closed convex sets. Define $C = C_1 \times C_2 = \{(x, y) : x \in C_1, y \in C_2\}$. Then,*

$$P_C(x, y) = (P_{C_1}(x), P_{C_2}(y)) \quad \forall (x, y) \in \Re^n \times \Re^m.$$

Proof: Let $(x, y) \in \Re^n \times \Re^m$ then,

$$
\begin{aligned}
P_C(x,y) &= \operatorname*{argmin}_{(z_1,z_2)\in C_1\times C_2} \|(x,y)-(z_1,z_2)\|^2 \\
&= \operatorname*{argmin}_{(z_1,z_2)\in C_1\times C_2} \|(x-z_1,y-z_2)\|^2 \\
&= \operatorname*{argmin}_{(z_1,z_2)\in C_1\times C_2} \|x-z_1\|^2+\|y-z_2)\|^2 \\
&= \left(\operatorname*{argmin}_{z_1\in C_1} \|x-z_1\|^2, \operatorname*{argmin}_{z_2\in C_2} \|y-z_2)\|^2\right) \\
&= (P_{C_1}(x), P_{C_2}(y))
\end{aligned}
$$

□

A simple generalization of the previous example is when we consider the intersection of $C_1 \cap C_2$ with an hyperspace.

Theorem A.1.9 *Let $C_1 \subseteq \Re^n, C_2 \subseteq \Re^m$ be closed convex sets and let $\alpha \in \Re^n, \beta \in \Re^m, \gamma \in \Re$. Define $C = (C_1 \times C_2) \cap \{(z,w) : \langle \alpha, z\rangle + \langle \beta, w\rangle = \gamma\}$. Then,*

$$P_C(x,y) = (P_{C_1}(x-\mu\alpha), P_{C_2}(y-\mu\beta)) \quad \forall x \in \Re^n, y \in \Re^m,$$

where μ is the solution to the following equation:

$$\langle \alpha, P_{C_1}(x-\mu\alpha)\rangle + \langle \beta, P_{C_2}(y-\mu\beta)\rangle = \gamma$$

Proof: The optimization problem associated with the projection problem is:

$$\min_{z\in C_1, w\in C_2, \langle\alpha,z\rangle+\langle\beta,w\rangle=\gamma} \|z-x\|^2+\|w-y\|^2.$$

By attaching a lagrange multiplier to the linear constraint $\langle \alpha, z\rangle + \langle \beta, w\rangle = \gamma$ we obtain the following dual function:

$$
\begin{aligned}
h(\mu) &= \min_{z\in C_1, w\in C_2} \|z-x\|^2+\|w-y\|^2+2\mu(\langle\alpha,z\rangle+\langle\beta,w\rangle-\gamma) \\
&= \min_{z\in C_1, w\in C_2} \|z-(x-\mu\alpha)\|^2+\|w-(y-\mu\beta)\|^2+2\mu(\langle x,\alpha\rangle+\langle\beta,y\rangle-\mu^2\|\alpha\|^2-\gamma) \\
&= \|P_{C_1}(x-\mu\alpha)-(x-\mu\alpha)\|^2+\|P_{C_2}(y-\mu\beta)-(y-\mu\beta)\|^2+2\mu(\langle x,\alpha\rangle+\langle\beta,y\rangle-\mu^2\|\alpha\|^2-\gamma)
\end{aligned}
$$

Thus, there exists $\mu \in \Re$ such that:

$$P_C(x,y) = (P_{C_1}(x-\mu\alpha), P_{C_2}(y-\mu\beta)).$$

Since $(P_{C_1}(x-\mu\alpha), P_{C_2}(y-\mu\beta)) \in C$ it has to satisfy the linear equation $\langle \alpha, P_{C_1}(x - \mu\alpha)\rangle + \langle \beta, P_{C_2}(y-\mu\beta)\rangle = \gamma$. $\square$

Example: Consider the case where C_1, C_2 are affine spaces. $C_1 = \{x : Ax = a\}, C_2 = \{x : Bx = b\}$. From the discussion in the next section we have that:

$$\begin{aligned} P_{C_1}(x-\mu\alpha) &= x - \mu\alpha - A^T(AA^T)^{-1}(A(x-\mu\alpha)-a), \\ P_{C_2}(y-\mu\beta) &= y - \mu\beta - B^T(BB^T)^{-1}(B(y-\mu\beta)-b). \end{aligned}$$

$$\langle \alpha, P_{C1}(x-\mu\alpha)\rangle + \langle \beta, P_{C_2}(y-\mu\beta)\rangle = \gamma$$

By substituting the expressions for the projections we have:

$$\begin{aligned} & \mu(\|\alpha\|^2 - \alpha^T A^T (AA^T)^{-1} A\alpha + \|\beta\|^2 - \beta^T B^T (BB^T)^{-1} B\beta) = \\ = \;& \langle \alpha, x - A^T(AA^T)^{-1}(Ax-a)\rangle + \langle \beta, y - B^T(BB^T)^{-1}(By-b)\rangle - \gamma, \end{aligned}$$

and thus we have the following expression for μ:

$$\mu = \frac{\langle \alpha, P_{C_1}(x)\rangle + \langle \beta, P_{C_2}(y)\rangle - \gamma}{\|\alpha\|^2 - \alpha^T A^T (AA^T)^{-1} A\alpha + \|\beta\|^2 - \beta^T B^T (BB^T)^{-1} B\beta}$$

A.2 Examples of Projections

A.2.1 Hyperplanes

A hyperplane is a convex set defined by $H = \{x \in \Re^n : \langle a, x\rangle = b\}$ where $0 \neq a \in \Re^n$ and $b \in \Re$.

Theorem A.2.1 (Projection on Hyperplanes) *Let $H = \{x \in \Re^n : \langle a, x\rangle = b\}$ be a hyperplane where $0 \neq a \in \Re^n$ and $b \in \Re$. Then,*

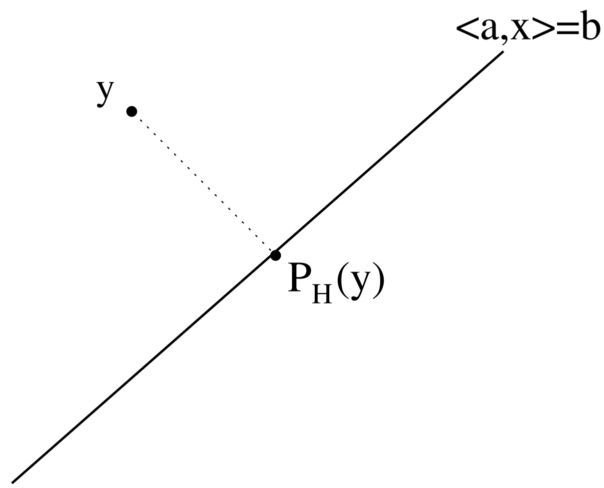

Figure A.4: Projection on Hyperplanes

$$P_H(x) = x - \frac{\langle a, x\rangle - b}{\|a\|^2} a$$

proof: Recall that $P_H(x) = \underset{\langle a,z\rangle = b}{\operatorname{argmin}} \|z - x\|^2$. Denote $z^* = P_H(x)$. By the KKT conditions we have that there is a $\lambda \in \Re$ such that:

$$\begin{aligned} z^* - x + \lambda a &= 0 \\ \langle a, z^*\rangle &= b \end{aligned}$$

From the first equation we have that

$$z^* = x - \lambda a. \tag{A.8}$$

Substituting this in the second equation we have:

$$\begin{aligned} b &= \langle a, z^*\rangle \\ &\overset{(A.8)}{=} \langle a, x - \lambda a\rangle \\ &= \langle a, x\rangle - \lambda \|a\|^2, \end{aligned}$$

and thus $\lambda = \frac{\langle a,x\rangle - b}{\|a\|^2}$ which yields:

$$z^* = x - \frac{\langle a, x\rangle - b}{||a||^2} a.$$

$\square$

A.2.2 Halfspaces

A halfspace is a convex set defined by $C = \{x \in \Re^n : \langle a, x\rangle \leq b\}$ where $0 \neq a \in \Re^n$ and $b \in \Re$. By Similar arguments to those in the case of hyperplanes we have the following theorem which describes the projection on halfspaces:

Theorem A.2.2 (Projection on Halfspaces) *Let $H = \{x \in \Re^n : \langle a, x\rangle \leq b\}$ be a hyperspace where $0 \neq a \in \Re^n$ and $b \in \Re$. Then,*

$$P_H(x) = \begin{cases} x - \frac{\langle a,x\rangle - b}{||a||^2} a & \langle a, x\rangle \geq b \\ x & \mathit{else} \end{cases}.$$

A.2.3 Affine Spaces

Let $C = \{x \in \Re^n : Ax = b\}$ be an affine space where $A \in \Re^{m\times n}, b \in \Re^m$. Notice that Every hyperplane is also an affine space.We assume that $m \leq n$ and that A is full rank (otherwise, we can delete redundant equations). With this assumption we have the following theorem:

Theorem A.2.3 (Projection on an Affine Space) *Let $C = \{x \in \Re^n : Ax = b\}$ be an affine space with $A \in \Re^{m\times n}, b \in \Re^m$. Then,*

$$P_C(x) = x - A^T(AA^T)^{-1}(Ax - b)$$

Proof: As in the case of hyperplanes we consider the optimization problem $P_C(x) = \underset{Az=b}{\operatorname{argmin}} ||z - x||^2$. Denote $z^* = P_C(x)$. The KKT conditions for the minimization problem imply that there exists $\lambda \in \Re^m$ such that:

$$z^* - x + A^T\lambda = 0, \quad \text{(A.9)}$$

$$Az^* = b. \quad \text{(A.10)}$$

(A.9) implies that,

$$z^* = x - A^T\lambda.$$

Substituting this in (A.10) we obtain:

$$A(x - A^T\lambda) = b,$$

which implies that $\lambda = (AA^T)^{-1}(Ax - b)$ and as a result we have that,

$$z^* = x - A^T\lambda = x - A^T(AA^T)^{-1}(Ax - b).$$

□

A.2.4 $\Re^n_+$

Theorem A.2.4 (Projection on the non-negative orthant) *Let* $\Re^n_+ = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0 \quad \forall i = 1, 2, \ldots, n\}$ *be the non-negative orthant of* $\Re^n$*. Then,*

$$P_{\Re^n_+}(x) = x_+,$$

where $(x_+)_i = \begin{cases} x_i & x_i \geq 0 \\ 0 & x_i < 0 \end{cases}.$

Proof: Notice that the optimization problem related to the projection problem has a separable structure:

$$\min_{z \in \Re^n_+} \|z - x\|^2 = \min_{z_1, z_2, \ldots, z_n \geq 0} \sum_{i=1}^{n} |z_i - x_i|^2 = \sum_{i=1}^{n} \min_{z_i \geq 0} |z_i - x_i|^2.$$

for every $1 \leq i \leq n$, if $x_i \geq 0$ then the scalar function $g(z_i) = |z_i - x_i|^2$ attains its minimum at $z_i = x_i$ (the minimum value in this case is 0). If $x_i < 0$ then the scalar function $g(z_i)$ is monotone increasing for every $z_i \geq 0$ and thus attains its minimum at $z_i = 0$, and the proof is completed. □

A.2.5 Polyhedral Sets

We consider the convex set $P = \{x \in \Re^n : Ax \leq b\}$ where $A \in \Re^{m\times n}, b \in \Re^m$. In general, there isn't any known analytic expression for the projection on a polyhedral set. Nevertheless, it is worth mentioning that there is a dual problem to the optimization problem related to the projection problem which is in general easier to solve. The optimization problem in this case is:

$$\min_{Az\leq b} \|z - x\|^2.$$

First, construct the Lagrangian:

$$\lambda \geq 0 \quad L(z,\lambda) = \|z - x\|^2 + 2\lambda^T(Az - b).$$

We obtain the dual problem by minimizing the Lagrangian with respect to z:

$$(D) \quad \sup_{\lambda\geq 0} -\lambda^T AA^T\lambda + 2\lambda^T(Ax - b).$$

Notice that (D) is a very simple problem in the sense that the feasible reason has a very simple structure (the non-negative orthant). Although, as already mentioned, there isn't any explicit expression for the solution of (D) iterative methods can be applied to this problem, for example, The gradient projection algorithm:

$$\lambda^{k+1} = P_{\Re^m_+}(\lambda^k - t(-AA^T\lambda^k + Ax)),$$

where the step size, t, is chosen appropriately.

A.2.6 Simplex

The simplex $X = \{x : x \geq 0, \sum_{i=1}^n x_i = 1\}$ is one of the most simple nontrivial polyhedral sets. From the optimality conditions of the related optimization problem we have the following theorem:

Theorem A.2.5 *Let $X = \{x : x \geq 0, \sum_{i=1}^n x_i = 1\}$ and denote z^* is equal to $P_X(x)$ iff there exists $\lambda \in \Re$ such that:*

$$\begin{aligned} z_i^* &= (x_i + \lambda)_+ \\ \sum_{i=1}^n z_i^* &= 1 \end{aligned}$$

Denote $g(\lambda) = \sum_{i=1}^n (x_i + \lambda)_+$. Then, the problem of finding $P_X(x)$ is equivalent to the problem of finding λ^* such that $g(\lambda^*) = 1$. $g(\lambda)$ is a nondecreasing piecewise linear function and thus finding λ^* is a simple task.

A.2.7 Second Order Cone

Let $\mathcal{L}^n = \{(x,t) \in \Re^{n+1} : \|x\| \leq t, x \in \Re^n, t \in \Re\}$ be the *Lorenz Cone* (or *the ice cream cone*). This cone is used frequently in second order cone optimization problems. Explicit expression for the projection on the Lorenz cone is given in the following theorem:

Theorem A.2.6 *Let $\mathcal{L}^n \subseteq \Re^{n+1}$ be the Lorenz cone. Then for every $x \in \Re^n, t \in \Re$:*

$$\begin{aligned} t \geq 0 \;\; P_{\mathcal{L}^n}(x,t) &= \begin{cases} (x,t) & \|x\| \leq t \\ \frac{\|x\|+t}{2}\left(\frac{x}{\|x\|}, 1\right) & else \end{cases}, \\ t < 0 \;\; P_{\mathcal{L}^n}(x,t) &= \begin{cases} (0,0) & \|x\| \leq -t \\ \frac{\|x\|+t}{2}\left(\frac{x}{\|x\|}, 1\right) & else \end{cases}. \end{aligned}$$

Proof: If $\|x\| \leq t$ then it is obvious that $P_{\mathcal{L}^n}(x,t) = (x,t)$. If $\|x\| > t$ then the minimization problem related to the problem is:

$$\min_{\|z\| \leq w} \|z - x\|^2 + (w - t)^2.$$

Since $(x,t) \notin \mathcal{L}^n$ we have that $P_{\mathcal{L}^n}(x,t) \in bd(\mathcal{L}^n)$ (theorem A.1.2). Thus, the minimization problem is reduced to

$$\min_{\|z\| = w} \|z - x\|^2 + (w - t)^2.$$

We will perform the minimization in two steps: first we take w to be constant and minimize with respect to z and then we will minimize with respect to w. By minimizing with respect z we obtain:

$$\begin{aligned}\min_{\|z\|=w}(\|z-x\|^2+(w-t)^2) &= \min_{\|z\|=w}(\|z\|^2-2\langle z,x\rangle+\|x\|^2+(w-t)^2)\\ &= \min_{\|z\|=w}(w^2-2\langle z,x\rangle+\|x\|^2+(w-t)^2).\end{aligned}$$

By the Cauchy-Schwartz inequality we have that the minimum is obtained at $z^* = \frac{x}{\|x\|}w$ and the minimal value is $w^2-2\|x\|w+\|x\|^2+(w-t)^2$ which is equal to $2w^2-2(\|x\|+t)w+\|x\|^2+t^2$. The optimal value of w denoted by w^* satisfies:

$$w^* = \operatorname*{argmin}_{w\geq 0}(2w^2-2(\|x\|+t)w+\|x\|^2+t^2) = \begin{cases} 0 & \|x\|\leq -t \\ \frac{\|x\|+t}{2} & \text{else}\end{cases}$$

□

A.2.8 S_+^n - The Cone of Positive Semidefinite Matrices

We consider the cone of positive semidefinite matrices $S_+^n = \{X \in \Re^{n\times n} : X \succeq 0\}$. S_+^n is a closed convex cone. The matrix norm we consider here is the frobenius norm defined by $\|A\|_F = \sqrt{tr(A^TA)} = \sqrt{\sum_{i,j} a_{ij}^2}$, which is a kind of a l_2 norm for matrices.

Theorem A.2.7 *Let $S_+^n = \{X \in \Re^{n\times n} : X \succeq 0\}$ be the cone of positive semidefinite matrices. Let $A \in \Re^{n\times n}$ be a symmetric matrix. Suppose that A has the following orthogonal diagonalization $A = U^TDU$ where $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and U is an orthogonal matrix. then:*

$$P_{S_+^n}(A) = U^TD_+U,$$

where $D_+ = diag((\lambda_1)_+, (\lambda_2)_+ \ldots, (\lambda_n)_+)$.

Proof: The minimization problem related to the projection problem is:

$$\min_{X\succeq 0}\|X-A\|_F^2$$

Using the properties of the trace operator we have:

$$\begin{aligned}
\min_{X\succeq 0} \|X-A\|_F^2 &= \min_{X\succeq 0} tr((X-A)^T(X-A)) \\
&= \min_{X\succeq 0} tr(U^T(X-A)^T(X-A)U) \\
&\overset{UU^T=I}{=} \min_{X\succeq 0} tr(U^T(X-A)^T UU^T(X-A)U) \\
&= \min_{X\succeq 0} tr((U^TXU-D)^T(U^TXU-D)) \\
&= \min_{X\succeq 0} \|U^TXU-D\|_F^2
\end{aligned}$$

Denote $Y = U^TXU$ and all that is left is to solve is the following minimization problem:

$$\min_{Y\succeq 0} \|Y-D\|_F^2. \tag{A.11}$$

We will prove that $Y = D_+$ is the solution to (A.11). First,

$$\min_{Y\succeq 0} \|Y-D\|_F^2 \leq \|D-D_+\|_F^2.$$

On the other hand, $\{Y : Y \succeq 0\} \subseteq \{Y : Y \geq 0\}$ where $Y \geq 0$ denotes the fact that every component of Y is non-negative. As a result, we have:

$$\min_{Y\succeq 0} \|Y-D\|_F^2 \geq \min_{Y\geq 0} \|Y-D\|_F^2$$

The problem $\min_{Y\geq 0} \|Y-D\|_F^2$ is exactly like the problem of projection of D on $\Re_+^m$ (where $m = n^2$) with the Euclidean norm. Thus, by our previous results we obtain that $\underset{Y\geq 0}{\operatorname{argmin}} \|Y-D\|_F^2 = D_+$. And so,

$$\min_{Y\succeq 0} \|Y-D\|_F^2 \geq \|D-D_+\|_F^2.$$

Finally, we have obtained that $Y = D_+$ is the solution to (A.11) and thus the solution to the projection problem is U^TD_+U. □

A.3 Application of Projections to Optimality conditions

We consider the following minimization problem:

$$(OP) \quad \min_{x \in C} f(x),$$

where C is a closed convex set and f is a differentiable function. The following definition is important for the study of local minima of function.

Definition A.3.1 *Let C be a closed convex set and let f be a differentiable function over C then x^* is called a* stationary point *if the following condition is satisfied:*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C.$$

Remarks:

1. if x^* is a local minimum of (OP) then it is also a stationary point of (OP).

2. For convex functions a point is a stationary point iff it is an optimal point.

Another way of defining stationary point is by using the projection operator. This is described in the following theorem:

Theorem A.3.1 *Consider the optimization problem (OP) where C is a closed convex set and f is a differentiable function. Then for every $t > 0$, x^* is a stationary point iff the following condition is satisfied:*

$$x^* = P_C(x^* - t\nabla f(x^*)).$$

Proof: By the definition of inverse operators we have that $x^* = P_C(x^* - t\nabla f(x^*))$ iff :

$$x^* - t\nabla f(x^*) \in P_C^{-1}(x^*).$$

By theorem A.1.7 we have that the last equation is equivalent to:

$$x^* - t\nabla f(x^*) \in x^* + N_C(x^*).$$

We can substract x^* from both sides and obtain:

$$-t\nabla f(x^*) \in N_C(x^*).$$

$N_C(x^*)$ is a cone and thus we have the following equivalent equation:

$$-\nabla f(x^*) \in N_C(x^*)$$

By the definition of the normal cone we have that this is equivalent to the following:

$$\langle -\nabla f(x^*), x - x^* \rangle \leq 0,$$

which proves the theorem. □

Appendix B

Mathematical Background

This appendix contains a list of definitions and useful mathematical results that are used throughout the thesis.

Theorem B.0.2 (Descent Lemma) *Under Assumptions 1 & 2 the following is satisfied:*

$$f(x+y) \leq f(x) + y^T \nabla f(x) + \frac{L}{2}\|y\|^2.$$

Theorem B.0.3 *Let $z : \Re \to \Re$ be a continuous function over a closed interval $[a,b]$ with directional derivatives in (a,b). Then, there is a $c \in (a,b)$ such that:*

$$\frac{z(b)-z(a)}{b-a} \in [z'_-(c), z'_+(c)].$$

Theorem B.0.4 *Let $z : \Re \to \Re$ be a continuously differentiable convex function. If z has directional derivatives in a point $x \in \Re$ then, $z''_-(x), z''_+(x) \geq 0$.*

Theorem B.0.5 *Let S be a closed convex set and let f be a convex function with ∇f which is Lipschitz continuous on S with constant L. then,*

$$\langle \nabla f(x) - \nabla f(y), (x-y) \rangle \geq \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2$$

Definition B.0.2 *f is a* strongly convex *function on a closed convex set S if there exists a $m > 0$ such that:*

$$f(x) - f(y) \geq (x-y)^T \nabla f(y) + \frac{1}{2}m\|x-y\|^2 \quad \forall x, y \in S$$

Theorem B.0.6 *Let f be a strongly convex function on a closed convex set S with a parameter $m > 0$. then,*

$$\langle \nabla f(x) - \nabla f(y), (x - y) \rangle \geq m\|x - y\|^2 \quad \forall x, y \in S$$

Definition B.0.3 *Let C be a closed convex set then the* normal cone *of C at a point $\bar{x} \in C$ is denoted by $N_C(\bar{x})$ and defined by:*

$$N_C(\bar{x}) = \{d : d^T(x - \bar{x}) \leq 0 \ \ \forall x \in C\}$$

.

Lemma B.0.1 *Let $f : \Re^n \to \Re \cup \{+\infty\}$ be convex and let $Q \succ 0$. Define:*

$$f_Q(x) = \inf_y \left\{ f(y) + \frac{1}{2}(y - x)^T Q(y - x) \right\}. \tag{B.1}$$

Then,
(i) f_Q is convex differentiable and finite everywhere.
(ii) $\nabla f_Q(x) = Q(x - T_Q(x))$. where T_Q is the unique minimizer of (B.1).
(iii) $\|\nabla f_Q(x_1) - \nabla f_Q(x_2)\| \leq \lambda_{max}(Q)\|x_1 - x_2\| \ \ \forall x_1, x_2$.

Lemma B.0.2 *Let $f : \Re^n \to \Re \cup \{+\infty\}$ be a proper, lsc and convex function with $X^* \neq \emptyset$. Then, for any $x \in \Re^n$, one has*

$$f(x) - \inf_{x \in S} f(x) \leq \rho d(x, X^*)$$

where $\rho = d(0, \partial f(x))$, and ∂f denotes the subdifferential of f.

Proof. See [38, Proposition 10.59, p. 469]. □.

Lemma B.0.3 (Hoffman's Lemma) *Let $S = \{Ax = b, Cx \leq d\} \subseteq \Re^n$,where A is an $m \times n$ matrix, C is a $k \times n$ matrix, b is an $m \times 1$ vector and d is a $k \times 1$ vector. Suppose that $S \neq \emptyset$. Then, there exists $\tau >$ such that for all $x \in \Re^n$ one has,*

$$d(x, S) \leq \tau(\|Ax - b\| + \|(Cx - d)_+\|).$$

Lemma B.0.4 *Let* $\{a_k\}_{k=0}^m$ *be a nonnegative sequence of real numbers.*
(i) Sublinear rate: If $\{a_k\}$ *is such that* $a_{k-1} - a_k \geq \gamma a_{k-1}^2$ *for some* $\gamma > 0$ *and for any* $k = 1, \ldots, m$, *then*

$$a_m \leq \frac{a_0}{1 + m\beta a_0} < (\gamma m)^{-1}.$$

(ii) Linear Rate: If $\{a_k\}$ *is such that* $a_{k-1} - a_k \geq \gamma_k a_{k-1}$ *for some* $\gamma_k \geq 0$, $\forall\ k = 1, \ldots, m$, *then*

$$a_m \leq a_0 e^{-\sum_{k=1}^m \gamma_k}.$$

Proof: (i)

$$\begin{aligned} \frac{1}{a_m} - \frac{1}{a_{m-1}} &= \frac{a_{m-1} - a_m}{a_m a_{m-1}} \geq \frac{\gamma a_{m-1}^2}{a_m a_{m-1}} \\ &= \gamma \frac{a_{m-1}}{a_m} \geq \gamma \end{aligned}$$

Thus, and $\frac{1}{a_m} \geq \frac{1}{a_0} + \gamma m = \frac{1 + a_0 \gamma m}{a_0}$ and the result then follows.
(ii) $a_k \leq (1 - \gamma_k) a_{k-1}$. Using the inequality $1 - \gamma_k \leq e^{-\gamma_k}$ we obtain,

$$a_m \leq a_0 \prod_{i=1}^m (1 - \gamma_i) \leq a_0 e^{-\sum_{i=1}^m \gamma_i}.$$

□

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