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A new semidefinite programming relaxation scheme for a class of quadratic matrix problems

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ABSTRACT

We consider a special class of quadratic matrix optimization problems which often arise in applications. By exploiting the special structure of these problems, we derive a new semidefinite relaxation which, under mild assumptions, is proven to be tight for a larger number of constraints than could be achieved via a direct approach. We show the potential usefulness of these results when applied to robust least-squares and sphere-packing problems.

1. Introduction

The class of nonconvex quadratically constrained quadratic programming (QCQP) problems plays a key role in subproblems arising in optimization algorithms such as trust region methods (see, for example, [9,12]) and is also a bridge to the analysis of many combinatorial optimization problems that can be formulated as such. In principle, nonconvex QCQP problems are hard to solve, and as a result many approximation techniques have been devised in order to tackle them. Many of these techniques rely on so-called semidefinite relaxation (SDR), which is a related convex problem over the matrix space that can be solved efficiently; see, e.g., [13,19].

A key issue in the analysis of QCQP problems is to determine under which conditions the semidefinite relaxation is tight, meaning that it has the same optimal value as the original QCQP problem. In these cases, one can construct the global optimal solution of the QCQP problem from the optimal solution of the SDR via a rank reduction procedure. There are several classes of QCQP problems which posses this "tight semidefinite relaxation" result; among them are the class of generalized trust region subproblems [12,14], which are QCQPs with a single quadratic constraint, problems with two constraints over the complex number field [6], and problems arising in the context of quadratic assignment problems [2,1].

Another class of QCQP problems is the class of *quadratic matrix programming* (QMP) problems whose general form is given by

$$\text{(QMP)}\quad \begin{array}{ll} \min\limits_{X\in\mathbb{R}^{n\times r}} & \mathrm{Tr}(X^TA_0X)+2\mathrm{Tr}(\tilde{B}_0^TX)+c_0 \\ \text{s.t.} & \mathrm{Tr}(X^TA_iX)+2\mathrm{Tr}(\tilde{B}_i^TX)+c_i\le\alpha_i,\quad i\in\mathcal{I}, \\ & \mathrm{Tr}(X^TA_jX)+2\mathrm{Tr}(\tilde{B}_j^TX)+c_j=\alpha_j,\quad j\in\mathcal{E}, \end{array}$$

where n, r are positive integers, $\mathcal{I}$ and $\mathcal{E}$ are sets of indices such that $\mathcal{I}\cap\mathcal{E}=\emptyset$, $A_i\in\mathbb{S}^n$, $\tilde{B}_i\in\mathbb{R}^{n\times r}$, and $c_i,\alpha_i\in\mathbb{R}$. This class of problems was introduced and studied in [5], where it was also shown that it encompasses a broad class of problems that are important both in theory and in applications. The main result in [5] is that problem (QMP) with at most r constraints has a tight SDR property. In the homogeneous case (i.e., when $\tilde{B}_i=0$ for all i), the question of the existence of a tight SDR was already studied by Barvinok [3,4] for the problem of determining the feasibility of this problem; Barvinok's results were then extended by Pataki [16] to include any homogeneous quadratic objective function. In both cases it was shown that it is possible to use semidefinite relaxation (SDR) to solve the original nonconvex problem when the number of constraints is at most $\binom{r+2}{2}-1$.

In this paper, we concentrate on a special type of QMP problems defined by

$$\text{(sQMP)}\quad \begin{array}{ll} \min\limits_{X\in\mathbb{R}^{n\times r}} & \mathrm{Tr}(X^TA_0X)+2\mathrm{Tr}(V^TB_0^TX)+c_0 \\ \text{s.t.} & \mathrm{Tr}(X^TA_iX)+2\mathrm{Tr}(V^TB_i^TX)+c_i\le\alpha_i, \\ & \quad i\in\mathcal{I}, \\ & \mathrm{Tr}(X^TA_jX)+2\mathrm{Tr}(V^TB_j^TX)+c_j=\alpha_j, \\ & \quad j\in\mathcal{E}, \end{array}\tag{1.1}$$

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with $A_i \in \mathbb{S}^n, B_i \in \mathbb{R}^{n\times s}$ $(i \in \{0\} \cup \mathcal{I} \cup \mathcal{E})$ and $0 \neq V \in \mathbb{R}^{s\times r}$, $s \leq r$. Essentially, this type of QMP problem is characterized by the property that the matrices $\tilde{B}_i$ are of the special form $\tilde{B}_i = B_i V$; for the case $n > r > s$, this means that the range spaces of the $n \times r$ matrices $\tilde{B}_i$, $(i \in \{0\} \cup \mathcal{I} \cup \mathcal{E})$ are all contained in the same s-dimensional subspace, which is the range space of V. Note that when $s = r$ and $V = I_r$ we are back to the original QMP setting.

At first glance, it seems that this property of the matrices $\tilde{B}_i$ is quite restrictive; however, it naturally appears in applications, as the example below demonstrates.

Example 1.1 (*Robust Least Squares*)**.** Consider the *robust least-squares problem* which seeks to minimize $\|Ax - b\|^2$ when the matrix $A \in \mathbb{R}^{r\times n}$ is perturbed by an unknown matrix $\Delta \in \mathcal{U}$. This problem was defined and studied in [11,10], and was later inspected via the QMP framework in [5]. The problem can be formulated as

$$\min_x \max_{\Delta\in\mathcal{U}} \|b - (A + \Delta^T)x\|^2, \tag{1.2}$$

where in the following we assume that the set $\mathcal{U}$ has the following form:

$$\mathcal{U} = \{\Delta \in \mathbb{R}^{n\times r} : \|L_i\Delta\|^2 \leq \rho_i,\ i = 1, \dots, m\}$$

for some $L_i \in \mathbb{R}^{k_i\times n}$, and where the norm used is the Frobenius norm. Under these assumptions, we can rewrite the robust least-squares problem (1.2) as follows:

$$(\text{RLS})\quad \begin{array}{ll} \displaystyle\min_x \max_{\Delta\in\mathbb{R}^{n\times r}} & \operatorname{Tr}(\Delta^T xx^t \Delta) + 2\operatorname{Tr}((b - Ax)x^T \Delta) \\ & \quad +\operatorname{Tr}((b - Ax)(b - Ax)^T) \\ \text{s.t.} & \operatorname{Tr}(\Delta^T L_i^T L_i \Delta) \leq \rho_i, \quad i = 1, \dots, m. \end{array}$$

The inner maximization problem is an sQMP with $s = 1$ since here we can take $V = (b - Ax)^T, B_0 = x, B_i = 0,\ i = 1, \dots, m$.

The main result of this paper, developed in Section 3, is that a specially devised SDR of problem (sQMP) is tight as long as the number of constraints does not exceed $\binom{r+2}{2} - \binom{s+1}{2} - 1$, which is an improvement of the result from [5] that allows only r constraints. To do so, we use a rank reduction argument which can be traced back to Barvinok and Pataki (see the beginning of the introduction). Further analysis of the robust least-squares example along with an additional sphere-packing application is given in Section 4.

Notation. We use the following notation. Suppose that (P) is an optimization problem that attains its optimal value (e.g., (P) $\min_{x\in C} f(x)$). Then we denote (P)'s optimal value by $\operatorname{val}(P)$. We use $\mathbb{S}^n$ to denote the set of $n\times n$ symmetric matrices over $\mathbb{R}$, and for two matrices $A, B, A \succeq B (A \succ B)$ means that $A - B$ is positive semidefinite (positive definite). The $n \times m$ matrix of zeros is denoted by $0_{n\times m}$, I_r is the $r \times r$ identity matrix, and $e_i \in \mathbb{R}^n, i = 1, \dots, n$, stands for the i-th canonical unit vector.

2. Preliminaries

We record here some results that will be useful in our analysis. We begin with a fundamental result on the existence of low-rank solutions to general SDP problems which was established by Pataki [16].

Consider the general SDP problem:

$$\begin{array}{ll} \displaystyle\min_{X\in\mathbb{S}^n} & \operatorname{Tr}(C_0X) \\ \text{s.t.} & \operatorname{Tr}(C_iX) \leq b_i, \quad i \in \mathcal{I}, \\ & \operatorname{Tr}(C_iX) = b_i, \quad i \in \mathcal{E}, \\ & X \succeq 0, \end{array} \tag{2.1}$$

where $C_i \in \mathbb{S}^n, i \in \{0\} \cup \mathcal{I} \cup \mathcal{E}$. We state here a slightly different (but equivalent) version of Pataki's result, which was given in [5, Theorem 3.1].

Theorem 2.1. *Suppose that the SDP problem (2.1) attains its optimal value. Then if $|\mathcal{I}|+|\mathcal{E}| \leq \binom{r+2}{2} - 1$, there exists an optimal solution $X^* \in \mathbb{S}^n$ satisfying $\operatorname{rank} X^* \leq r$.*

The next result recalls the so-called Schur complement lemma; see e.g., [7].

Lemma 2.2. *Consider a square matrix in block form:*

$$M = \begin{pmatrix} F & G^T \\ G & H \end{pmatrix},$$

where F is a square matrix assumed to be positive definite. Then,

$$M \succeq 0\ (\succ 0) \quad \textit{if and only if} \quad H - GF^{-1}G^T \succeq 0\ (\succ 0).$$

Finally, we need the following result, which plays an important role in the forthcoming analysis.

Lemma 2.3. *Let $A, B \in \mathbb{R}^{m\times n}$ be two matrices satisfying $AA^T = BB^T$. Then there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $A = BQ$.*

Proof. Since $AA^T = BB^T$, it follows that A and B have the same singular values. Let U be an orthogonal matrix diagonalizing $AA^T = BB^T$, namely, $U^TAA^TU = U^TBB^TU$ is diagonal. The matrices A and B have the following singular value decomposition (SVD):

$$A = U\Sigma V_1^T, \qquad B = U\Sigma V_2^T,$$

where $V_1, V_2 \in \mathbb{R}^{n\times n}$ are orthogonal matrices and Σ is an $m \times n$ diagonal matrix containing the singular values of A (which are also the singular values of B). Thus, $A = BV_2V_1^T$, and the result is established with $Q = V_2V_1^T$. $\square$

3. A tight SDR result for (sQMP)

Consider the problem (sQMP) (given in (1.1)). For $i \in \{0\}\cup\mathcal{I}\cup\mathcal{E}$, define

$$M_i = \begin{pmatrix} A_i & B_i \\ B_i^T & \frac{c_i}{\operatorname{Tr}VV^T} I_s \end{pmatrix} \in \mathbb{S}^{n+s},$$

and consider the following homogenized program:

$$(\text{sQMP}_2)\quad \begin{array}{ll} \displaystyle\min_{Z\in\mathbb{S}^{n+s}} & \operatorname{Tr}(M_0Z) \\ \text{s.t.} & \operatorname{Tr}(M_iZ) \leq \alpha_i, \quad i \in \mathcal{I}, \\ & \operatorname{Tr}(M_jZ) = \alpha_j, \quad j \in \mathcal{E}, \\ & Z \succeq 0, \\ & \operatorname{rank} Z \leq r, \\ & Z_{n+i,n+j} = (VV^T)_{i,j}, \quad i, j = 1, \dots, s, \end{array}$$

where the last set of constraints essentially state that the bottom right $s \times s$ submatrix of Z is VV^T. Note that these constraints can be expressed using $\binom{s+1}{2}$ trace constraints. As the following lemma shows, (sQMP) and (sQMP$_2$) are essentially the same problem.

Lemma 3.1. *Problem (sQMP) attains its optimal value if and only if (sQMP$_2$) attains its optimal value. Furthermore, if either val(sQMP) or val(sQMP$_2$) is finite, then val(sQMP) = val(sQMP$_2$).*

Proof. We will show that any feasible point for one problem can be transformed into a feasible point for the other problem without affecting the objective value.

Suppose that X is feasible for (sQMP). Then define

$$Z = \begin{pmatrix} XX^T & XV^T \\ VX^T & VV^T \end{pmatrix}.$$

Since

$$Z = \begin{pmatrix} X \\ V \end{pmatrix} \begin{pmatrix} X^T & V^T \end{pmatrix},$$

we get that rank $Z \leq r$. In addition,

$$\mathrm{Tr}(M_i Z) = \mathrm{Tr}(A_i XX^T) + 2\mathrm{Tr}(B_i^T XV^T) + c_i, \quad i \in \{0\} \cup \mathcal{I} \cup \mathcal{E}, \tag{3.1}$$

which immediately implies that Z is feasible for (sQMP$_2$)and has the same objective function value as X for (sQMP). In the reverse direction, suppose that Z is feasible for (sQMP$_2$). Since the rank of Z is at most r and Z is positive semidefinite, there exists a matrix $W \in \mathbb{R}^{(n+s)\times r}$ such that $Z = WW^T$. Denote the first n rows of W by $Y \in \mathbb{R}^{n\times r}$ and the last s rows of W by $U \in \mathbb{R}^{s\times r}$ (i.e., $W = (Y; U)$ in Matlab notation); we can therefore write

$$Z = \begin{pmatrix} YY^T & YU^T \\ UY^T & UU^T \end{pmatrix}.$$

From the constraints on Z, we obtain that $UU^T = VV^T$, and thus it follows from Lemma 2.3 that there exists an orthogonal matrix $Q \in \mathbb{S}^r$ such that $U = VQ$. Now, define $X = YQ^T$. Then, since $YU^T = XQQ^TV^T = XV^T$, we get

$$Z = \begin{pmatrix} XX^T & XV^T \\ VX^T & VV^T \end{pmatrix}$$

and therefore, following the same argument as in the first part of the proof, X is feasible for (sQMP) and achieves the same objective value. □

We now omit the hard rank constraint and consider the SDP relaxation of (sQMP$_2$)given by

$$(\text{sQMP-R}) \quad \begin{array}{ll} \min\limits_{Z \in \mathbb{S}^{n+s}} & \mathrm{Tr}(M_0 Z) \\ \text{s.t.} & \mathrm{Tr}(M_i Z) \leq \alpha_i, \quad i \in \mathcal{I} \\ & \mathrm{Tr}(M_j Z) = \alpha_j, \quad j \in \mathcal{E} \\ & Z \succeq 0 \\ & Z_{n+i,n+j} = (VV^T)_{i,j}, \quad i,j = 1,\ldots,s. \end{array}$$

Remark 3.2. Note that when $n + s \leq r$ the relaxation (sQMP-R) is exact, since the rank constraint in (sQMP$_2$)is trivially satisfied.

We now proceed to give a condition, similar to Theorem 3.2 in [5], under which (sQMP) can be solved via (sQMP-R). Note that the number of trace constraints in (sQMP-R) is $|\mathcal{I}| + |\mathcal{E}| + \binom{s+1}{2}$ instead of $|\mathcal{I}|+|\mathcal{E}|+\binom{r+1}{2}$ in the corresponding setting of Theorem 3.2 in [5]. This property of the new SDP relaxation allows us to improve and extend the result of Theorem 3.2 in [5] as follows.

Theorem 3.3. *Suppose that problem (sQMP-R) attains its optimal value, and that either $n + s \leq r$ or $|\mathcal{I}| + |\mathcal{E}| \leq \binom{r+2}{2} - \binom{s+1}{2} - 1$. Then* val*(sQMP) is finite and* val*(sQMP)* = val*(sQMP-R).*

Proof. Suppose that problem (sQMP-R) attains its optimal value and that $|\mathcal{I}| + |\mathcal{E}| \leq \binom{r+2}{2} - \binom{s+1}{2} - 1$. Then the number of constraints in (sQMP-R) is $\binom{r+2}{2} - 1$. Hence, by Theorem 2.1, problem (sQMP-R) has a an optimal solution with rank at most r. This solution is therefore feasible and optimal for (sQMP$_2$), and, by Lemma 3.1, val(sQMP) = val(sQMP$_2$) = val(sQMP-R).

When $n+s \leq r$, the claim follows immediately from Lemma 3.1 and Remark 3.2. □

In particular, note that when $s = r$ the SDP relaxation is tight when the number of constraints is at most r; thus we recover [5, Theorem 3.2].

In the following, we need the dual of (sQMP-R), which is given by

$$(\text{sQMP-D}) \quad \begin{array}{ll} \max\limits_{\lambda_i, \Phi \in \mathbb{S}^s} & -\sum\limits_{i \in \mathcal{I}\cup\mathcal{E}} \lambda_i \alpha_i - \mathrm{Tr}(VV^T \Phi) \\ \text{s.t.} & M_0 + \sum\limits_{i \in \mathcal{I}\cup\mathcal{E}} \lambda_i M_i + \begin{pmatrix} 0_{n\times n} & 0_{n\times s} \\ 0_{s\times n} & \Phi \end{pmatrix} \succeq 0, \\ & \Phi \in \mathbb{S}^s, \\ & \lambda_i \geq 0, \quad i \in \mathcal{I}. \end{array}$$

From the conic duality theorem [7], if (sQMP-D) is strictly feasible and bounded from above, then (sQMP-R) and (sQMP-D) have the same optimal value. The next claim immediately follows.

Corollary 3.4. *Suppose that (sQMP-D) is strictly feasible and bounded from above. Then if either $n+s \leq r$ or $|\mathcal{I}|+|\mathcal{E}| \leq \binom{r+2}{2}-\binom{s+1}{2}-1$, we have* val*(sQMP)* = val*(sQMP-D).*

A simple condition given in [5, Lemma 3.2] that ensures the strict feasibility and boundedness of (sQMP-D) is the following: there exist numbers $\lambda_i \in \mathbb{R}$, $i \in \{0\} \cup \mathcal{I} \cup \mathcal{E}$, for which

$$A_0 + \sum_{i \in \mathcal{I}\cup\mathcal{E}} \lambda_i A_i \succ 0 \quad \text{and} \quad \lambda_i \geq 0 \ \forall i \in \mathcal{I}.$$

4. Applications

4.1. Robust least squares

Consider the robust least-squares problem (RLS) discussed in Example 1.1. Recall that the problem is formulated as

$$(\text{RLS}) \quad \begin{array}{ll} \min\limits_{x} \max\limits_{\Delta \in \mathbb{R}^{n\times r}} & \mathrm{Tr}(\Delta^T xx^t \Delta) + 2\mathrm{Tr}((b - Ax)x^T \Delta) \\ & \quad +\mathrm{Tr}((b - Ax)(b - Ax)^T) \\ \text{s.t.} & \mathrm{Tr}(\Delta^T L_i^T L_i \Delta) \leq \rho_i, \quad i = 1,\ldots,m. \end{array}$$

We begin our analysis by deriving the dual of the inner maximization problem in (RLS). Suppose that $Ax = b$. Then in this case the inner maximization problem in (RLS) is a homogeneous quadratic problem; performing the standard SDP relaxation technique for homogeneous problems, and taking the dual, we reach the following problem:

$$(\text{RLS-D}') \quad \begin{array}{ll} \min\limits_{\lambda_i, t} & \sum\limits_{i=1}^m \lambda_i \rho_i \\ \text{s.t.} & -xx^T + \sum\limits_{i=1}^m \lambda_i L_i^T L_i \succeq 0, \\ & \lambda_i \geq 0, \quad i = 1,\ldots,m. \end{array}$$

When $Ax \neq b$, the inner maximization problem in (RLS) is of the form of problem (sQMP) with $s = 1$ and

$A_0 = -xx^T$,
$B_0 = -\|b - Ax\|x$,
$c_0 = -\|b - Ax\|^2$,
$A_i = L_i^T L_i, \quad i = 1,\ldots,m$,
$B_i = 0, \quad i = 1,\ldots,m$,
$c_i = 0, \quad i = 1,\ldots,m$,
$\alpha_i = \rho_i, \quad i = 1,\ldots,m$,

$$V = \frac{1}{\|b - Ax\|}(b - Ax)^T.$$

By taking the dual form (sQMP-D), we get

$$
\text{(RLS-D)}\quad
\begin{aligned}
\min_{\lambda_i,t}\quad & \sum_{i=1}^m \lambda_i\rho_i + t\\
\text{s.t.}\quad & \begin{pmatrix} -xx^T + \sum_{i=1}^m \lambda_i L_i^T L_i & -\|b-Ax\|x \\ -\|b-Ax\|x^T & -\|b-Ax\|^2 + t \end{pmatrix} \succeq 0\\
& \lambda_i \ge 0,\quad i=1,\ldots,m.
\end{aligned}
$$

Note that, if we set $Ax = b$ in (RLS-D), the optimal value for t becomes 0, and we are left with problem (RLS-D$'$); hence, (RLS-D) can be used as the dual problem for both cases.

Now, if we further assume that (RLS-D) is strictly feasible and bounded (e.g., when $\sum_{i=1}^m \lambda_i L_i^T L_i \succ 0$ for some $\lambda_i \ge 0$) and that either $r \ge n+1$ or that the number of constraints satisfies $m \le \binom{r+2}{2} - 2$, then (RLS-D) and the inner maximization problem in (RLS) have the same optimal solution for every x. Therefore, in order to solve (RLS) it is sufficient to solve the following problem:

$$
(\text{RLS}_2)\quad
\begin{aligned}
\min_{x,\lambda_i,t}\quad & \sum_{i=1}^m \lambda_i\rho_i + t\\
\text{s.t.}\quad & \mathcal{A}(x) = \begin{pmatrix} -xx^T + \sum_{i=1}^m \lambda_i L_i^T L_i & -\|b-Ax\|x \\ -\|b-Ax\|x^T & -\|b-Ax\|^2 + t \end{pmatrix} \succeq 0,\\
& \lambda_i \ge 0,\quad i=1,\ldots,m.
\end{aligned}
$$

We will now show that it is possible to rewrite (RLS$_2$)as a standard SDP problem.

Proposition 4.1. *The point (x^*, λ^*, t^*) is an optimal solution for (RLS$_2$) if and only if it is an optimal solution for the following SDP problem:*

$$
\begin{aligned}
\min_{x,\lambda_i,t}\quad & \sum_{i=1}^m \lambda_i\rho_i + t\\
\text{s.t.}\quad & \begin{pmatrix} 1 & x^T & (b-Ax)^T \\ x & \sum_{i=1}^m \lambda_i L_i^T L_i & 0 \\ b-Ax & 0 & tI_r \end{pmatrix} \succeq 0,\\
& \lambda_i \ge 0,\quad i=1,\ldots,m.
\end{aligned}
$$

Proof. The matrix inequality in (RLS$_2$)is given by

$$\mathcal{A}(x) \succeq 0,$$

which is equivalent to

$$\begin{pmatrix} \mathcal{A}(x) & 0_{n\times(r-1)} \\ 0_{(r-1)\times n} & I_{r-1} \end{pmatrix} \succeq 0.$$

The latter inequality can be rewritten as

$$\begin{pmatrix} -xx^T + \sum_{i=1}^m \lambda_i L_i^T L_i & -\|b-Ax\|xe_1^T \\ -\|b-Ax\|e_1x^T & -\|b-Ax\|^2 e_1e_1^T + tI_r \end{pmatrix} \succeq 0. \tag{4.1}$$

Now, let $Q \in \mathbb{S}^r$ be an orthogonal matrix such that $Qe_1 = \frac{b-Ax}{\|b-Ax\|}$ (when $Ax = b$, one can choose $Q = I_m$). Then

$$
\begin{aligned}
& \begin{pmatrix} I_n & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} -xx^T + \sum_{i=1}^m \lambda_i L_i^T L_i & -xe_1^T\|b-Ax\| \\ -\|b-Ax\|e_1x^T & -\|b-Ax\|^2 e_1e_1^T + tI_r \end{pmatrix}\\
&\times \begin{pmatrix} I_n & 0 \\ 0 & Q^T \end{pmatrix}\\
&= \begin{pmatrix} -xx^T + \sum_{i=1}^m \lambda_i L_i^T L_i & -x(b-Ax)^T \\ -(b-Ax)x^T & -(b-Ax)(b-Ax)^T + tI_r \end{pmatrix}.
\end{aligned}
$$

Hence, (4.1) is equivalent to

$$\begin{pmatrix} -xx^T + \sum_{i=1}^m \lambda_i L_i^T L_i & -x(b-Ax)^T \\ -(b-Ax)x^T & -(b-Ax)(b-Ax)^T + tI_r \end{pmatrix} \succeq 0. \tag{4.2}$$

Finally, writing the last constraint in the form

$$\begin{pmatrix} \sum_{i=1}^m \lambda_i L_i^T L_i & 0 \\ 0 & tI_r \end{pmatrix} - \begin{pmatrix} x \\ b-Ax \end{pmatrix}\begin{pmatrix} x \\ b-Ax \end{pmatrix}^T \succeq 0,$$

and applying the Schur complement lemma (see Lemma 2.2), we obtain the desired equivalent SDP formulation. □

Note that the dimension of the matrix constraint is $n+r+1$ instead of $nr+r+1$ in the standard formulation [5]. Thus, assuming strong duality holds, this new formulation can handle much more complex sets of uncertainty, with $\binom{r+2}{2} - 2$ constraints if $r \le n$ and an arbitrary number of quadratic constraints if $r \ge n+1$.

4.2. *The sphere-packing problem*

In the *sphere-packing problem*, we are interested in determining a feasible configuration of non-overlapping spheres bounded within a given shape. This problem has been extensively studied in various settings over the years. See, for example, [8,15,17,18] amongst others.

Consider the problem of finding a packing of n spheres with given radii within the intersection of k balls with known centers and radii in $\mathbb{R}^d$ ($k \le d+1$). This problem can be formulated as determining whether the following set of constraints is feasible:

$$
\begin{aligned}
& \|X^Te_i - c_j\| \le R_j - r_i,\quad i=1,\ldots,n,\ j=1,\ldots,k,\\
& \|X^Te_i - X^Te_j\| \ge r_i + r_j,\quad i,j=1,\ldots,n,\\
& X \in \mathbb{R}^{n\times d},
\end{aligned}
$$

where $c_1,\ldots,c_k \in \mathbb{R}^d$ are the centers of the containing balls, $R_1,\ldots,R_k > 0$ are the respective radii, and $r_1,\ldots,r_n > 0$ are the radii of the inner spheres. The radii are assumed to satisfy the relation $\min_{j=1,\ldots,k} R_j \ge \max_{i=1,\ldots,n} r_i$, which is necessary in order to make the problem feasible. The rows of the decision variables matrix X represent the centers of the spheres to be determined.

Since we can assume without loss of generality that $c_1 = 0$, by choosing

$$V = \begin{pmatrix} c_2^T \\ \vdots \\ c_k^T \end{pmatrix} = \sum_{j=2}^k e_{j-1}c_j^T,$$

and for the first kn constraints taking

$$
\begin{aligned}
& B_{i,1} = 0_{d\times k-1},\quad i=1,\ldots,n\\
& B_{i,j} = \underbrace{e_i}_{\in\mathbb{R}^{d\times 1}}\ \underbrace{e_{j-1}^T}_{\in\mathbb{R}^{1\times(k-1)}},\quad i=1,\ldots,n,\ j=2,\ldots,k,
\end{aligned}
$$

it can be readily seen that this problem is of the form (sQMP) discussed above with $s = k-1$ and $kn + \binom{n}{2}$ constraints. According to Theorem 3.3, the SDP relaxation is tight when $kn + \binom{n}{2} \le$

$\binom{d+2}{2} - \binom{k}{2} - 1$ or when $d \geq n + k - 1$. The first condition is equivalent to

$$n \leq -k + \frac{1}{2} + \sqrt{d^2 + 3d + \frac{1}{4}},$$

and since

$$\begin{aligned} d - k + 1 &= -k + \frac{1}{2} + \sqrt{d^2 + d + \frac{1}{4}} \\ &< -k + \frac{1}{2} + \sqrt{d^2 + 3d + \frac{1}{4}}, \end{aligned}$$

it follows that the validity of the second condition implies the validity of the first condition. Thus we have proved the following.

Proposition 4.2. *The problem of finding the feasibility of packing n spheres in the intersection of k balls in d dimensions can be solved by an SDP problem when $n \leq d - k + 1$.*

Note that the standard homogenization scheme can be applied when $kn + \binom{n}{2} \leq d$. Hence for a fixed k only $O(\sqrt{d})$ spheres can be handled this way, and thus the technique presented in this paper provides a major improvement.

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