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On Minimizing Quadratically Constrained Ratio of Two Quadratic Functions*

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Dedicated to Hedy Attouch on the occasion of his 60th birthday.

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We consider the nonconvex problem of minimizing the ratio of two quadratic functions over finitely many nonconvex quadratic inequalities. Relying on the homogenization technique we establish a sufficient condition that warrants the attainment of an optimal solution. Our result allows to extend and recover known conditions for some interesting special instances of the problem and to derive further results on its algorithmic and modeling aspects.

1. Introduction

In this paper we consider the general problem of minimizing a ratio of two quadratic functions subject to a constraint set defined by finitely many quadratic inequality constraints:

$$\text{(QCRQ)} \qquad \inf_{\mathbf{x}\in\mathbb{R}^n} \left\{ \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} : g_i(\mathbf{x}) \le 0,\ i = 1, \dots m \right\},$$

where all the data functions are quadratic functions on $\mathbb{R}^n$, see Section 2 for a precise formulation. Note that even if we assume the convexity of the constraints, with no further assumptions on f_i, the problem remains a nonconvex one. Problem (QCRQ) encompasses a variety of fundamental problems arising in both optimization theory/algorithmic development itself, and in many important scientific applications that will be discussed below. Optimization problems involving ratio in the objective function are commonly called fractional programs, and provide an important sub-class of global optimization which has attracted intensive research activities, see [19], and the large bibliography therein. Despite these extensive studies, the specific generic model (QCRQ) has received limited attention.

For any given optimization problem, a key question is to characterize and warrant the existence and infimum attainment of an optimal solution. This issue is not only

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theoretical. It has a major impact on both the ability to determine/identify the condition(s) under which a given nonconvex problem can be adequately transformed (e.g., as a convex one), as well as on the algorithmic design, see for instance [1]. This paper focuses on this question and its potential implications for (QCRQ). A common approach to tackle this question is via techniques of asymptotic cones and functions in nonconvex analysis, [2]. Here, we depart from this approach and tackle the problem by following an elementary and well known homogenization technique. This is developed in Section 3 where we derive a sufficient condition under which the optimal solution of the problem (QCRQ) is attained. In addition, we show that this condition ensures the equivalence of the original problem to a nonconvex but homogenous quadratic problem. Our result allows to extend and recover known conditions for some interesting special instances of the QCRQ problem and to provide new algorithmic and practical insights in the analysis of (QCRQ). Using the homogenous formulation of the problem, we construct in Section 4 a semidefinite relaxation (SDR) of the problem and we show by simulations that for random problems, the SDR has a high probability to produce the *global* optimal solution. Finally, we address the question of representation of two-sided linear constraints and prove that it is always better to represent this kind of constraints by a quadratic constraint.

Notation. Vectors are denoted by boldface lowercase letters, *e.g.,* $\mathbf{y}$, and matrices are denoted by boldface uppercase letters *e.g.,* $\mathbf{A}$. For any symmetric matrix $\mathbf{A}$ and symmetric positive definite matrix $\mathbf{B}$ we denote the corresponding minimum generalized eigenvalue by $\lambda_{\min}(\mathbf{A},\mathbf{B})$; the minimum generalized eigenvalue has several equivalent formulations (see e.g., [18]):

$$\begin{aligned}\lambda_{\min}(\mathbf{A},\mathbf{B}) &= \max\{\lambda : \mathbf{A}-\lambda\mathbf{B}\succeq \mathbf{0}\} = \min\{\mathbf{x}^T\mathbf{A}\mathbf{x} : \mathbf{x}^T\mathbf{B}\mathbf{x}=1\}\\ &= \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{B}\mathbf{x}} = \lambda_{\min}(\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}),\end{aligned}$$

where we use the notation $\mathbf{A}\succeq 0$ ($\mathbf{A}\succ 0$) for a positive semidefinite (positive definite) matrix $\mathbf{A}$. The value of the optimal objective function of an optimization problem:

$$(\mathrm{P}):\quad \inf\{f(\mathbf{x}) : \mathbf{x}\in C\}$$

is denoted by val(P), and with the usual notation "min" in place of "inf", whenever attained.

2. Motivation, Problem Formulation and Examples

2.1. Motivation

Many problems in data fitting and estimation give rise to an overdetermined system of linear equations $\mathbf{A}\mathbf{x}\approx\mathbf{b}$, where both the matrix $\mathbf{A}\in\mathbb{R}^{m\times n}$ and the vector $\mathbf{b}\in\mathbb{R}^m$ are contaminated by noise. The Total Least Squares (TLS) approach to this problem [12, 13, 20] is to seek a perturbation matrix $\mathbf{E}\in\mathbb{R}^{m\times n}$ and a perturbation vector $\mathbf{r}\in\mathbb{R}^m$ that minimize $\|\mathbf{E}\|^2+\|\mathbf{r}\|^2$ subject to the consistency equation $(\mathbf{A}+\mathbf{E})\mathbf{x}=\mathbf{b}+\mathbf{r}$. It is well known that for ill posed problems the TLS solution might give rise to a poor quality solution, usually with a huge norm. For that reason, several regularization techniques were devised in order to stabilize the solution including truncation methods [10, 16] and Tikhonov regularization [4]. Another standard technique to regularize the solution is to incorporate some a priori knowledge on the unknown vector $\mathbf{x}$. Therefore, assuming that the "true" vector $\mathbf{x}$ belongs to some set X, the regularized TLS (RTLS) vector is the solution of the problem:

$$\begin{array}{ll} \inf\limits_{\mathbf{E},\mathbf{r},\mathbf{x}} & \|\mathbf{E}\|^2 + \|\mathbf{r}\|^2 \\ \text{s.t.} & (\mathbf{A}+\mathbf{E})\mathbf{x} = \mathbf{b}+\mathbf{r}, \\ & \mathbf{x} \in X. \end{array} \tag{1}$$

Fixing $\mathbf{x}$ and minimizing the above problem with respect to $\mathbf{E}$ and $\mathbf{w}$, the problem transforms to (for details see [13, 5])

$$\text{(RTLS)} \qquad \begin{array}{ll} \inf\limits_{\mathbf{x}} & \frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2}{\|\mathbf{x}\|^2+1} \\ \text{s.t.} & \mathbf{x} \in X. \end{array} \tag{2}$$

The case in which X is given by a single convex quadratic and homogenous constraint, i.e., $X_1 = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{L}\mathbf{x}\|^2 \leq \rho\}$ was extensively studied in [23, 14, 11, 6, 5]. In particular, a fixed point iteration based on quadratic eigenvalue solvers was devised in [23]. A proof of superlinear global rate of convergence for the fixed point scheme can be found in [6]. Another globally convergent algorithm, proposed in [5], relies on a combination of Dinklbach's observation for fractional programming [9] and the hidden convexity result of [8]. This algorithm is also capable of handling two sided constraints of the form $\eta \leq \|\mathbf{L}\mathbf{x}\|^2 \leq \rho$. Finally, it was shown in [6] that the RTLS problem with $X = X_1$ can be recast as a semidefinite programming (SDP) problem; therefore, the RTLS problem is essentially *equivalent* to a convex optimization problem. This latter result gives rise to yet another method for solving the RTLS problem with $X = X_1$.

In contrast to the relatively large amount of works dealing with the RTLS problem with a single quadratic constraint (i.e., $X = X_1$), it seems that more involved choices of the set of admissible vectors X are not treated in the literature. The main reason for that is probably the fact that the objective function is not convex and therefore this problem does not render itself to the powerful algorithms/theory of convex optimization.

2.2. Problem Formulation

In our analysis we consider the more general problem of minimizing a ratio of two general quadratic functions over multiple quadratic inequalities. This problem will be called the *quadratically constrained ratio quadratic* (QCRQ) problem:

$$\text{(QCRQ)}: \qquad \begin{array}{ll} \inf\limits_{\mathbf{x}} & \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})} \\ \text{s.t.} & \mathbf{x} \in X. \end{array} \tag{3}$$

Here $f_i(\mathbf{x}) = \mathbf{x}^T\mathbf{A}_i\mathbf{x} + 2\mathbf{b}_i^T\mathbf{x} + c_i$ and $\mathbf{A}_i = \mathbf{A}_i^T \in \mathbb{R}^{n\times n},\ \mathbf{b}_i \in \mathbb{R}^n,\ c_i \in \mathbb{R},\ i = 1,2$ and where the set of admissible vectors X is chosen as the intersection of several level sets of quadratic functions, i.e.,

$$X = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0,\ i = 1,\ldots,m\}, \tag{4}$$

where

$$g_i(\mathbf{x}) = \mathbf{x}^T\mathbf{B}_i\mathbf{x} + 2\mathbf{d}_i^T\mathbf{x} + \alpha_i \tag{5}$$

with $\mathbf{B}_i = \mathbf{B}_i^T \in \mathbb{R}^{n\times n}$, $\mathbf{d}_i \in \mathbb{R}^n$, $\alpha_i \in \mathbb{R}$. This form of X is quite general and encompasses a wide variety of structures.

Throughout the paper we assume that:

(i) the feasible set X is nonempty,

(ii)

$$\begin{pmatrix} \mathbf{A}_2 & \mathbf{b}_2 \\ \mathbf{b}_2^T & c_2 \end{pmatrix} \succ \mathbf{0}, \tag{6}$$

which implies that $f_2(\mathbf{x}) > 0$ for every $\mathbf{x} \in \mathbb{R}^n$, and in particular that the problem is well defined with an objective function bounded below.

Note that $\mathbf{A}_1$ is not assumed to be positive semidefinite so that the function f_1 is not necessarily convex (as in the case of the RTLS problem (2)).

2.3. Examples

The general model QCQR includes interesting particular instances of the RTLS problem:

1. **Single constrained RTLS**

$$\inf\left\{\frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2}{\|\mathbf{x}\|^2+1} : \|\mathbf{L}\mathbf{x}\|^2 \le \rho\right\}, \tag{7}$$

where $\mathbf{L} \in \mathbb{R}^{d\times n}$ and $\rho > 0$. As noted in the introduction, this is the RTLS problem considered in [23, 14, 11, 6, 5]. The matrix $\mathbf{L}$ is usually chosen as an approximation of the first or second order derivative [11, 15, 17].

2. **Box constrained RTLS**

$$\min\left\{\frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2}{\|\mathbf{x}\|^2+1} : l \le x_i \le u,\ i=1,\dots,n\right\},$$

where $l < u$. We note that in some situations box constraints are more suitable than a single weighted constraint of the form $\|\mathbf{L}\mathbf{x}\|^2 \le \rho$. For example, if the unknown vector $\mathbf{x}$ stands for an image in a BMP format, then each component (or pixel) x_i is bounded below and above by 0 and 255 respectively; therefore, in this case $l = 0,\ u = 255$.

The above two RTLS problems will also serve as guiding prototype models that will be used to illustrate our results. The corresponding sets of admissible vectors for the two prototype problems are denoted by

$$X_1 = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{L}\mathbf{x}\|^2 \le \rho\},$$
$$X_\infty = \{\mathbf{x} \in \mathbb{R}^n : l \le x_i \le u\}.$$

Modeling of the box constraints

There are two approaches for modeling the box constraints within the general structure (4) of X. Specifically, we can describe X_∞ with $m = 2n$ linear constraints where $g_{2i-1}(\mathbf{x}) = x_i - u,\ g_{2i}(\mathbf{x}) = -x_i + l,\ i = 1,\dots,n$. The second option is to model X_∞ via n quadratic constraints: $g_i(\mathbf{x}) = (x_i - u)(x_i - l) \le 0,\ i = 1,\dots,n$. A natural question that arises is whether one approach is "better" than the other. The answer to the latter question is affirmative and and will be considered in Section 4.2.

3. Homogenization and Attainment of an Optimal Solution

The attainment of the solution of the QCRQ problem is not always guaranteed. Of course, when the feasible set X is compact, attainability is ensured, but here X is not compact. For some special cases, sufficient conditions for attainment have been obtained. Specifically, for the unconstrained TLS problem (problem (2) with $X = \mathbb{R}^n$), the condition

$$\lambda_{\min}\begin{pmatrix} \mathbf{A}^T\mathbf{A} & -\mathbf{A}^T\mathbf{b} \\ -\mathbf{b}^T\mathbf{A} & \|\mathbf{b}\|^2 \end{pmatrix} < \lambda_{\min}(\mathbf{A}^T\mathbf{A}) \tag{8}$$

is enough to guarantee the attainability of the TLS solution [12, 20]. Moreover, in [6] it was shown that (QCRQ) with $X = X_1$ attains its optimal solution if

$$\lambda_{\min}(\mathbf{N}_1, \mathbf{N}_2) < \lambda_{\min}(\mathbf{F}^T\mathbf{A}_1\mathbf{F}, \mathbf{F}^T\mathbf{A}_2\mathbf{F}), \tag{9}$$

where

$$\mathbf{N}_1 = \begin{pmatrix} \mathbf{F}^T\mathbf{A}_1\mathbf{F} & \mathbf{F}^T\mathbf{b}_1 \\ \mathbf{b}_1^T\mathbf{F} & c_1 \end{pmatrix},$$
$$\mathbf{N}_2 = \begin{pmatrix} \mathbf{F}^T\mathbf{A}_2\mathbf{F} & \mathbf{F}^T\mathbf{b}_2 \\ \mathbf{b}_2^T\mathbf{F} & c_2 \end{pmatrix}$$

and $\mathbf{F}$ is a matrix whose columns form an orthonormal basis for the null space of $\mathbf{L}$. This result was established in [6] by using techniques of asymptotic cones and functions in nonconvex analysis, [2]. Here, we depart from this approach and tackle the problem via an elementary homogenization approach to derive a sufficient condition under which the optimal solution of the QCRQ problem is attained. The condition is expressed via the optimal values of two nonconvex homogenous quadratic problems. We will show that the sufficient condition also guarantees that the problem is equivalent to a nonconvex homogenous quadratic problem. This fact will be important in Section 4 where a semidefinite relaxation will be constructed based on the homogenized problem. As an application of Theorem 3.2, we obtain explicit conditions which are generalizations of conditions (8) and (9).

3.1. Homogenization of (QCRQ)

We use the following notation: for any quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$, the *homogenized version* of f is the function $f^H : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by $f^H(\mathbf{y}, t) = \mathbf{y}^T\mathbf{A}\mathbf{y} + 2\mathbf{b}^T\mathbf{y}t + ct^2$ which is, of course, a homogenous quadratic function.

Substituting $\mathbf{x} = \mathbf{y}/t$ with $\mathbf{y} \in \mathbb{R}^n, t \neq 0$, problem (3) becomes

$$\begin{array}{ll} \inf\limits_{\mathbf{y},t} & \frac{f_1^H(\mathbf{y},t)}{f_2^H(\mathbf{y},t)} \\ \text{s.t.} & g_i^H(\mathbf{y}, t) \leq 0, \quad i = 1, \ldots, m, \\ & t \neq 0. \end{array} \tag{10}$$

We will first consider a slightly different problem, namely

$$\begin{array}{ll} \inf\limits_{\mathbf{y},t} & \frac{f_1^H(\mathbf{y},t)}{f_2^H(\mathbf{y},t)} \\ \text{s.t.} & g_i^H(\mathbf{y}, t) \leq 0, \quad i = 1, \ldots, m, \\ & (\mathbf{y}, t) \neq (\mathbf{0}_n, 0). \end{array} \tag{11}$$

Problem (11) is different from problem (10) in the sense that a feasible vector $(\mathbf{y}, t)$ of (11) can not be identically equal to the all-zeros vector, but it can satisfy $t = 0$. In contrast, a feasible solution $(\mathbf{y}, t)$ of (10) must satisfy $t \neq 0$. Our motivation for considering the "mutated" problem (11) is that it can be proven to be equivalent to the following nonconvex homogeneous quadratic problem:

$$\text{(H):} \quad \begin{array}{ll} \min_{\mathbf{z},s} & f_1^H(\mathbf{z}, s) \\ \text{s.t.} & f_2^H(\mathbf{z}, s) = 1, \\ & g_i^H(\mathbf{z}, s) \leq 0, \quad i = 1, \ldots, m. \end{array} \tag{12}$$

Note that by assumption (6), the feasible set of (12) is compact; therefore, the optimal solution of (12) is attained. Lemma 3.1 below proves the equivalence between problems (11) and (12).

Lemma 3.1. *Let v_1 and v_2 be the optimal values of problems* (11) *and* (12) *respectively. Then,*

(*i*) $v_1 = v_2$.
(*ii*) *The solution of problem* (11) *is attained.*
(*iii*) *If $(\mathbf{z}^*, s^*)$ is an optimal solution of* (12), *then it is also an optimal solution of* (11).
(*iv*) *If $(\mathbf{y}^*, t^*)$ is an optimal solution of* (11), *then $(\mathbf{z}^*, s^*) = \frac{1}{\sqrt{f_2^H(\mathbf{y}^*, t^*)}}(\mathbf{y}^*, t^*)$ is an optimal solution of* (12).

Proof. Let $(\mathbf{z}^*, s^*)$ be an optimal solution of (12). For every $(\mathbf{y}, t)$ in the feasible set of (11) we have that $(\mathbf{z}, s) = \frac{(\mathbf{y},t)}{\sqrt{f_2(\mathbf{y},t)}}$ is feasible for (12); thus, by the optimality of $(\mathbf{z}^*, s^*)$, we have

$$v_2 = f_1^H(\mathbf{z}^*, s^*) \leq f_1^H(\mathbf{z}, s) = \frac{f_1^H(\mathbf{y}, t)}{f_2^H(\mathbf{y}, t)}.$$

Minimizing over all feasible solutions $(\mathbf{y}, t)$ of (11), the latter inequality implies $v_2 \leq v_1$. On the other hand,

$$v_1 \leq \frac{f_1^H(\mathbf{z}^*, s^*)}{f_2^H(\mathbf{z}^*, s^*)} = f_1^H(\mathbf{z}^*, s^*).$$

Thus, combining the latter inequality with $v_2 \leq v_1$, we obtain

$$v_1 \leq f_1^H(\mathbf{z}^*, s^*) = v_2 \leq v_1.$$

Therefore, $(\mathbf{z}^*, s^*)$ is an optimal solution of problem (11) and $v_1 = v_2$, proving parts (*i*), (*ii*) and (*iii*) of the lemma. To prove part (*iv*), let $(\mathbf{y}^*, t^*)$ be an optimal solution of (11). Then $(\mathbf{z}^*, s^*) = \frac{1}{\sqrt{f_2^H(\mathbf{y}^*, t^*)}}(\mathbf{y}^*, t^*)$ is feasible for (12) and satisfies

$$f_1^H(\mathbf{z}^*, s^*) = \frac{f_1^H(\mathbf{y}^*, t^*)}{f_2^H(\mathbf{y}^*, t^*)} = v_1 = v_2,$$

establishing the optimality of $(\mathbf{z}^*, s^*)$. □

3.2. Attainment of the Optimal Solution of (QCRQ)

We now return to the original problem (10). While the solution of the closely related problem (11) is attained, the solution of (10) is not necessarily attained. Theorem 3.2 below establishes a sufficient condition for the attainability of the solution of (10). This condition is expressed in terms of the optimal values of (H) and of the following problem constructed from (H) by restricting s to be zero:

$$(\mathrm{H}_0): \quad \begin{array}{ll} \min & f_1^H(\mathbf{z},0) \\ \text{s.t.} & f_2^H(\mathbf{z},0)=1, \\ & g_i^H(\mathbf{z},0)\le 0, \quad i=1,\dots,m, \\ & \mathbf{z}\in\mathbb{R}^n. \end{array} \tag{13}$$

Theorem 3.2. *Suppose that*

$$\mathrm{val}\,(\mathrm{H}) < \mathrm{val}\,(\mathrm{H}_0). \tag{14}$$

Let $(\mathbf{z}^*, s^*)$ *be an optimal solution of* (H) *(problem* (12)*). Then* $s^* \neq 0$ *and* $\mathbf{x}^* = \frac{1}{s^*}\mathbf{z}^*$ *is an optimal solution of* (QCRQ). *In particular, the optimal solution of* (QCRQ) *is attained.*

Proof. Suppose in contradiction that $s^* = 0$. Then

$$f_1^H(\mathbf{z}^*,0) = \mathrm{val}\,(\mathrm{H}). \tag{15}$$

On the other hand,

$$f_1(\mathbf{z}^*,0) \ge \min_{\mathbf{z}}\{f_1^H(\mathbf{z},0) : f_2^H(\mathbf{z},0)=1,\ g_i^H(\mathbf{z},0)\le 0\} = \mathrm{val}\,(\mathrm{H}_0).$$

Combining the latter inequality with (15) we obtain that $\mathrm{val}\,(\mathrm{H}) \ge \mathrm{val}\,(\mathrm{H}_0)$, which is a contradiction to (14). Thus, $s^* \neq 0$. By Lemma 3.1(*iii*), $(\mathbf{z}^*, s^*)$ is an optimal solution of (11) and since $s^* \neq 0$ we also have that $(\mathbf{z}^*, s^*)$ is an optimal solution of (10). Finally, by the construction of (10) it follows that $\mathbf{x}^* = \frac{1}{s^*}\mathbf{z}^*$ is an optimal solution of (QCRQ). □

Remark 3.3. Since (H_0) is constructed from (H) by restricting s to be zero, weak inequality $\mathrm{val}\,(\mathrm{H}) \le \mathrm{val}\,(\mathrm{H}_0)$ is always satisfied.

The following corollary is a direct consequence of Theorem 3.2.

Corollary 3.4. *Consider the QCRQ problem* (3) *and assume that*

$$\exists \gamma_1,\dots,\gamma_m > 0: \quad \textstyle\sum_{i=1}^m \gamma_i \mathbf{B}_i \succ \mathbf{0}. \tag{16}$$

Then condition (14) *is satisfied.*

Proof. By condition (16), the feasible set of (H_0) is empty which implies that $\mathrm{val}\,(\mathrm{H}_0) = \infty$, proving the validity of (14). □

The condition (16) is satisfied for example in the case of box constraints of the form $(x_i - l)(x_i - u) \le 0,\ i = 1,\dots,n$.

Remark 3.5. Note that under condition (16), the feasible set of (QCRQ) is compact so attainment of the optimal solution is not really an issue. It is still important to note that in this case condition (14) is satisfied since it also guarantees that the QCRQ problem is equivalent to the homogenized problem (H).

In order to validate condition (14), one requires to solve two nonconvex homogeneous quadratic problems, which seems to be a difficult task. However, in some important cases, the values of the corresponding problems ((H) and (H_0)) can be either computed exactly or bounded. Specifically, suppose that the constraint set is of the form

$$X_B = \{\mathbf{x} \in \mathbb{R}^n : \ell_i \leq \|\mathbf{L}_i\mathbf{x}\|^2 \leq u_i,\ i = 1, \dots, p\}, \tag{17}$$

where $0 \leq \ell_i \leq u_i < \infty$ for $i = 1, \dots, p$ and $\mathbf{L}_i \in \mathbb{R}^{n_i \times n}$ ($n_1, \dots, n_p$ are positive integers). We assume that X_B is nonempty – (an assumption that is automatically satisfied when $\ell_i = 0$ for all i). Then, an application of Theorem 3.2, allows us to write an explicit condition for the attainment of the optimal solution, as stated in the following theorem.

Theorem 3.6. *Consider problem* (QCRQ) *(problem* (3)*) with a nonempty feasible set $X = X_B$ given in* (17)*. Suppose that*

$$\lambda_{\min}(\mathbf{M}_1, \mathbf{M}_2) < \lambda_{\min}(\mathbf{F}^T\mathbf{A}_1\mathbf{F}, \mathbf{F}^T\mathbf{A}_2\mathbf{F}), \tag{18}$$

where

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{F}^T\mathbf{A}_1\mathbf{F} & \mathbf{F}^T(\mathbf{A}_1\mathbf{x}_0 + \mathbf{b}_1) \\ (\mathbf{A}_1\mathbf{x}_0 + \mathbf{b}_1)^T\mathbf{F} & \mathbf{x}_0^T\mathbf{A}_1\mathbf{x}_0 + 2\mathbf{b}_1^T\mathbf{x}_0 + c_1 \end{pmatrix}, \tag{19}$$

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{F}^T\mathbf{A}_2\mathbf{F} & \mathbf{F}^T(\mathbf{A}_2\mathbf{x}_0 + \mathbf{b}_2) \\ (\mathbf{A}_2\mathbf{x}_0 + \mathbf{b}_2)^T\mathbf{F} & \mathbf{x}_0^T\mathbf{A}_2\mathbf{x}_0 + 2\mathbf{b}_2^T\mathbf{x}_0 + c_2 \end{pmatrix} \tag{20}$$

and $\mathbf{F} \in \mathbb{R}^{n\times k}$ is a matrix whose columns form an orthonormal basis for $\bigcap_{i=1}^p$ Null($\mathbf{L}_i$) (k being the dimension of the intersection of the null spaces), and $\mathbf{x}_0$ is an arbitrary point in X_B. Then the optimal solution of problem (QCRQ) *is attained.*

Proof. Problem (H_0) corresponding to the constraint set X_B can be written as

$$\min\{\mathbf{z}^T\mathbf{A}_1\mathbf{z} : \mathbf{z}^T\mathbf{A}_2\mathbf{z} = 1,\ \|\mathbf{L}_i\mathbf{z}\|^2 = 0,\ i = 1, \dots, p\},$$

which is the same as

$$\min\{\mathbf{z}^T\mathbf{A}_1\mathbf{z} : \mathbf{z}^T\mathbf{A}_2\mathbf{z} = 1,\ \mathbf{z} \in V\}, \tag{21}$$

where $V = \bigcap_{i=1}^p \mathrm{Null}(\mathbf{L}_i)$. Making the change of variables $\mathbf{z} = \mathbf{F}\mathbf{w}$, (21) becomes:

$$\min_{\mathbf{w}}\{\mathbf{w}^T\mathbf{F}^T\mathbf{A}_1\mathbf{F}\mathbf{w} : \mathbf{w}^T\mathbf{F}^T\mathbf{A}_2\mathbf{F}\mathbf{w} = 1\},$$

and we thus conclude that $\mathrm{val}\,(\mathrm{H}_0) = \lambda_{\min}(\mathbf{F}^T\mathbf{A}_1\mathbf{F}, \mathbf{F}^T\mathbf{A}_2\mathbf{F})$.
Denote:

$$Y = \{(\mathbf{z}, s) : f_2^H(\mathbf{z}, s) = 1,\ \ell_i s^2 \leq \|\mathbf{L}_i\mathbf{z}\|^2 \leq u_i s^2\},$$
$$Z = \{(\mathbf{z}, s) : f_2^H(\mathbf{z}, s) = 1,\ \ell_i s^2 \leq \|\mathbf{L}_i\mathbf{z}\|^2 \leq u_i s^2,\ \mathbf{z} = s\mathbf{x}_0 + \mathbf{F}\mathbf{w} \text{ for some } \mathbf{w} \in \mathbb{R}^k\},$$

where $\mathbf{x}_0$ is some vector in X_B as defined in the premise of the theorem. Evidently $Z \subseteq Y$ and as a result

$$\begin{aligned}\text{val (HMP)} &= \min_{\mathbf{z},s}\{f_1^H(\mathbf{z},s) : (\mathbf{z},s) \in Y\} \\ &\leq \min_{\mathbf{z},s}\{f_1^H(\mathbf{z},s) : (\mathbf{z},s) \in Z\}. \end{aligned} \tag{22}$$

Note that by $\mathbf{x}_0 \in X_B$, every $\mathbf{z}$ of the form $\mathbf{z} = s\mathbf{x}_0 + \mathbf{F}\mathbf{w}$ readily satisfies the constraints $\ell_i s^2 \leq \|\mathbf{L}_i\mathbf{z}\|^2 \leq u_i s^2$ and as a result these constraints can be omitted in problem (22). Thus, problem (22) reduces to

$$\min_{\mathbf{w},s}\{f_1^H(s\mathbf{x}_0 + \mathbf{F}\mathbf{w}, s) : f_2(s\mathbf{x}_0 + \mathbf{F}\mathbf{w}, s) = 1\},$$

whose optimal solution is equal to $\lambda_{\min}(\mathbf{M}_1, \mathbf{M}_2)$, where $\mathbf{M}_1$ and $\mathbf{M}_2$ are defined in (19) and (20) respectively. Therefore,

$$\begin{aligned}\text{val (H)} &\leq \min_{\mathbf{z},s}\{f_1^H(\mathbf{z},s) : (\mathbf{z},s) \in Z\} = \lambda_{\min}(\mathbf{M}_1, \mathbf{M}_2) \\ &< \lambda_{\min}(\mathbf{F}^T\mathbf{A}_1\mathbf{F}, \mathbf{F}^T\mathbf{A}_2\mathbf{F}) = \text{val}\,(\text{H}_0),\end{aligned}$$

where the strict inequality follows from (18). Finally, invoking Theorem 3.2, we conclude that the minimum of problem (QCRQ) is attained. □

Remark 3.7. Weak inequality is always satisfied in (18): the matrix in the right-hand side of (18) is a principal submatrix of the one in the left-hand side. Hence, by the interlacing theorem of eigenvalues [26, Theorem 7.8], weak inequality holds.

Remark 3.8. It can be shown that the expression in the left-hand side of (18) does not depend on the specific choice of the feasible vector $\mathbf{x}_0 \in X_B$.

Example 3.9. If $\ell_i = 0$ for $i = 1, \ldots, p$, then $\mathbf{x}_0$ can be chosen to be $\mathbf{0}$ and condition (18) reduces to $\lambda_{\min}(\mathbf{M}_1, \mathbf{M}_2) < \lambda_{\min}(\mathbf{F}^T\mathbf{A}\mathbf{F}, \mathbf{F}^T\mathbf{A}_2\mathbf{F})$ where

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{F}^T\mathbf{A}_1\mathbf{F} & \mathbf{F}^T\mathbf{b}_1 \\ \mathbf{b}_1^T\mathbf{F} & c_1 \end{pmatrix}, \qquad \mathbf{M}_2 = \begin{pmatrix} \mathbf{F}^T\mathbf{A}_2\mathbf{F} & \mathbf{F}^T\mathbf{b}_2 \\ \mathbf{b}_2^T\mathbf{F} & c_2 \end{pmatrix},$$

which is a generalization of the condition (9) derived in [6] for problem (QCRQ) with $X = X_1$.

Example 3.10. The unconstrained TLS problem is the QCRQ problem with $X = X_B$ and

$$p = 1, \qquad u_1 = 1, \qquad \ell_1 = 0, \qquad \mathbf{L} = \mathbf{0},$$

$$\mathbf{A}_1 = \mathbf{A}^T\mathbf{A}, \qquad \mathbf{b}_1 = -\mathbf{A}^T\mathbf{b}, \qquad c_1 = \|\mathbf{b}\|^2, \qquad \mathbf{A}_2 = \mathbf{I}, \qquad \mathbf{b}_2 = \mathbf{0}, \qquad c_2 = 1.$$

In this case the condition (18) reduces to the well known condition (8) for attainability of the TLS solution.

4. Algorithmic and Modeling Applications

4.1. A Convex Semidefinite Relaxation of QCRQ

In the previous section, Theorem 3.2 warrants that under condition (14) the QCRQ problem can be written as the nonconvex quadratic optimization problem (H) (problem (12)). Generally speaking, nonconvex quadratic problems are difficult to solve.

However, they can be approximated by their semidefinite relaxation (SDR) [7, 25]. For the sake of completeness we first briefly describe the well-known construction of the SDR. First, we make the change of variables $\mathbf{w} = (\mathbf{z}^T, s)^T$ and represent (12) as

$$\begin{array}{ll} \min_{\mathbf{w}} & \mathbf{w}^T \mathcal{M}(f_1)\mathbf{w} \\ \text{s.t.} & \mathbf{w}^T \mathcal{M}(f_2)\mathbf{w} = 1, \\ & \mathbf{w}^T \mathcal{M}(g_i)\mathbf{w} \le 0, \quad i = 1, \ldots, m, \end{array} \tag{23}$$

where for a given quadratic function $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$, the associated matrix is defined by

$$\mathcal{M}(f) \equiv \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & c \end{pmatrix}.$$

Using the fact that a $p \times p$ positive semidefinite matrix $\mathbf{W}$ has rank one if and only if $\mathbf{W} = \mathbf{w}\mathbf{w}^T$ for some $\mathbf{w} \in \mathbb{R}^p$, we conclude that problem (23) can be rewritten as

$$\begin{array}{ll} \min_{\mathbf{W}} & \mathrm{Tr}(\mathcal{M}(f_1)\mathbf{W}) \\ \text{s.t.} & \mathrm{Tr}(\mathcal{M}(f_2)\mathbf{W}) = 1, \\ & \mathrm{Tr}(\mathcal{M}(g_i)\mathbf{W}) \le 0, \quad i = 1, \ldots, m, \\ & \mathrm{rank}(\mathbf{W}) = 1. \end{array}$$

Dropping the rank-one constraint we obtain the semidefinite relaxation of the QCRQ problem (3)

$$(\mathrm{SDR}): \quad \begin{array}{ll} \min_{\mathbf{W}} & \mathrm{Tr}(\mathcal{M}(f_1)\mathbf{W}) \\ \text{s.t.} & \mathrm{Tr}(\mathcal{M}(f_2)\mathbf{W}) = 1, \\ & \mathrm{Tr}(\mathcal{M}(g_i)\mathbf{W}) \le 0, \quad i = 1, \ldots, m, \\ & \mathbf{W} \succeq \mathbf{0}. \end{array} \tag{24}$$

The above problem is a convex relaxation of the QCRQ problem and its optimal value provides a lower bound, i.e., $\mathrm{val}\,(\mathrm{SDR}) \le \mathrm{val}\,(\mathrm{QCRQ})$; however, as opposed to the QCRQ problem (3), problem SDR is a well structured convex problem that can be solved efficiently via e.g., interior point methods [7, 21].

As immediate consequence of Theorem 3.2 we obtain the following corollary which describes how to extract the optimal solution of (QCRQ) from the SDR problem whenever the semidefinite relaxation is tight.

Corollary 4.1. *Suppose that the semidefinite relaxation* (SDR) *has an optimal solution* $\mathbf{W}$ *with rank one and that the sufficient condition* (14) *holds. Then the exact solution of the original* QCRQ *problem is* $\frac{\mathbf{v}}{t}$ *where* $(\mathbf{v}^T, t)^T \in \mathbb{R}^{n+1}$ *is an eigenvector of the matrix* $\mathbf{W}$ *associated with the maximum eigenvalue.*

Motivated by the "tight case", we are now ready to define an algorithm for computing an approximate solution of the QCRQ problem.

Algorithm QCRQ-SDR.

1. Solve the SDR problem (24) and obtain a solution $\mathbf{W} \in \mathbb{R}^{(n+1)\times(n+1)}$.
2. Compute the eigenvector $\begin{pmatrix} \mathbf{v} \\ t \end{pmatrix}$ associated with the maximum eigenvalue of $\mathbf{W}$.
3. The output of the algorithm is the vector $\mathbf{v}/t$.

In the next example we show that, at least for random instances of the RTLS problem with box constraints, the semidefinite relaxation tends to be tight and consequently the QCRQ-SDR algorithm provides an *exact* solution of the QCRQ problem.

Example 4.2. Consider randomly generated instances of the RTLS problem with box constraints. The experiments were performed in MATLAB and the semidefinite programs were solved using SeDuMi [24]. The admissible set of vectors is given by[1]

$$X = \{\mathbf{x} \in \mathbb{R}^n : x_i^2 \leq 1,\ i = 1, \ldots, n\}.$$

By Corollary 3.4, the sufficient condition (14) is satisfied in this case. The exact random linear system is

$$\mathbf{A}_{\mathrm{T}}\mathbf{x}_{\mathrm{T}} = \mathbf{b}_{\mathrm{T}}, \tag{25}$$

where $\mathbf{A}_{\mathrm{T}}, \mathbf{x}_{\mathrm{T}}$ are generated by the MATLAB commands rand(m,n), rand(n,1) and $\mathbf{b}_{\mathrm{T}}$ is defined by the relation (25). The observed matrix and vector $\mathbf{A}$ and $\mathbf{b}$ are generated by adding white nose:

$$\mathbf{A} = \mathbf{A}_{\mathrm{T}} + \sigma\mathbf{E}, \qquad \mathbf{b} = \mathbf{b}_{\mathrm{T}} + \sigma\mathbf{w},$$

with E=randn(m,n), w=randn(m,1) and σ being the standard deviation. We tested the QCRQ-SDR algorithm for several sizes:

$$(m, n) = (10, 10), (15, 10), (50, 50), (75, 50), (100, 100), (150, 100)$$

and for three choices of noise level: $\sigma = 10^{-1}, 10^{-2}, 10^{-3}$. In the following table each entry is the number of runs out of 100 realizations in which the SDR was tight. Here by "tight" we mean a run in which the sum of the eigenvalues excluding the maximum eigenvalue is smaller than 10^{-10}.

m	n	σ		
		10^{-3}	10^{-2}	10^{-1}
10	10	100	96	86
50	50	99	97	54
100	100	96	96	29
10	15	100	100	100
50	75	100	100	100
100	150	100	100	100

Note that for the lower noise levels $\sigma = 10^{-3}, 10^{-2}$, the SDR was almost always tight and it was *always* tight in the rectangular instances.
When the SDR is not tight the output of the SDR-QCQR algorithm can be used as a good starting point for general-purpose optimization algorithms.

Remark 4.3 (Case $m = 1$). When $m = 1$ the SDR (24) always has an optimal rank-one solution. This result follows from the rank reduction algorithm of Pataki [22]

[1]Note that here we used the quadratic representation $x_i^2 \leq 1$ rather than the linear representation $-1 \leq x_i \leq 1$. In Section 4.2 we will show that indeed the quadratic representation is better in some sense.

which shows in particular that an optimal solution of a semidefinite program with two constraints can be transformed into a rank-one optimal solution, see also algorithm RED in [3]. A related result was proven in [6] where it was shown that the QCRQ problem with $X = X_1$ can be solved using an associated SDP which is in fact the dual problem of the SDR described in this section.

4.2. Modeling of Linear Constraints

We now address the question of representation of the linear constraints. More specifically, suppose that one of the constraints defining the set X are double-sided linear constraints of the form

$$\ell \leq \mathbf{a}^T\mathbf{x} \leq u, \tag{26}$$

where $\ell < u$ and $\mathbf{a} \in \mathbb{R}^n$ is a nonzero vector. The two constraints (26) can be written in the following quadratic form:

$$\left(\mathbf{a}^T\mathbf{x} - \frac{\ell+u}{2}\right)^2 \leq \frac{(u-\ell)^2}{4}. \tag{27}$$

The natural question that arises is whether or not one of the representations (26) or (27) is better. We will show that using the quadratic representation (27) is preferable in two senses:

(i) the sufficient condition (14) is more likely to be satisfied for the quadratic representation and,

(ii) the SDR problem provides a tighter lower bound on the QCRQ problem.

Consider then the QCRQ problem (3) with the following set of admissible of vectors:

$$X_{\mathrm{L}} = \{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0,\ \ell \leq \mathbf{a}^T\mathbf{x} \leq u,\ i = 1,\dots,p\}, \tag{28}$$

where g_i are given by (5). The alternative representation of the above set is

$$X_{\mathrm{Q}} = \left\{\mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq 0,\ \left(\mathbf{a}^T\mathbf{x} - \frac{\ell+u}{2}\right)^2 \leq \frac{(u-\ell)^2}{4},\ i = 1,\dots,p\right\} \tag{29}$$

Of course, $X_{\mathrm{L}} = X_{\mathrm{Q}}$, but as we shall see the specific representation has great influence on issues such as the validity of the sufficient condition and the optimal value of the SDR.

We recall that our solution approach is to first homogenize the problem and then use the semidefinite relaxation. In Section 3 we derived the sufficient condition (14) that guarantees both the attainment of the solution and the equivalence of the problem to a corresponding nonconvex homogenous problem. The sufficient condition for the QCRQ problem with $X = X_{\mathrm{L}}$ is

$$\mathrm{val}(\mathrm{H}^{\mathrm{L}}) < \mathrm{val}(\mathrm{H}_0^{\mathrm{L}}), \tag{30}$$

where $(\mathrm{H}^{\mathrm{L}})$ and $(\mathrm{H}_0^{\mathrm{L}})$ are the problems

$$(\mathrm{H}^{\mathrm{L}}): \quad \begin{array}{ll} \min_{\mathbf{z},s} & f_1^H(\mathbf{z},s) \\ \text{s.t.} & f_2^H(\mathbf{z},s) = 1, \\ & g_i^H(\mathbf{z},s) \leq 0, \\ & \mathbf{a}^T\mathbf{z}s \leq us^2, \\ & \mathbf{a}^T\mathbf{z}s \geq \ell s^2. \end{array} \qquad (\mathrm{H}_0^{\mathrm{L}}): \quad \begin{array}{ll} \min_{\mathbf{z}} & f_1^H(\mathbf{z},0) \\ \text{s.t.} & f_2^H(\mathbf{z},0) = 1, \\ & g_i^H(\mathbf{z},0) \leq 0. \end{array}$$

Similarly, the sufficient condition for the QCRQ problem with $X = X_{\mathrm{Q}}$ is

$$\mathrm{val}(\mathrm{H}^{\mathrm{Q}}) < \mathrm{val}(\mathrm{H}_0^{\mathrm{Q}}), \tag{31}$$

where $(\mathrm{H}^{\mathrm{Q}})$ and $(\mathrm{H}_0^{\mathrm{Q}})$ are the problems

$$(\mathrm{H}^{\mathrm{Q}}): \quad \begin{array}{ll} \min_{\mathbf{z},s} & f_1^H(\mathbf{z},s) \\ \text{s.t.} & f_2^H(\mathbf{z},s) = 1, \\ & g_i^H(\mathbf{z},s) \le 0, \\ & \left(\mathbf{a}^T\mathbf{z} - \frac{u+\ell}{2}s\right)^2 \le \frac{u-\ell}{2}s^2. \end{array} \qquad (\mathrm{H}_0^{\mathrm{Q}}): \quad \begin{array}{ll} \min_{\mathbf{z}} & f_1^H(\mathbf{z},0) \\ \text{s.t.} & f_2^H(\mathbf{z},0) = 1, \\ & g_i^H(\mathbf{z},0) \le 0, \\ & \mathbf{a}^T\mathbf{z} = 0. \end{array}$$

The following result shows that if the sufficient condition is satisfied for the linear representation, then it surely holds for the quadratic representation.

Theorem 4.4. *Suppose that* $\mathrm{val}(H^{\mathrm{L}}) < \mathrm{val}(H_0^{\mathrm{L}})$ *holds. Then* $\mathrm{val}(H^{\mathrm{Q}}) < \mathrm{val}(H_0^{\mathrm{Q}})$ *is also satisfied.*

Proof. Suppose that condition $\mathrm{val}(\mathrm{H}^{\mathrm{L}}) < \mathrm{val}(\mathrm{H}_0^{\mathrm{L}})$ holds. Note that problem $(\mathrm{H}_0^{\mathrm{Q}})$ is essentially problem $(\mathrm{H}_0^{\mathrm{L}})$, but with an additional constraint: $\mathbf{a}^T\mathbf{z} = 0$, thus,

$$\mathrm{val}(\mathrm{H}_0^{\mathrm{L}}) \le \mathrm{val}(\mathrm{H}_0^{\mathrm{Q}}). \tag{32}$$

On the other hand, a straightforward computation shows that the inequality $(\mathbf{a}^T\mathbf{z} - \frac{u+\ell}{2}s)^2 \le \frac{u-\ell}{2}s^2$ follows from the pair of inequalities $\mathbf{a}^T\mathbf{z}s \le us^2$, $\mathbf{a}^T\mathbf{z}s \ge \ell s^2$, showing that the feasible set of $(\mathrm{H}^{\mathrm{L}})$ is contained in the feasible set $(\mathrm{H}^{\mathrm{Q}})$. Consequently,

$$\mathrm{val}(\mathrm{H}^{\mathrm{Q}}) \le \mathrm{val}(\mathrm{H}^{\mathrm{L}}). \tag{33}$$

Finally, combining (32) and (33) along with (30) we obtain:

$$\mathrm{val}(\mathrm{H}^{\mathrm{Q}}) \le \mathrm{val}(\mathrm{H}^{\mathrm{L}}) < \mathrm{val}(\mathrm{H}_0^{\mathrm{L}}) \le \mathrm{val}(\mathrm{H}_0^{\mathrm{Q}}),$$

proving the validity of condition (31). □

Example 4.7 at the end of the section demonstrates the fact that the reverse result does not hold, i.e., it is possible that the sufficient condition is satisfied for the quadratic representation but not for the linear representation.

The SDR associated with the linear representation (28) is

$$(\text{SDR-L}): \quad \begin{array}{ll} \min_{\mathbf{W}} & \mathrm{Tr}(\mathcal{M}(f_1)\mathbf{W}) \\ \text{s.t.} & \mathrm{Tr}(\mathcal{M}(f_2)\mathbf{W}) = 1, \\ & \mathrm{Tr}(\mathcal{M}(g_i)\mathbf{W}) \le 0, \quad i = 1,\ldots,p, \\ & \mathrm{Tr}(\mathbf{D}_1\mathbf{W}) \le 0, \\ & \mathrm{Tr}(\mathbf{D}_2\mathbf{W}) \le 0, \\ & \mathbf{W} \succeq \mathbf{0}, \end{array}$$

where

$$\mathbf{D}_1 = \begin{pmatrix} \mathbf{0} & \frac{1}{2}\mathbf{a} \\ \frac{1}{2}\mathbf{a}^T & -u \end{pmatrix}, \qquad \mathbf{D}_2 = \begin{pmatrix} \mathbf{0} & -\frac{1}{2}\mathbf{a} \\ -\frac{1}{2}\mathbf{a}^T & \ell \end{pmatrix},$$

and the SDR associated with the quadratic representation (29) is

$$\text{(SDR-Q)}: \quad \begin{array}{ll} \min_{\mathbf{W}} & \mathrm{Tr}(\mathcal{M}(f_1)\mathbf{W}) \\ \text{s.t.} & \mathrm{Tr}(\mathcal{M}(f_2)\mathbf{W}) = 1, \\ & \mathrm{Tr}(\mathcal{M}(g_i)\mathbf{W}) \leq 0, \quad i = 1, \ldots, p, \\ & \mathrm{Tr}(\mathbf{D}\mathbf{W}) \leq 0, \\ & \mathbf{W} \succeq \mathbf{0}, \end{array}$$

where $\mathbf{D} = \begin{pmatrix} \mathbf{a}\mathbf{a}^T & -\frac{\ell+u}{2}\mathbf{a} \\ -\frac{\ell+u}{2}\mathbf{a}^T & u\ell \end{pmatrix}$.

The optimal values of (SDR-L) and (SDR-Q) are both lower bounds on the optimal value of the QCRQ problem. In the next result we show that (SDR-Q) consistently provides a *tighter* lower bound.

Theorem 4.5. val (SDR-L) $\leq$ val (SDR-Q).

Proof. To prove the result, it suffice to show that the feasible set of (SDR-Q) is contained in the feasible set of (SDR-L). Suppose that $\mathbf{W}$ is a feasible solution of (SDR-Q). The first $p+1$ constraints in (SDR-L) are satisfied since they are the same as the first $p+1$ constraints of (SDR-Q). It is easy to show that

$$\mathbf{D} \succeq (u-\ell)\mathbf{D}_1, \qquad \mathbf{D} \succeq (u-\ell)\mathbf{D}_2. \tag{34}$$

Combining (34) along with the last constraints in (SDR-Q): $\mathbf{W} \succeq \mathbf{0}, \mathrm{Tr}(\mathbf{D}\mathbf{W}) \leq 0$, we have

$$\mathrm{Tr}(\mathbf{D}_i\mathbf{W}) \leq \frac{1}{u-\ell}\mathrm{Tr}(\mathbf{D}\mathbf{W}) \leq 0, \quad i = 1, 2.$$

Therefore, $\mathbf{W}$ is a feasible solution of (SDR-L), proving the result. □

Remark 4.6. In the presence of several double-sided linear constraints, repeated application of Theorems 4.4 and 4.5 shows that it is best to represent all of them as quadratic constraints in the sense that (i) the sufficient condition is more likely to be satisfied and (ii) the SDR arising from the quadratic representation provides a tight lower bound than the one associated with the linear representation.

Example 4.7 (Quadratic versus Linear: satisfiability of the sufficient condition. Consider the RTLS problem (2) with $m = 3,\ n = 2$ and

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}.$$

The set of admissible vectors is given by either

$$X_{\mathrm{L}} = \{(x_1, x_2) : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$$

or

$$X_{\mathrm{Q}} = \left\{(x_1, x_2) : x_1^2 \leq 1,\ x_2^2 \leq 1\right\}.$$

The sufficient condition is satisfied by Corollary 3.4. We solved the corresponding SDR using SeDuMi and it turned out that it is tight. Therefore, algorithm QCRQ-SDR produces the exact solution which is $(-1, 0.4111)^T$.

The sufficient condition is not satisfied for the linear representation. To see this, note that problem (H^{L}) is given by

$$\min\left\{\|\mathbf{A}\mathbf{x}-s\mathbf{b}\|^2 : x_1^2+x_2^2+s^2=1,\ -s^2\le x_i s\le s^2,\ i=1,2\right\}. \tag{35}$$

The solution of the semidefinite relaxation of the above problem was solved and the optimal solution of the SDR is the matrix (given with four digits of accuracy):

$$\begin{pmatrix} 0.6161 & -0.4863 & 0 \\ -0.4863 & 0.3839 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is of rank one. Consequently, the optimal solution of (35) is the unit-norm maximum eigenvector of the above matrix: $(-0.7849, 0.6196, 0)^T$. Since the last component of the optimal solution of (H^{L}) is zero, it follows that $\mathrm{val}(\mathrm{H}^{\mathrm{L}}) = \mathrm{val}(\mathrm{H}_0^{\mathrm{L}})$, showing that the sufficient condition is not satisfied. We note that this example is extreme in the sense that the quadratic representation provides us with the global optimal solution while the linear representation is useless!

Example 4.7 (Quadratic versus Linear: quality of the SDR). Consider the RTLS problem with the same matrix and vector as in Example 4.7. The set of admissible vectors is given by either

$$X_{\mathrm{L}} = \left\{(x_1,x_2) : x_1^2+x_2^2\le 1.2,\ -1\le x_1\le 1,\ -1\le x_2\le 1\right\}$$

or

$$X_{\mathrm{Q}} = \left\{(x_1,x_2) : x_1^2+x_2^2\le 1.2,\ x_1^2\le 1,\ x_2^2\le 1\right\}.$$

By Corollary 3.4, the sufficient condition holds for both representations. The optimal solution of the SDR associated with the linear representation was computed and is not of rank one. The corresponding optimal value is 2.4405, which is only a lower bound. The output of algorithm QCRQ-SDR is $(-1.0167, 0.4079)^T$. This is of course just an approximation of the exact solution. The optimal solution of the SDR associated with quadratic representation is of rank-one and its value is 2.4785 (this is the value of the global optimal solution). The output of algorithm QCRQ-SDR is $(-1, 0.411)^T$, which is the *exact* solution of the RTLS problem.

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