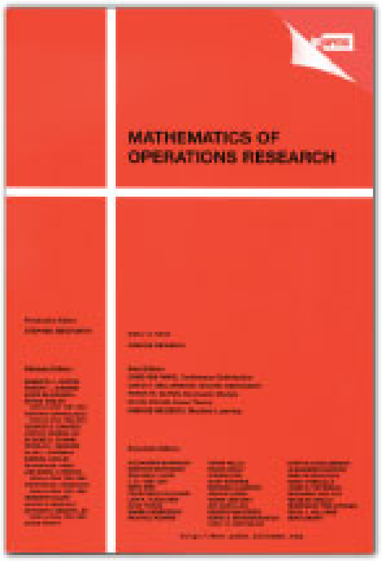

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Amir Beckhttp://iew3.technion.ac.il/Home/Users/becka0.html, Marc Teboullehttp://www.math.tau.ac.il/~teboulle/,

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A Linearly Convergent Dual-Based Gradient Projection Algorithm for Quadratically Constrained Convex Minimization

Amir Beck
Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa 32000, Israel,
becka@ie.technion.ac.il, http://iew3.technion.ac.il/Home/Users/becka0.html

Marc Teboulle
School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv 69978, Israel,
teboulle@post.tau.ac.il, http://www.math.tau.ac.il/~teboulle/

This paper presents a new dual formulation for quadratically constrained convex programs. The special structure of the derived dual problem allows us to apply the gradient projection algorithm to produce a simple explicit method involving only elementary vector-matrix operations, proven to converge at a linear rate.

Key words: quadratically constrained problems; gradient projection algorithm; convex minimization; rate of convergence analysis
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1. Introduction. Gradient methods are probably the most basic and fundamental class of optimization algorithms. For constrained problems, these methods, known as gradient projection algorithms (GPA), have been extensively studied in the literature; see for example Goldstein [12], Levitin and Polyak [13], Bertsekas [4], Dunn [10], Calamai and Moré [9], Dunn [11], Luo and Tseng [17], and Wang and Xiu [27]. More details and further references on the GPA can be found in Bertsekas [5]. The main disadvantage of these methods is that without imposing strong assumptions on the problem's data, they exhibit a slow convergence rate, e.g., sublinear, and thus do not seem to be competitive with the modern interior point methods (IPM) which, for convex optimization problems, have attractive polynomial time complexity (Nesterov and Nemirovski [22]). However, the power of IPM has some drawbacks and limits. Indeed, these IPM require sophisticated and heavy computational tasks to be performed at each iteration, e.g., solving Newton's type systems. For very large-scale problems in the decision variables, a single iteration of such a polynomial time algorithm is often too expensive to be of practical use. This has led to a revived interest in the study of simple algorithms, such as gradient-based methods. Despite their apparent lack of efficiency, gradient methods appear to remain legitimate and affordable candidate algorithms for large-scale applications, which in particular do not often require highly accurate solutions. This has been substantiated by some recent theoretical and computational studies on gradient-based algorithms; see, e.g., Auslender and Teboulle [1], Beck and Teboulle [2], Bienstock [7], Ben-Tal et al. [3], and Nemirovski [20]. In particular, we mention the successful computational experiments recently reported by Bienstock [7] for approximating linear programs, which also uses a simple algorithm (that relies on the Frank-Wolfe method applied to an appropriate potential function); the work of Ben-Tal et al. [3] for solving very large-scale image reconstruction problems, which uses a general mirror descent method (see Nemirovski and Yudin [21]), and which has been recently proven by Beck and Teboulle [2] to be equivalent to a specific GPA; and the work of Nemirovski [20] for approximating large instances of the Lovasz capacity of a graph. All the aforementioned works have clearly shown that such simple gradient-type algorithms can be a viable alternative in practical solutions to large-scale problems, especially when high accuracy is not required, and that their theoretical efficiency deserves to be further studied.

The main advantage of GPA is its simplicity, provided that the orthogonal projection on the feasible set and the gradient of the objective function can be easily computed. Indeed, for minimizing an objective function f over a set of constraints $S \subset \mathbb{R}^n$, the basic GPA simply consists of iterating the formula

$$x^{k+1} = P_S(x^k - t\nabla f(x^k)), \quad x^0 \in \mathbb{R}^n, \tag{1}$$

where P_S denotes the orthogonal projection map and $t > 0$ is some appropriately chosen step size. For example, if the set of constraints is an affine space, then the resulting projection map at each iteration of GPA involves only matrix-vector multiplications. One of the main drawbacks of GPA, as just mentioned, is that its convergence rate is in general only sublinear, unless some further and often restrictive assumptions on the problem's data are made (e.g., strong convexity). A natural question is thus to identify classes of problems for which on the one hand, the rate of convergence can be improved, say to linear, under weaker or reasonable assumptions, while, on the other hand, the simplicity of the algorithm of GPA will be preserved, namely, the projections and gradients, can be easily and analytically computed.

In this paper, we prove that for a class of quadratically constrained convex minimization problems (QCQP), both requirements can be achieved. We present a new and simple dual-based projected gradient method, namely, we derive a GPA-based algorithm which involves only simple matrix-vector multiplications producing a simple and provably linearly convergent algorithm. To achieve this goal requires us to find an adequate equivalent formulation of (QCQP) and a new line of analysis for proving its linear rate of convergence. Interestingly, the class (QCQP) enlightens the difficulties encountered in the double task of deriving a simple algorithm with a linear rate of convergence. Indeed, and to the best of our knowledge, none of the known results available in the current literature can be directly applied to our problem to produce our declared tasks; see §3.

Our first objective is to construct a new dual problem on which the GPA can be applied, namely, where the projections can be computed explicitly, and with an objective with a Lipschitz continuous gradient that can be easily computed. It turns out that one can construct such a dual problem with the desirable affine constraints, thus rendering the computation of the projections a trivial task. The dual objective function we derive possesses an interesting structure, and is proven to be continuously differentiable with a Lipschitz continuous gradient that is easy to compute. This is developed in §2, where we also state the resulting explicit dual gradient projection (DGP) algorithm.

Our second and main contribution will be to prove that the proposed DGP algorithm is linearly convergent. The latter task requires us to develop a specific and novel analysis, which we develop in §3. We end the paper with a short appendix that includes some simple technical results that are used throughout this paper.

2. A dual approach for (QCQP). We consider the minimization of a strictly convex quadratic function under strictly convex quadratic inequalities:

$$\text{(QCQP)}\quad \text{minimize}\{x^T Q_0 x + 2b_0^T x\colon\ x^T Q_i x + 2b_i^T x \le c_i,\ i = 1, 2, \ldots, m\},$$

where $Q_0, Q_1, \ldots, Q_m$ are $n \times n$ positive definite matrices, $b_0, b_1, \ldots, b_m \in \mathbb{R}^n$ and $c_1, \ldots, c_m \in \mathbb{R}$.

A classical problem that can be cast as (QCQP) is the problem of projection onto the intersection of ellipsoids. For a recent work considering this problem and its applications to nonlinear programming, we refer the reader to the work of Lin and Han [14], and the references therein.

Throughout, we assume that (QCQP) is strictly feasible; as a result, we have $c_i + b_i^T Q_i^{-1} b_i > 0\ \forall\, i$. Because we assume that Slater's condition is satisfied, we thus have that strong duality is satisfied and the optimal value of (QCQP) is equal to the attained optimal value of the dual problem. To solve (QCQP), we would like to derive a dual problem that can be easily solved via the GPA, namely, with an analytical iteration formula.

2.1. Standard dual formulations. A standard dual formulation of (QCQP) can be easily shown to be given by

$$\begin{aligned} \text{maximize}\quad & -\left(b_0 + \sum_{i=1}^m b_i\lambda_i\right)^T \left(Q_0 + \sum_{i=1}^m \lambda_i Q_i\right)^{-1} \left(b_0 + \sum_{i=1}^m b_i\lambda_i\right) - \sum_{i=1}^m \lambda_i c_i \\ \text{s.t.}\quad & \lambda_i \ge 0, \quad i = 1, 2, \ldots, m. \end{aligned}$$

The trouble with this formulation is that each function value or gradient calculation of the dual objective function consists of inverting a matrix. Thus, for very large-scale problems in the design variables n (even with small m), a gradient-based algorithm would require us to compute the inverse of a huge matrix (and in general with no specific structure) at each iteration, a task that is practically intractable. Our goal is to define an algorithm that consists only of matrix-vector multiplications and does not involve any matrix inversion at each iteration (excluding a one-time preprocessing calculation that might involve matrix inversion). To achieve this task, we use a decomposition approach (see, e.g., Bertsekas and Tsitsiklis [6]), where we duplicate the variables to obtain simpler expressions for the dual problem. An equivalent primal problem to (QCQP) is

$$\begin{aligned} \text{minimize}\quad & \frac{1}{m}\sum_{i=1}^m x_i^T Q_0 x_i + 2b_0^T x \\ \text{s.t.}\quad & x_i^T Q_i x_i + 2b_i^T x_i \le c_i, \quad i = 1, 2, \ldots, m, \\ & x_i = x, \quad i = 1, \ldots, m. \end{aligned}$$

Assigning a Lagrange multiplier $\lambda_i \in \mathbb{R}^n$ for each linear equality constraint, we obtain the following dual:

$$\text{maximize}\left\{\sum_{i=1}^m g_i(\lambda_i)\colon \sum_{i=1}^m \lambda_i = b_0\right\},$$

where for each i, $g_i(\lambda_i) = \min\{(1/m)x_i^T Q_0 x_i + 2\lambda_i^T x_i\colon x_i^T Q_i x_i + 2b_i^T x_i \le c_i\}$. The difficulty here is that the function g_i does not have an explicit expression. The only case where it is possible to find an explicit expression for g_i is the case where $Q_0 = \alpha Q_i$ for some $\alpha > 0$ and for every i. The motivation behind the proposed new formulation given below is, thus, somehow to enforce the situation where Q_0 is equal to αQ_i for some $\alpha > 0$. It turns out that this can be done by adding a redundant constraint.

2.2. A new dual formulation for (QCQP). One of the key arguments in establishing the new dual formulation is to guarantee that we can write Q_0 as a positive linear combination of the matrices Q_i, i.e., that $Q_0 = \sum_{i=1}^m \beta_i Q_i$ where $\beta_1, \beta_2, \ldots, \beta_m > 0$. Of course, there is no guarantee that there exists such a linear combination. This is the reason why we will add a redundant constraint to the original problem (QCQP), which will enforce the validity of such a linear combination. The following lemma allows us to do that. In the sequel, $\lambda_{\min}(Q)$ ($\lambda_{\max}(Q)$) denotes the minimum (maximum) eigenvalue of Q.

Lemma 2.1. *Let $Q_0, \ldots, Q_m$ be $n \times n$ positive definite matrices, $b_1, \ldots, b_m \in \mathbb{R}^n$ and $c_1, \ldots, c_m \in \mathbb{R}$. Let $\beta_1, \ldots, \beta_m$ be m positive real numbers that satisfy the following inequality*:

$$\sum_{i=1}^m \beta_i \lambda_{\max}(Q_i) < \lambda_{\min}(Q_0). \tag{2}$$

Then, the following set of quadratic inequalities

$$x^T Q_i x + 2b_i^T x \le c_i, \quad i = 1, 2, \ldots, m, \tag{3}$$

imply the inequality

$$x^T Q_{m+1} x \le c_{m+1},$$

where

$$Q_{m+1} = Q_0 - \sum_{i=1}^m \beta_i Q_i, \qquad c_{m+1} = \lambda_{\max}(Q_{m+1}) \min_{i=1,\ldots,m}\left(\frac{1}{\sqrt{\lambda_{\min}(Q_i)}}\sqrt{c_i + b^T Q_i^{-1} b_i} + \|Q_i^{-1} b_i\|\right)^2.$$

Proof. By Lemma A.1, the system of inequalities (3) implies that $\|x\|^2 \le \alpha$, where

$$\alpha = \min_{i=1,\ldots,m}\left(\frac{1}{\sqrt{\lambda_{\min}(Q_i)}}\sqrt{c_i + b^T Q_i^{-1} b_i} + \|Q_i^{-1} b_i\|\right)^2.$$

Let $\beta_1, \ldots, \beta_m$ be m positive real numbers such that

$$\sum_{i=1}^m \beta_i \lambda_{\max}(Q_i) < \lambda_{\min}(Q_0).$$

This inequality implies that $Q_{m+1} = Q_0 - \sum_{i=1}^m \beta_i Q_i$ is a positive definite matrix. Thus, $x^T Q_{m+1} x \le \lambda_{\max}(Q_{m+1})\|x\|^2 \le \lambda_{\max}(Q_{m+1})\alpha$. □

An immediate consequence of Lemma 2.1 is that (QCQP) is equivalent to the following minimization problem:

$$\text{minimize}\{x^T Q_0 x + 2b_0^T x\colon x^T Q_i x + 2b_i^T x \le c_i,\ i = 1, 2, \ldots, m+1\},$$

where Q_{m+1}, c_{m+1} are as defined in Lemma 2.1 and $b_{m+1} = 0$. Note that by the construction of Q_{m+1}, it follows that there are positive numbers $\beta_1, \ldots, \beta_{m+1}$ such that

$$Q_0 = \sum_{i=1}^{m+1} \beta_i Q_i, \tag{4}$$

where $\beta_1, \ldots, \beta_m > 0$ are chosen to satisfy (2) and $\beta_{m+1} = 1$. Given the eigenvalues of the matrices, finding such parameters is a trivial task. We can now use a decomposition technique to find the desired dual problem.

The decomposition is obtained by duplicating the variables $x \in \mathbb{R}^n$, so that the resulting problem is equivalent to (QCQP) in the variables (x, x_i), $i = 1, 2, \ldots, m+1$:

$$\begin{aligned} \text{minimize} \quad & x^T Q_0 x + 2b_0^T x \\ \text{s.t.} \quad & x_i^T Q_i x_i + 2b_i^T x_i \le c_i, \quad i = 1, 2, \ldots, m+1, \\ & x_i = x, \quad i = 1, \ldots, m+1. \end{aligned} \tag{5}$$

Substituting (4), we have that (5) is equivalent to

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m+1} \beta_i x_i^T Q_i x_i + 2b_0^T x \\ \text{s.t.} \quad & x_i^T Q_i x_i + 2b_i^T x_i \le c_i, \quad i = 1, 2, \ldots, m+1, \\ & x_i = x, \quad i = 1, \ldots, m+1, \end{aligned}$$

where $x_1, \ldots, x_{m+1}$ are vectors in $\mathbb{R}^n$. We associate a Lagrange multiplier $\lambda_i \in \mathbb{R}^n$ for every constraint $x_i = x$ and form the Lagrangian

$$\begin{aligned} L(x, x_1, \ldots, x_{m+1}, \lambda_1, \ldots, \lambda_{m+1}) &= \sum_{i=1}^{m+1} \beta_i x_i^T Q_i x_i + 2b_0^T x + \sum_{i=1}^{m+1} 2\lambda_i^T (x_i - x) \\ &= \sum_{i=1}^{m+1} (\beta_i x_i^T Q_i x_i + 2\lambda_i^T x_i) + 2\left(b_0 - \sum_{i=1}^{m+1} \lambda_i\right)^T x. \end{aligned}$$

Consequently, the dual problem of (QCQP) is $\max\{h(\lambda_1, \ldots, \lambda_{m+1})\}$, where

$$\begin{aligned} h(\lambda_1, \ldots, \lambda_{m+1}) &= \inf_{x_i^T Q_i x_i + 2b_i^T x_i \le c_i} L(x, x_1, \ldots, x_{m+1}, \lambda_1, \ldots, \lambda_{m+1}) \\ &= \sum_{i=1}^{m+1} \inf_{x_i^T Q_i x_i + 2b_i^T x_i \le c_i} (\beta_i x_i^T Q_i x_i + 2\lambda_i^T x_i) + \inf_x \left(2\left(b_0 - \sum_{i=1}^{m+1} \lambda_i\right)^T x \right). \end{aligned} \tag{6}$$

To find an explicit expression for $h(\lambda_1, \ldots, \lambda_{m+1})$, we will solve each of the minimization problems in (6). The next lemma enables us to find the required expression.

Lemma 2.2. *Let Q be an $n \times n$ positive definite matrix, $\lambda \in \mathbb{R}^n$, and let $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ such that $c + b^T Q^{-1} b > 0$. Then,*

$$\min_{x^T Q x + 2b^T x \le c} (x^T Q x + 2\lambda^T x) = -\gamma g_Q(z) - 2\sqrt{\gamma} z^T Q^{-1} b - b^T Q^{-1} b,$$

where

$$g_Q(z) = \begin{cases} z^T Q^{-1} z & \text{if } z^T Q^{-1} z \le 1, \\ 2\sqrt{z^T Q^{-1} z} - 1 & \text{if } z^T Q^{-1} z > 1, \end{cases} \tag{7}$$

$$z = \frac{\lambda - b}{\sqrt{\gamma}},$$

$$\gamma = c + b^T Q^{-1} b.$$

Proof. A direct result of the KKT optimality conditions. □

Now, using the separable structure of the minimization problem (6), with the following notations,

$$z_i = \frac{(1/\beta_i)\lambda_i - b_i}{\sqrt{\gamma_i}}, \quad \gamma_i = c_i + b_i^T Q_i^{-1} b_i, \quad i = 1, \ldots, m+1,$$

it follows that the dual problem to (QCQP) is given by

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{m+1} (-\beta_i \gamma_i g_{Q_i}(z_i) - 2\sqrt{\gamma_i} z_i^T Q_i^{-1} b_i - b_i^T Q_i^{-1} b_i) \\ \text{s.t.} \quad & \sum_{i=1}^{m+1} \sqrt{\gamma_i} \beta_i z_i = b_0 - \sum_{i=1}^{m+1} \beta_i b_i. \end{aligned}$$

We summarize our development in the following theorem:

Theorem 2.1 (A Dual Problem for (QCQP)). *A dual problem for* (QCQP) *is given by* (DQCQP) *defined by*

$$\text{(DQCQP)} \quad \text{maximize}\left\{\sum_{i=1}^{m+1}(-\delta_i g_{Q_i}(z_i)+h_i^T z_i)+p\colon \sum_{i=1}^{m+1}\alpha_i z_i = e\right\},$$

where

- g_Q *is defined by* (7).
- $\beta_1,\dots,\beta_m$ *are* m *positive real numbers such that* $\sum_{i=1}^m \beta_i\lambda_{\max}(Q_i) < \lambda_{\min}(Q_0)$.
- $$Q_{m+1} = Q_0 - \sum_{i=1}^m \beta_i Q_i,$$
  $$c_{m+1} = \lambda_{\max}(Q_{m+1}) \min_{i=1,\dots,m}\left(\frac{1}{\sqrt{\lambda_{\min}(Q_i)}}\sqrt{c_i + b^T Q_i^{-1} b_i} + \|Q_i^{-1}b_i\|\right)^2,$$
  $$b_{m+1} = 0, \qquad \beta_{m+1} = 1,$$
  $$p = -\sum_{i=1}^{m+1} b_i^T Q_i^{-1} b_i, \qquad e = b_0 - \sum_{i=1}^{m+1}\beta_i b_i.$$
- *For every* $i = 1, 2, \dots, m+1$,
  $$\delta_i = \beta_i(c_i + b_i^T Q_i^{-1} b_i) > 0, \qquad h_i = -2\sqrt{c_i + b_i^T Q_i^{-1} b_i}\, Q_i^{-1} b_i, \qquad \alpha_i = \sqrt{c_i + b_i^T Q_i^{-1} b_i}\,\beta_i.$$

Following the analysis of the derivation of the dual problem, we can easily obtain the relation between the optimal solution of (QCQP) and the optimal solution of (DQCQP).

Lemma 2.3. *Suppose that* $(z_1, z_2, \dots, z_{m+1})$ *is the solution of* (DQCQP)*. Define the following variables for* $i = 1, 2, \dots, m+1$:

$$x_i = \begin{cases} -\sqrt{\dfrac{\gamma_i}{z_i^T Q_i^{-1} z_i}}\, Q_i^{-1} z_i - Q_i^{-1} b_i & \text{if } z_i^T Q_i^{-1} z_i \ge 1,\\[2ex] -Q_i^{-1}(\sqrt{\gamma_i}\, z_i + b_i) & \text{if } z_i^T Q_i^{-1} z_i < 1.\end{cases}$$

Then, $x_1 = x_2 = \cdots = x_{m+1}$ *and their common value* x *is the solution to* (QCQP)*.*

We will now show that the objective function in (DQCQP),

$$h(z_1,\dots,z_{m+1}) = \sum_{i=1}^{m+1}(-\delta_i g_{Q_i}(z_i) + h_i^T z_i) + p,$$

is a concave function with a Lipschitz continuous gradient.

Theorem 2.2. *The objective function* h *of* (DQCQP) *satisfies the following properties*:

(i) h *is concave and everywhere finite on* $\mathbb{R}^{(m+1)n}$, *and*

(ii) h *is continuously differentiable and has a Lipschitz continuous gradient with Lipschitz constant* $L_h = 2\max_{1\le i\le m+1}\{\delta_i/\lambda_{\min}(Q_i)\}$.

Proof. The concavity of the function h follows by construction as a direct result of duality, which proves the first part of (i). To prove (ii) and the remaining part of (i), we first note that from the separable structure of the function h (linear combination of g_{Q_i}), it is sufficient to show that for $Q \succ 0$, the function g_Q has a Lipschitz continuous gradient ∇g_Q. In fact, this property follows directly from a general result on proximal regularization of convex functions (Rockafellar [26]). Indeed, let us show that

$$g_Q(u) = 2\inf_{v\in\mathbb{R}^n}\left\{\|v\|_{Q^{-1}} + \frac{1}{2}\|v-u\|^2_{Q^{-1}}\right\}, \tag{8}$$

where $\|z\|_{Q^{-1}} := \sqrt{z^T Q^{-1} z}$, $Q \succ 0$. This implies that g_Q is differentiable and finite everywhere and has a Lipschitz gradient with Lipschitz constant $2\lambda_{\max}(Q^{-1})$; see, e.g., Rockafellar [26].

To prove (8), let $h_1(v) = (1/2)\|v-u\|^2_{Q^{-1}}$ and $h_2(v) = \|v\|_{Q^{-1}}$. Then, $g_Q(u) = 2\inf_{v\in\mathbb{R}^n}\{h_1(v)+h_2(v)\}$. Invoking the Fenchel Duality Theorem (Rockafellar [25]), we then obtain $g_Q(u) = 2\sup_z\{-h_1^*(z) - h_2^*(z)\}$, where the conjugates of h_1, h_2 are respectively given by

$$h_1^*(z) = \frac{1}{2}z^TQz + z^Tu, \qquad h_2^*(z) = \begin{cases} 0 & \text{if } \|z\|_Q \le 1, \\ +\infty & \text{otherwise.}\end{cases}$$

Therefore, $g_Q(u) = \sup\{-z^TQz - 2z^Tu \colon \|z\|_Q \le 1\}$. Invoking Lemma 2.2, it follows that (7) and (8) coincide, and a simple computation shows that ∇h has a Lipschitz constant $L_h = 2\max_{1\le i\le m+1}\{\delta_i/\lambda_{\min}(Q_i)\}$. □

2.3. The dual gradient projection algorithm for (DQCQP). The derived dual problem (DQCQP) shares the two basic ingredients needed to apply a GPA: a concave objective with computable Lipschitz gradient, and an affine constraint set for which the orthogonal projection can be computed analytically.

From now on, and for convenience reasons, the dual problem (DQCQP) will be rewritten as a convex minimization problem (we also omit the constant term p in the objective function (cf. Theorem 2.1)):

$$\text{(DQCQP)}\quad \text{minimize}\left\{\sum_{i=1}^{m+1}\delta_i g_{Q_i}(\eta_i) - h_i^T\eta_i\colon \sum_{i=1}^{m+1}\alpha_i\eta_i = e\right\},$$

where g_Q is defined by (7). The objective function of (DQCQP) is denoted by

$$f(\eta) = \sum_{i=1}^{m+1}(\delta_i g_{Q_i}(\eta_i) - h_i^T\eta_i), \tag{9}$$

where $\eta = (\eta_i)_{i=1}^{m+1}$ and $\eta_i \in \mathbb{R}^n$ for every $i = 1,\ldots,m+1$. It was proved in §2.2 that f has a Lipschitz continuous gradient. The feasible set is denoted by S and defined by the affine set $S = \{\eta\colon \sum_{i=1}^{m+1}\alpha_i\eta_i = e\}$, so that (DQCQP) can be written as the convex minimization problem

$$\text{(DQCQP)}\quad \min\{f(\eta)\colon \eta\in S\}.$$

Note that S has a very special structure that enables us to find a simple and explicit expression for the projection operator P_S, thus yielding to a simple algorithm for solving (DQCQP).

Lemma 2.4. *Let $y = (y_1,\ldots,y_{m+1})$. Then, $P_Sy = (y_j - \alpha_j w)_{j=1}^{m+1}$, where*

$$w = \left(\sum_{i=1}^{m+1}\alpha_i^2\right)^{-1}\left(\sum_{i=1}^{m+1}\alpha_i y_i - e\right).$$

Proof. Follows by direct calculation. □

Using Lemma 2.4, we can now give the formal description of the dual gradient projection (DGP) algorithm, which results as a direct application of GPA (cf. (1)) when applied to (DQCQP).

DGP Algorithm for (DQCQP). Start with $\eta_0^0, \eta_1^0, \ldots, \eta_{m+1}^0 \in \mathbb{R}^n$ and let $t\in(0, 2L^{-1})$, where $L = 2\max_{1\le i\le m+1}\{\delta_i/\lambda_{\min}(Q_i)\}$. Generate the sequence $(\eta_0^k, \eta_1^k, \ldots, \eta_{m+1}^k)$ as follows:

$$y_i^k = \begin{cases} \eta_i^k - t(2\delta_i Q_i^{-1}\eta_i^k + h_i), & \eta_i^k Q_i^{-1}\eta_i^k \le 1, \\ \eta_i^k - t\left(2\delta_i\dfrac{Q_i^{-1}\eta_i^k}{\sqrt{\eta_i^k Q_i^{-1}\eta_i^k}} + h_i\right), & \text{else,}\end{cases} \qquad i = 1,2,\ldots,m+1,$$

$$\eta_i^{k+1} = y_i^k - \alpha_i w^k, \quad i = 1,2,\ldots,m+1,$$

where $w^k = (\sum_{i=1}^{m+1}\alpha_i^2)^{-1}(\sum_{i=1}^{m+1}\alpha_i y_i^k - e)$.

3. Linear rate of convergence of DGP. This section covers the second and main contribution of this paper, namely, to prove that the DGP algorithm is linearly convergent. For that purpose, we first need some results on the general GPA.

3.1. Preliminaries. In this subsection, we use the same notation (f, S) as in (DQCQP), but for a general objective and constraints set, as defined below. For a closed and convex set S, P_S denotes the orthogonal projection on the set S and $d(x, S) \equiv \min\{\|y - x\|: y \in S\}$ is the usual point to set distance. Consider the following convex optimization problem:

$$(\mathrm{P}) \quad \inf\{f(x): x \in S\},$$

satisfying the following assumption:

Assumption A.
- *$S \subseteq \mathbb{R}^n$ is a closed convex set.*
- *The optimal set $X^* = \{x^* \in S: f(x^*) \equiv f^* = \inf_{x\in S} f(x)\}$ is nonempty.*
- *$f: \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable convex function and ∇f is Lipschitz continuous on $\mathbb{R}^n$, that is,*

$$\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\| \quad \textit{for every } x, y \in \mathbb{R}^n,$$

where $L > 0$ is the Lipschitz constant.

A standard result (see, e.g., Levitin and Polyak [13, Theorem 5.1]) on the convergence of the GPA applied to (P), i.e.,

$$x^0 \in \mathbb{R}^n, \qquad x^{k+1} = P_S(x^k - t\nabla f(x^k)), \quad t \in (0, 2L^{-1}), \quad k = 1, 2, \ldots,$$

states that if $S \subset \mathbb{R}^n$ is closed convex and bounded, then the sequence $\{x^k\}$ converges to some $x^* \in X^*$ at a sublinear rate in function values, i.e.,

$$f(x^k) - f^* \le \frac{C}{k}$$

for every $k \ge 1$ and some constant $C > 0$.

To prove linear convergence of the function values or the sequence, stronger assumptions than Assumption A must be imposed. For instance, it is known that strong convexity implies linear convergence of the sequence generated by GPA. However, in (DQCQP) the objective is not a strongly convex function. To overcome this difficulty, we need some kind of weaker hypothesis. A standard way to achieve this is through the use of the theory of error bounds. Indeed, it is widely known that the existence of error bounds is a key ingredient in proving convergence rates of iterative methods. Major contributions on developing and using error bounds to derive rate of convergence results of iterative descent algorithms have been developed in a series of papers by Luo and Tseng [16, 17, 18, 19] and Luo [15]; for a comprehensive survey on error bounds, their applications, and further references, we refer the reader to Pang [23].

Linear rate of convergence results for the GPAs were proven in the aforementioned papers, under various type of error bound assumptions and for several classes of optimization problems. Here we follow the works of Luo and Tseng, and consider a slightly modified error bound, which we call the gradient error bound (GREB), that will be useful to analyze the special structure of problem (DQCCP). In the rest of this paper, t is a fixed positive number. Let T be the map defined by

$$T(x) = \|P_S(x - t\nabla f(x)) - x\|.$$

Assumption 3.1 (GREB). *For every closed bounded set $B \subseteq \mathbb{R}^n$, there exists $\sigma_B > 0$ such that*

$$d(x, X^*) \le \sigma_B T(x) \quad \textit{for every } x \in B \cap S.$$

The next result shows that under Assumption A and the GREB hypothesis, the sequence generated by GPA converges at a linear rate.

Theorem 3.1 (Linear Rate of Convergence of $d(x^k, X^*)$). *Let f be a convex function with a Lipschitz continuous gradient. Suppose that GREB is satisfied. Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $t \in (0, 2/L)$. Then, there exists $\eta \in (0, 1)$ such that*

$$d(x^{k+1}, X^*) \le \eta d(x^k, X^*), \quad k \ge 0.$$

Proof. Let $x^* \in X^*$. By using the Lipschitz continuity of ∇f on $\mathbb{R}^n$ and the argument in Polyak [24, p. 207], the sequence $\{x^k\}$ produced by the GPA satisfies

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - (1 - Lt/2)\|x^{k+1} - x^k\|^2 = \|x^k - x^*\|^2 - (1 - Lt/2)T(x^k)^2. \tag{10}$$

Therefore, under the GREB assumption, it follows that there exists $\sigma > 0$ such that $d(x^k, X^*) \le \sigma T(x^k)$, $k \ge 0$, and the global linear rate follows immediately:

$$d^2(x^{k+1}, X^*) \le (1 - \sigma^{-2}(1 - Lt/2))d^2(x^k, X^*). \quad \square$$

The linear rate of convergence of the distance of the sequence from the optimal set implies also the linear rate of convergence of the function values of the sequence.

Corollary 3.1 (Linear Rate of Convergence of the Function Values). *Let f be a convex function with Lipschitz continuous gradient. Suppose that GREB is satisfied. Let $\{x^k\}$ be a sequence generated by GPA with constant stepsize $t \in (0, 2/L)$. Then, there is $\gamma \in (0, 1)$ and $C > 0$ such that*

$$f(x^k) - f^* \le C\gamma^k.$$

Proof. By the mean-value theorem, we have

$$f(x^k) - f^* = \nabla f(z^k)^T(x^k - y^k) \le \|\nabla f(z^k)\| d(x^k, X^*),$$

where $y^k = \arg\min\{\|y - x^k\|: y \in X^*\}$ and $z^k = (1 - w_k)x^k + w_k y^k$ for some $w_k \in [0, 1]$. Because by (10) $\{x^k\} \subseteq S$ is bounded, then so is $\{z^k\}$, so $\|\nabla f(z^k)\| \le l$ for some $l > 0$. Invoking Theorem 3.1, the result immediately follows. $\square$

The main task that thus remains is to prove that GREB is fulfilled for (DQCQP) so that by Theorem 3.1 the linear rate of convergence of distances from the optimal set of the sequence produced by GPA on (DQCQP) will follow. Furthermore, because strong duality holds for the pair (QCQP) and (DQCQP), then as a consequence of Corollary 3.1 this will prove the linear convergence of the sequence of the function values for both primal and dual problems. We end these preliminaries by introducing an assumption that is slightly different from GREB, but, nonetheless, will be proven to be equivalent to GREB.

Assumption 3.2. *For every bounded set $B \subseteq \mathbb{R}^n$, there exist $\sigma_B > 0$ and $\epsilon > 0$ such that*

$$d(x, X^*) \le \sigma_B T(x) \quad \textit{for every } x \in B \cap X^*_\epsilon \cap S,$$

where $X^*_\epsilon = \{x: d(x, X^*) \le \epsilon\}$.

The following lemma states that GREB is equivalent to Assumption 3.2.

Lemma 3.1. *Let $f: \mathbb{R}^n \to \mathbb{R}$. Then, Assumption 3.2 is equivalent to GREB.*

Proof. First, assume that GREB is fulfilled. Then, Assumption 3.2 holds true because $B \cap X^*_\epsilon$ is also a bounded set. Now, assume that Assumption 3.2 is fulfilled. To prove that GREB holds true, consider a bounded set $B \subseteq \mathbb{R}^n$. Define the function $h(x) \equiv d(x, X^*)/T(x)$ for every $x \in \mathrm{cl}(B - X^*_\epsilon) \cap S$. The function $h(x)$ is continuous over the closed and bounded set $\mathrm{cl}(B - X^*_\epsilon) \cap S$. Thus, by the Weierstrass theorem, $h(x)$ is bounded over $\mathrm{cl}(B - X^*_\epsilon) \cap S$ so that there is $\tau > 0$ such that $h(x) \le \tau$ for every $x \in \mathrm{cl}(B - X^*_\epsilon) \cap S$. In other words, we have

$$d(x, X^*) \le \tau T(x) \quad \text{for every } x \in \mathrm{cl}(B - X^*_\epsilon) \cap S,$$

which, combined with Assumption 3.2, proves the result. $\square$

3.2. A sufficient condition for GREB. As was already noted, proving the validity of GREB is, generally speaking, a hard task. For that reason, in this subsection we develop an alternative condition that implies GREB. The new condition is expressed in terms of one-dimensional functions, which are restrictions of the multivariate function to certain line segments. The linear space associated with the affine set of constraints S of (DQCQP) is denoted by W and is defined by $W = \{\eta: \sum_{i=1}^{m+1} \alpha_i \eta_i = 0\}$. We adopt the following terminology: A vector $d \in W$ will be called a *feasible direction*. The following result establishes a relation between the projection on S and the projection on W.

Lemma 3.2. *Let W be the linear space associated with S. Then, there exists $b \in S$ such that*

$$P_S\eta = P_W\eta + b \quad \textit{for every } \eta \in \mathbb{R}^{(m+1)n}.$$

Proof. Let $s \in S$ and $\eta \in \mathbb{R}^{(m+1)n}$. Then,

$$\begin{aligned} P_S(\eta) &= \arg\min_{x\in S} \|x-\eta\| = \arg\min_{x\in S} \|x-s+s-\eta\| \\ &\overset{y=x-s}{=} \arg\min_{y\in W} \|y-(\eta-s)\| + s = P_W(\eta-s)+s = P_W(\eta) - P_W s + s. \end{aligned}$$

By setting $b = s - P_W s \in S$, the lemma is proved. □

The following technical lemma, which introduces an equivalent condition to Assumption 3.2, is a key argument in proving the sufficient condition for GREB to hold. For any set $C \subset \mathbb{R}^n$, we use the notation $N_C(\cdot)$ for the normal cone to C and $\mathrm{bd}(C)$ for its boundary; see, e.g., Rockafellar [25].

Lemma 3.3. *Assumption 3.2 is equivalent to the following condition: For every bounded set $B \subseteq \mathbb{R}^n$ and $\eta^* \in X^*$, there exist $\epsilon > 0$ and $\sigma_B > 0$ such that*

$$\frac{\|P_W \nabla f(\eta^* + \beta d)\|}{\beta} \ge \frac{\epsilon}{t\sigma_B}$$

for every $d \in W \cap N_{X^*}(\eta^*)$, $\beta \in (0,1]$, and $\eta^* \in \mathrm{bd}(X^*)$ such that $\|d\| = \epsilon$ and $\eta^* + \beta d \in B$.

Proof. By Assumption 3.2, there exists $\sigma_B > 0$ such that

$$d(\eta, X^*) \le \sigma_B \|\eta - P_S(\eta - t\nabla f(\eta))\| \quad \text{for every } \eta \in S \cap B \text{ such that } d(\eta, X^*) \le \epsilon.$$

By Lemma 3.2, we have

$$\eta - P_S(\eta - t\nabla f(\eta)) = tP_W \nabla f(\eta).$$

Thus, Assumption 3.2 is equivalent to

$$d(\eta, X^*) \le \sigma_B t \|P_W \nabla f(\eta)\| \quad \text{for every } \eta \in S \cap B \text{ such that } d(\eta, X^*) \le \epsilon.$$

Denote $\eta^* = P_{X^*}(\eta)$ and make the change of variables $\eta = \eta^* + \beta d$.

Note that as a consequence of the relation $\eta^* = P_{X^*}(\eta)$, we have that $d \in N_{X^*}(\eta^*)$. Thus, Assumption 3.2 holds true if and only if

$$\|\beta d\| \le \sigma_B t \|P_W \nabla f(\eta^* + \beta d)\|$$

for every $d \in W \cap N_{X^*}(\eta^*)$, $\beta \in (0,1]$, and $\eta^* \in \mathrm{bd}(X^*)$ such that $\|d\| = \epsilon$ and $\eta^* + \beta d \in B$. Dividing by β yields the desired result. □

For every optimal solution $\eta^* \in \mathrm{bd}(X^*)$ and every feasible direction $d \in W$, we investigate the following scalar function (recall that f is the dual objective given in (9)):

$$h_{d,\eta^*}(\beta) = f(\eta^* + \beta d), \quad \beta \in [0,1], \tag{11}$$

and find a condition in terms of the one-dimensional function $h_{d,\eta^*}(\beta)$ that implies GREB.

Lemma 3.4 (A Sufficient Condition for GREB). *The following condition implies GREB: For every bounded set B, there exist $\epsilon > 0$ and $s_B > 0$ for which*

$$\frac{h'_{d,\eta^*}(\beta)}{\beta} \ge s_B$$

for every $d \in W \cap N_{X^*}(\eta^*)$, $\beta \in (0,1]$, and $\eta^* \in \mathrm{bd}(X^*)$ such that $\|d\| = \epsilon$ and $\eta^* + \beta d \in B$.

Proof. Let $\{\psi_1, \psi_2, \ldots, \psi_k\}$ be an orthonormal basis for W. Then, every $d \in W$ has the following representation as a linear combination of the orthonormal basis:

$$d = \sum_{j=1}^{k} \langle d, \psi_j \rangle \psi_j. \tag{12}$$

Recall that P_W is a projection on the linear space W. Thus, for all η, $P_W\eta = \sum_{j=1}^k \langle \eta, \psi_j\rangle \psi_j$, so that $\|P_W\eta\|^2 = \sum_{j=1}^k \langle \eta, \psi_j\rangle^2$. Now we compute $h'_{d,\eta^*}(\beta)$ using the directional derivative formula:

$$h'_{d,\eta^*}(\beta) = \langle d, \nabla f(\eta^* + \beta d)\rangle \overset{(12)}{=} \sum_{j=1}^k \langle d, \psi_j\rangle \langle \psi_j, \nabla f(\eta^* + \beta d)\rangle \le \sum_{j=1}^k |\langle d, \psi_j\rangle| \cdot |\langle \psi_j, \nabla f(\eta^* + \beta d)\rangle|.$$

By the Cauchy-Schwartz inequality, one has for all $j = 1, 2, \ldots, k$, $|\langle d, \psi_j\rangle| \le \overbrace{\|d\|}^{\epsilon} \cdot \overbrace{\|\psi_j\|}^{1} = \epsilon$. Also, from the equivalence of norms in finite-dimensional spaces, we obtain that there exists $N > 0$ such that $\|x\|_1 \le N\|x\|$, where $\|\cdot\|_1$ is the standard l_1 norm. Therefore, from the latter inequality on $h'_{d,\eta^*}(\beta)$, we have

$$\begin{aligned} h'_{d,\eta^*}(\beta) &\le \sum_{j=1}^k |\langle d, \psi_j\rangle| \cdot |\langle \psi_j, \nabla f(\eta^* + \beta d)\rangle| \le \epsilon \sum_{j=1}^k |\langle \psi_j, \nabla f(\eta^* + \beta d)\rangle| \\ &\le \epsilon N \sqrt{\sum_{j=1}^k \langle \psi_j, \nabla f(\eta^* + \beta d)\rangle^2} = \epsilon N \|P_W \nabla f(\eta^* + \beta d)\|. \end{aligned} \tag{13}$$

It follows from the premise of the lemma that there exist $s_B > 0$ and $\epsilon > 0$ for which

$$\frac{h'_{d,\eta^*}(\beta)}{\beta} \ge s_B \tag{14}$$

for every $d \in W \cap N_{X^*}(\eta^*)$, $\beta \in (0, 1]$, and $\eta^* \in \mathrm{bd}(X^*)$ such that $\|d\| = \epsilon$ and $\eta^* + \beta d \in B$. Combining (14) with (13), we obtain that

$$\frac{\|P_W \nabla f(\eta^* + \beta d)\|}{\beta} \ge \frac{s_B}{\epsilon N}$$

for every $d \in W \cap N_{X^*}(\eta^*)$, $\beta \in (0, 1]$, and $\eta^* \in \mathrm{bd}(X^*)$ such that $\|d\| = \epsilon$ and $\eta^* + \beta d \in B$. Invoking Lemma 3.3, the latter relation is equivalent to Assumption 3.2, which thus implies that GREB holds true. □

Before proving that Assumption 3.2 is satisfied for problem (DQCQP), we will prove that X^* is a polyhedral set. This structure of X^* will play a crucial part in the sequel.

3.3. X^* is polyhedral. We will need to consider the following index sets. For every vector $u = (u_j)_{j=1}^{m+1} \in \mathbb{R}^{(m+1)n}$, we consider the partition of the set $\{1, 2, \ldots, m+1\}$ into the three sets:

$$I_u = \{j\colon \|u_j\|_{Q_j^{-1}} < 1\}, \qquad J_u = \{j\colon \|u_j\|_{Q_j^{-1}} > 1\}, \qquad \text{and} \qquad K_u = \{j\colon \|u_j\|_{Q_j^{-1}} = 1\}. \tag{15}$$

We are now ready to show that the optimal solution set X^* of (DQCQP) is a polyhedral set.

THEOREM 3.2. *X^* is a polyhedral set. More precisely, let $\zeta^* = (\zeta_j^*)_{j=1}^{m+1}$ be an optimal solution of* (DQCQP), *i.e., $\zeta^* \in X^*$. Then, $\eta = (\eta_j)_{j=1}^{m+1} \in X^*$ if and only if*

$$\eta_j = (a_j + 1)\zeta_j^*, \quad j = 1, 2, \ldots, m+1, \tag{16}$$

where $a_j \in \mathbb{R}$ satisfies the following set of linear equalities and inequalities:

$$\sum_{j=1}^{m+1} \alpha_j \zeta_j^* a_j = 0, \tag{17}$$

$$a_j = 0, \quad j \in I_{\zeta^*}, \tag{18}$$

$$a_j \ge \frac{1}{\|\zeta_j^*\|_{Q_j^{-1}}} - 1, \quad j \in J_{\zeta^*}, \tag{19}$$

$$a_j \ge 0, \quad j \in K_{\zeta^*}, \tag{20}$$

$$\sum_{j=1}^{m+1} (2\delta_j \|\zeta_j^*\|_{Q_j^{-1}} - h_j^T \zeta_j^*) a_j = 0, \tag{21}$$

where I_{ζ^*}, J_{ζ^*}, and K_{ζ^*} are the corresponding index sets defined by (15).

Proof. Fix any $\eta = (\eta_j)_{j=1}^{m+1} \in X^*$. Let $\eta^\beta = (\eta_j^\beta)_{j=1}^{m+1}$ for $\beta \in (0,1]$, where

$$\eta_j^\beta = (1-\beta)\zeta_j^* + \beta\eta_j, \quad j = 1, 2, \ldots, m+1.$$

For all $\beta \in (0,1]$ sufficiently small, we have $I_{\eta^\beta} \supseteq I_{\zeta^*}$ and $J_{\eta^\beta} \supseteq J_{\zeta^*}$. Then, for $j \in I_{\eta^\beta} \cup K_{\eta^\beta}$, we have $j \in I_{\zeta^*} \cup K_{\zeta^*}$ and the strict convexity of $\|\cdot\|^2_{Q_j^{-1}}$ yields

$$g_{Q_j}\left(\frac{\eta_j^\beta + \zeta_j^*}{2}\right) = \left\|\frac{\eta_j^\beta + \zeta_j^*}{2}\right\|^2_{Q_j^{-1}} \le \frac{\|\eta_j^\beta\|^2_{Q_j^{-1}} + \|\zeta_j^*\|^2_{Q_j^{-1}}}{2} = \frac{g_{Q_j}(\eta_j^\beta) + g_{Q_j}(\zeta_j^*)}{2},$$

with the inequality strict whenever $\eta_j^\beta \neq \zeta_j^*$. For $j \in J_{\eta^\beta}$, we have $j \in I_{\zeta^*} \cup K_{\zeta^*}$ and thus

$$g_{Q_j}\left(\frac{\eta_j^\beta + \zeta_j^*}{2}\right) = 2\left\|\frac{\eta_j^\beta + \zeta_j^*}{2}\right\|_{Q_j^{-1}} - 1 \le \|\eta_j^\beta\|_{Q_j^{-1}} + \|\zeta_j^*\|_{Q_j^{-1}} - 1 = \frac{g_{Q_j}(\eta_j^\beta) + g_{Q_j}(\zeta_j^*)}{2},$$

where the inequality is true due to the triangle inequality and is therefore strictly satisfied whenever η_j^β and ζ_j^* do not lie on the same ray from the origin. It follows from (9) that

$$\begin{aligned} f\left(\frac{\eta^\beta + \zeta^*}{2}\right) &= \sum_{j=1}^{m+1} \delta_j g_{Q_j}\left(\frac{\eta_j^\beta + \zeta_j^*}{2}\right) - h_j^T\left(\frac{\eta_j^\beta + \zeta_j^*}{2}\right) \\ &\le \sum_{j=1}^{m+1} \delta_j \frac{g_{Q_j}(\eta_j^\beta) + g_{Q_j}(\zeta_j^*)}{2} - h_j^T\left(\frac{\eta^\beta + \zeta^*}{2}\right) \\ &= \frac{f(\eta^\beta) + f(\zeta^*)}{2}, \end{aligned}$$

with the inequality strict whenever $\eta_j^\beta \neq \zeta_j^*$ for some $j \in I_{\eta^\beta} \cup K_{\eta^\beta}$ or η_j^β and ζ_j^* do not lie on the same ray from the origin for some $j \in J_{\eta^\beta}$. Because X^* is convex so that $\eta^\beta \in X^*$ and $(\eta^\beta + \zeta^*)/2 \in X^*$, the left-hand side must equal the right-hand side. This implies that $\eta_j^\beta = \zeta_j^*$ for all $j \in I_{\eta^\beta} \cup K_{\eta^\beta}$ and η_j^β and ζ_j^* lie on the same ray from the origin for all $j \in J_{\eta^\beta}$. This in turn implies that (i) $\eta_j = \zeta_j^*$ for all $j \in I_{\eta^\beta} \cup K_{\eta^\beta}$, and (ii) $\eta_j = (1 + a_j)\zeta_j^*$ for all $j \in J_{\eta^\beta}$, where $a_j \in \mathbb{R}$. Because $I_{\beta^*} \supseteq I_{\zeta^*}$, (i) implies that $\eta_j = \zeta_j^*$ for all $j \in I_{\zeta^*}$. By switching the role of η and ζ^* in the above argument, we also have $\zeta_j^* = \eta_j$ for all $j \in I_\eta$. Hence, $I_{\zeta^*} = I_\eta$. Then, $J_{\eta^\beta} \subseteq J_{\zeta^*} \cup K_{\zeta^*} = J_\eta \cup K_\eta$, so (ii) implies

$$\|\eta_j\|_{Q_j^{-1}} = (1 + a_j)\|\zeta_j^*\|_{Q_j^{-1}} \ge 1$$

for all $j \in J_{\eta^\beta}$ ($a_j \ge -1$ because otherwise we would have $\|\eta_j^\tau\|_{Q_j^{-1}} = |(1-\tau) + \tau(1 + a_j)|\|\zeta_j^*\|_{Q_j^{-1}} < 1$ for some $\tau \in (0,1]$, implying $j \in I_{\eta^\tau} = I_{\zeta^*}$). Thus,

$$\eta_j = (1 + a_j)\zeta_j^*, \quad j = 1, 2, \ldots, m+1,$$

with

$$a_j = 0 \quad \forall\, j \in I_{\eta^\beta} \cup K_{\eta^\beta}, \qquad a_j \ge \frac{1}{\|\zeta_j^*\|_{Q_j^{-1}}} - 1 \quad \forall\, j \in J_{\eta^\beta}.$$

Because $I_{\eta^\beta} \supseteq I_{\zeta^*}$ and $J_{\eta^\beta} \supseteq J_{\zeta^*}$, this proves (16) and (18)–(20). (17) and (21) follow from plugging (16) into the equations for $\eta - \zeta^* \in W$ and $f(\eta) = f(\zeta^*)$. The converse follows by using $\zeta^* \in S$ and (16), (17) to show $\eta \in S$, and using (16) and (18)–(20) to show that $g_{Q_j}(\eta_j) = g_{Q_j}(\zeta_j^*) + 2a_j\|\zeta_j^*\|_{Q_j^{-1}}$ for $j = 1, 2, \ldots, m+1$, so that (16) and (21) yield $f(\eta) = f(\zeta^*)$. ☐

We denote by $X_j^* \subseteq \mathbb{R}^n$ the set of all jth components of the optimal solution set X^*, i.e.,

$$X_j^* \equiv \{\eta_j^* \in \mathbb{R}^n : (\eta_j^*)_{j=1}^{m+1} \in X^* \text{ for some } \eta_1^*, \ldots, \eta_{j-1}^*, \eta_{j+1}^*, \ldots, \eta_{m+1}^* \in \mathbb{R}^n\}.$$

It is clear that X_j^* is a closed and convex set for every $j = 1, 2, \ldots, m+1$. A direct consequence of Theorem 3.2 is that the index set I_{η^*} is independent of the choice of $\eta^* \in X^*$ and that for $j \in I_{\eta^*}$, the set X_j^* is a singleton. This is summarized in Lemma 3.5.

Lemma 3.5. *Let X^* be the optimal solution set of* (DQCQP) *and let $\eta^* \in X^*$. If*

$$\|\eta_j^*\|_{Q_j^{-1}} < 1$$

for some j $(1 \le j \le m+1)$, then

$$X_j^* = \{\eta_j^*\}.$$

From now on, ζ^* will denote a fixed optimal solution. We end this subsection by proving that for two optimal solutions with the same set of active inequalities (i.e., inequalities that are satisfied as equalities) in the linear system given by (17)–(21), the corresponding partitions of the index set (15) are the same. This result will be useful in the sequel.

Lemma 3.6. *Let $\eta^1 = (\eta_j^1)_{j=1}^{m+1}$ and $\eta^2 = (\eta_j^2)_{j=1}^{m+1}$ be two optimal solutions (i.e., $\eta^1, \eta^2 \in X^*$) and let $d^i = (d_j^i)_{j=1}^{m+1}$, $(a_j^i)_{j \in J_{\zeta^*} \cup K_{\zeta^*}}$, $i = 1, 2$ be the additional vectors/scalars given by Theorem 3.2 for which the linear system (17)–(21) is satisfied for $\eta = \eta^i$, $d = d^i$, $a_j = a_j^i$, $j \in J_{\zeta^*} \cup K_{\zeta^*}$, $i = 1, 2$. Assume that the sets of active inequalities corresponding to the two solutions η^i, d^i, $(a_j^i)_{j \in J_{\zeta^*} \cup K_{\zeta^*}}$, $i = 1, 2$ of the linear system (17)–(21) are the same. Then, $I_{\eta^1} = I_{\eta^2}$, $J_{\eta^1} = J_{\eta^2}$, and $K_{\eta^1} = K_{\eta^2}$.*

Proof. First note that by Lemma 3.5, we have $I_{\eta^1} = I_{\eta^2}$. All that is left to prove is that $K_{\eta^1} = K_{\eta^2}$. (The equality $J_{\eta^1} = J_{\eta^2}$ will follow from the fact that I_{η^i}, J_{η^i}, K_{η^i} is a partition for $i = 1, 2$.) Let $j \in K_{\eta^1}$. Then, the following statement holds true:

(A) Either $j \in K_{\zeta^*}$ and the inequality $a_j \ge 0$ (see (20)) is active for a_j^1, i.e., $a_j^1 = 0$, or $j \in J_{\zeta^*}$ and the inequality $a_j \ge 1/\|\zeta_j^*\|_{Q_j^{-1}} - 1$ (see (19)) is active for a_j^1, i.e, $a_j^1 = 1/\|\zeta_j^*\|_{Q_j^{-1}} - 1$.

Because by the premise of the lemma the sets of the active inequalities of the two solutions are identical, then it follows that statement (A) is equivalent to saying that $j \in K_{\eta^2}$, which proves the result. □

3.4. Proving GREB for (DQCQP). We begin this section by developing a necessary condition on a set of representative points from the optimal set of (DQCQP). Using this condition, we will then prove the validity of the GREB assumption for (DQCQP).

We now recall the concept of a face and derive some basic properties of faces of convex sets needed for our analysis.

Definition 3.1. Let C be a closed convex set. A closed convex set $F \subseteq C$ is called a *face* if there is a supporting hyperplane H of C such that $H \cap C = F$.

Example. A set that contains one extreme point of C is a face.

We use the notation $\mathrm{ri}(S)$ for the relative interior of a set S (see, e.g., Rockafellar [25]). We will need the following result on faces, proven by Burke and Moré [8, Theorem 2.3].

Lemma 3.7 (Burke and Moré [8]). *Let C be a closed convex set and let F be a face of C. If $x, y \in \mathrm{ri}\, F$, then*

$$N_C(x) = N_C(y).$$

First, we recall that X^* is a polyhedral set and thus has a finite number of faces. Denote the faces of X^* by $F_1, F_2, \ldots, F_k$ and let $v^1, \ldots, v^k$ be arbitrary chosen representatives of the relative interiors of the faces, i.e.,

$$v^i \in \mathrm{ri}\, F_i, \quad i = 1, 2, \ldots, k.$$

If a relative interior of some face has several possible sets of active inequalities in the linear system (17)–(21), then we will take several representatives of the same relative interior; each corresponds to a possible set of active inequalities (hence the same face might appear more than once). Overall, each possibility of a set of active inequalities in the linear system has a representative in the set $\{v^i\}_{i=1}^k$. This process does not ruin the finiteness of the representatives set.

Definition 3.2. Let C be a closed convex set. A direction d is an *exterior direction* of C at a point $x \in C$ if $x + \beta d \notin C$ for all $\beta > 0$. The set of all exterior directions of C at x is denoted by $E_C(x)$.

Remark 3.1. (i) For every $x \in \mathrm{bd}(C)$, $N_C(x) \subseteq E_C(x) \cup \{0\}$.

(ii) For every $x \in \mathrm{int}(C)$, $E_C(x) = \varnothing$, $N_C(x) = \{0\}$.

Lemma 3.8. *Let $\eta^* \in X^*$, $d \in W \cap E_{X^*}(\eta^*)$ and let $s > 0$. Then, $h_{d,\eta^*}(\beta)$ is not linear on the interval* $[0, s]$.

Proof. By contradiction. Suppose that there exists a feasible direction $d \in W$ and $s > 0$ such that $h_{d,\eta^*}(\beta)$ is linear on $[0, s]$. There are two possible cases:

(i) The slope of the line is zero. In this case, for every $\beta \in [0, s]$, the point $\eta = \eta^* + \beta d$ is in the feasible set S and has the same objective function value as η^*. From this it follows that $\eta \in X^*$, which contradicts the fact that $d \in E_{X^*}$.

(ii) The slope of the line is not zero. η^* is a minimizer of (DQCQP) and h_{d,η^*} is a differentiable function, thus $h'_{d,\eta^*}(0)=0$ by Fermat's theorem. On the other hand, $h_{d,\eta^*}(\beta)$ is linear on $[0,s]$ with a nonzero slope, and as a result $h'_{d,\eta^*}(0)\neq 0$, which is a contradiction. □

We will use the following notation. For every positive definite matrix Q,

$$\|\eta\|_Q=\sqrt{\eta^T Q\eta}.$$

The next theorem states that a certain linear system admits only the trivial solution. This property will be the key argument proving the main result below.

THEOREM 3.3. *For every $i=1,2,\ldots,k$, the following system in the variables $\theta_1,\ldots,\theta_{m+1}$ does not have a solution*:

$$(\mathrm{NLS}_i)\qquad \begin{cases} \displaystyle\sum_{j\in J_{v^i}\cup K_{v^i}} \theta_j\alpha_j v_j^i=0,\\ \theta_j\geq 0, \quad j\in K_{v^i},\\ \theta_j=0, \quad j\in I_{v^i},\\ (\theta_j v_j^i)_{j=1}^{m+1}\in E_{X^*}(v^i). \end{cases}$$

PROOF. Fix some i and assume by contradiction that (NLS_i) does have a solution. Define

$$d_j=\theta_j v_j^i, \quad j=1,2,\ldots,m+1.$$

Thus, $d=(d_1,\ldots,d_{m+1})$ is a feasible direction (i.e., $\sum_{j=1}^{m+1}\alpha_j d_j=0$ so $d\in W$). Also, one has $d\in E_{X^*}(v^i)$. Define

$$B=\min_{j:\,\theta_j<0}\left\{\frac{1}{\theta_j}\left(\frac{1}{\|v_j^i\|_{Q_j^{-1}}}-1\right)\right\}.$$

We will now show that the function $h_{d,v^i}(\beta)=f(v^i+\beta d)$ is linear on $[0,B]$ in contradiction to Lemma 3.8. Because $0\leq\beta\leq B$, the following inequalities are satisfied:

$$\|v_j^i+\beta d_j\|_{Q_j^{-1}}\geq 1, \quad j\in J_{v^i}\cup K_{v^i}, \tag{22}$$

$$1+\beta\theta_j\geq 0, \quad j\in J_{v^i}\cup K_{v^i}. \tag{23}$$

Thus, for every $d\in W$ and every $\beta\in[0,B]$, using the definition of g_Q and f (cf. (7), (9)), we have

$$\begin{aligned} h_{d,v^i}(\beta) &= f(v^i+\beta d)\\ &= \sum_{j=1}^{m+1}\big(\delta_j g_{Q_j}(v_j^i+\beta d_j)-h_j^T(v_j^i+\beta d_j)\big)\\ &= \sum_{j\in J_{v^i}\cup K_{v^i}}\big(\delta_j g_{Q_j}(v_j^i+\beta d_j)-h_j^T(v_j^i+\beta d_j)\big)+\underbrace{\sum_{j\in I_{v^i}}\big(\delta_j g_{Q_j}(v_j^i)-h_j^T v_j^i\big)}_{\text{constant}}\\ &= \sum_{j\in J_{v^i}\cup K_{v^i}}\big(\delta_j g_{Q_j}(v_j^i+\beta\theta_j v_j^i)-h_j^T(v_j^i+\beta\theta_j v_j^i)\big)+\text{constant}\\ &\overset{(22)}{=} \sum_{j\in J_{v^i}\cup K_{v^i}}\Big(2\delta_j|1+\beta\theta_j|\sqrt{(v_j^i)^T Q_j^{-1}v_j^i}-\beta\theta_j h_j^T v_j^i-h_j^T v_j^i\Big)+\text{constant}\\ &\overset{(23)}{=} \underbrace{\sum_{j\in J_{v^i}\cup K_{v^i}}\Big(2\delta_j(1+\beta\theta_j)\sqrt{(v_j^i)^T Q_j^{-1}v_j^i}-\beta\theta_j h_j^T v_j^i-h_j^T v_j^i\Big)+\text{constant}}_{\text{linear in }\beta}. \end{aligned}$$

To summarize, we have obtained that $h_{d,v^i}(\beta)$ is a linear function of β on $[0,B]$ in contradiction to Lemma 3.8. Thus, (NLS_i) does not have a solution. □

In the following, we denote $B_\epsilon = \{x: \|x\| = \epsilon\}$.

THEOREM 3.4 (NECESSARY CONDITION ON THE REPRESENTATIVE POINTS OF X^*). *Let $1 \le i \le k$. Then, for every $\epsilon > 0$, there exists $\xi > 0$ (not depending on i) such that the following system of inequalities*

$$\begin{cases} d((\theta_j v_j^i)_{j=1}^{m+1}, N_{X^*}(v^i) \cap B_\epsilon) \le \epsilon/2, \\ \theta_j \ge 0, \quad j \in K_{v^i}, \\ \theta_j = 0, \quad j \in I_{v^i}, \end{cases} \tag{24}$$

implies

$$\left\| \sum_{j \in J_{v^i} \cup K_{v^i}} \alpha_j \theta_j v_j^i \right\| \ge \xi.$$

PROOF. Assume that there are variables $\theta_1, \ldots, \theta_{m+1}$ that satisfy (24). We will show that $(\theta_j v_j^i)_{j=1}^{m+1} \in E_{X^*}(v^i)$. Denote $w = (\theta_j v_j^i)_{j=1}^{m+1}$. Now, $d((\theta_j v_j^i)_{j=1}^{m+1}, N_{X^*}(v^i) \cap B_\epsilon) \le \epsilon/2$, and thus there is a direction $d \in N_{X^*}(v^i)$ such that $\|d\| = \epsilon$ and $\|d - w\| \le \epsilon/2$. Therefore, with $\|d\| = \epsilon$, we obtain

$$\|w\| \le \|d\| + \|d - w\| = \frac{3}{2}\epsilon, \qquad \|w\| \ge \|d\| - \|d - w\| = \frac{\epsilon}{2}.$$

The inequality $\|d - w\| \le \epsilon/2$ is equivalent to $\|d - w\|^2 \le \epsilon^2/4$, which after some simple algebraic manipulation together with the bounds on $\|w\|$ implies

$$\langle d, w\rangle \ge \frac{\|d\|^2 + \|w\|^2 - \epsilon^2/4}{2} = \frac{(3/4)\epsilon^2 + \|w\|^2}{2} \ge \frac{(3/4)\epsilon^2 + (1/4)\epsilon^2}{2} = \frac{1}{2}\epsilon^2 > 0. \tag{25}$$

To summarize, we have that $\langle d, w\rangle > 0$ for some $d \in N_{X^*}(v^i)$. Furthermore, $w \in E_{X^*}(v^i)$ because otherwise there would exist $\beta > 0$ such that $\bar{x} = v^i + \beta w \in X^*$, but from the definition of the normal cone we have that

$$d \in N_{X^*}(v^i) \;\Rightarrow\; \langle \bar{x} - v^i, d\rangle \le 0.$$

Substituting $\bar{x} = v^i + \beta w$, we derive that $\beta\langle w, d\rangle \le 0$, in contradiction to (25).

Now, consider the following minimization problem:

$$\begin{aligned} \text{minimize} \quad & \left\| \sum_{j \in J_{v^i} \cup K_{v^i}} \alpha_j \theta_j v_j^i \right\| \\ \text{s.t.} \quad & d((\theta_j v_j^i)_{j=1}^{m+1}, N_{X^*}(v^i) \cap B_\epsilon) \le \epsilon/2, \\ & \theta_j \ge 0, \quad j \in K_{v^i}, \\ & \theta_j = 0, \quad j \in I_{v^i}. \end{aligned}$$

Here, we minimize a continuous function on a closed and bounded set. Thus, the minimum is attained. Denoting the value of the minimum by ξ_i, one has $\xi_i > 0$ because otherwise the minimizing vector would be a solution for (NLS$_i$), which is a contradiction to Theorem 3.3. The result follows by setting $\xi = \min_{i=1,\ldots,k} \xi_i$. □

We are now ready to prove the main result of this section.

THEOREM 3.5 (GREB IS FULFILLED FOR (DQCQP)). *For every bounded set B, the inequality*

$$\frac{h'_{d,\eta^*}(\beta)}{\beta} \ge \gamma$$

holds true for every $d \in W \cap N_{X^*}(\eta^*)$, $\beta \in (0, 1]$, and $\eta^* \in \mathrm{bd}(X^*)$ such that $\|d\| = \epsilon$ and $\eta^* + \beta d \in B$. The positive numbers ϵ and γ are chosen so that

$$\epsilon < \min\left\{ \min_{j \in I_{\eta^*}} \frac{\left|1 - \|\eta_j^*\|_{Q_j^{-1}}\right|}{\|Q_j^{-1}\|^{1/2}}, \min_{j \in K_{\eta^*}} \frac{1}{\|Q_j^{-1}\|^{1/2}} \right\}, \tag{26}$$

$$\gamma < \min_{j=1,\ldots,m+1} \left\{ 1, \frac{1}{4N_j}, \left(\frac{\epsilon}{4C}\right)^4, \left(\frac{\xi}{D}\right)^4 \right\}. \tag{27}$$

Here, ξ is as defined in Theorem 3.4 and C, D are given by

$$C=\sqrt{(m+1)\cdot\max_{j=1,\ldots,m+1}\lambda_{\max}(Q_j)\cdot\max_{j=1,\ldots,m+1}\left\{\sqrt{N_j},N_j,\frac{1}{2\delta_j}\right\}},\tag{28}$$

$$D=\sqrt{m+1}\cdot\max_{j=1,\ldots,m+1}\left\{|\alpha_j|\sqrt{\lambda_{\max}(Q_j)}\right\}\cdot\max_{j=1,\ldots,m+1}\left\{\sqrt[4]{N_j},\sqrt{N_j},\frac{1}{\sqrt{2\delta_j}}\right\},\tag{29}$$

$$N_j=\|Q_j^{-1}\|^{3/2}\frac{(N+2\epsilon)^3}{2\delta_j},\tag{30}$$

where N is a positive number for which the inclusion

$$B\subseteq\{x\in\mathbb{R}^n:\ \|x\|\le N\}\tag{31}$$

holds true.

Proof. Let B be some fixed bounded set and let ϵ and γ be positive numbers that satisfy (26) and (27). Assume by contradiction that there exists an optimal solution η^*, a feasible direction d, and $\beta\in(0,1]$ such that

$$\frac{h'_{d,\eta^*}(\beta)}{\beta}<\gamma,$$

where d and η satisfy

$$d\in W\cap N_{X^*}(\eta^*),\quad \|d\|=\epsilon,\quad \eta^*\in\mathrm{bd}(X^*),\quad\text{and}\quad \eta^*+\beta d\in B.$$

Because $\eta^*\in\mathrm{bd}(X^*)$, it follows that

$$\eta^*\in\mathrm{ri}(F_p)$$

for some $1\le p\le k$. By the construction of the representative set $\{v^i\}_{i=1}^k$, we can assume that v^p and η^* have the same set of active inequalities in the linear system (17)–(21). Because $v^p\in\mathrm{ri}\,(F_p)$, we have by Lemma 3.7 that

$$N_{X^*}(\eta^*)=N_{X^*}(v^p),$$

and thus $d\in N_{X^*}(v^p)$. For simplicity of notation, in the sequel we set $h_d(\cdot)\equiv h_{d,\eta^*}(\cdot)$.

Let $z_i(\alpha)\equiv g_{Q_i}(\eta_i^*+\alpha d_i)$. Then, the function h_d in (11) can be written as

$$h_d(\alpha)=\sum_{i=1}^{m+1}(\delta_i z_i(\alpha)-h_i^T(\eta_i^*+\alpha d_i)).\tag{32}$$

By the optimality of η^*, we have that $h'_d(0)=0$. Now, h'_d is a continuous function over $[0,\beta]$. By the piecewise twice-differentiable property of z_i, h'_d has one-sided derivatives on $(0,\beta)$. Thus, by the mean value theorem (see Theorem A.1), there exists $c\in(0,\beta)$ such that

$$\gamma>\frac{h'_d(\beta)}{\beta}=\frac{h'_d(\beta)-h'_d(0)}{\beta-0}\in[(h_d)''_+(c),(h_d)''_-(c)].$$

Here, we assume without loss of generality that $(h_d)''_+(c)<(h_d)''_-(c)$. Consequently, we have that $(h_d)''_+(c)<\gamma$. By (32), we have

$$(h_d)''_+(c)=\sum_{i=1}^{m+1}\delta_i(z_i)''_+(c).$$

From the convexity of z_j and Theorem A.2, it follows that $(z_j)''_+(c)\ge 0$ for all $j=1,2,\ldots,m+1$ and $c\in(0,1)$. As a consequence (recall that $\delta_j>0$ for every j),

$$\gamma>(h_d)''_+(c)=\sum_{i=1}^{m+1}\delta_i(z_i)''_+(c)\ge\delta_j(z_j)''_+(c),\quad j=1,2,\ldots,m+1.$$

Thus,

$$(z_j)''_+(c)<\frac{\gamma}{\delta_j}.\tag{33}$$

We will divide the investigation of inequality (33) into several cases. Before doing so, we note that by Lemma 3.6 we have $I_{\eta^*}=I_{v^p}$, $J_{\eta^*}=J_{v^p}$, and $K_{\eta^*}=K_{v^p}$, and hence for convenience we omit the subscripts in the index sets defined in (15) and use the notation I, J, K. For every feasible direction $d\in W$, we partition K into two disjoint sets: $K=K_1^d\cup K_2^d$, where

$$K_1^d=\{j\colon \|\eta_j^*\|_{Q_j^{-1}}=1,\ d_j^TQ_j^{-1}\eta_j^*>0\},$$
$$K_2^d=\{j\colon \|\eta_j^*\|_{Q_j^{-1}}=1,\ d_j^TQ_j^{-1}\eta_j^*\le 0\}.$$

We also need the following quantity that will play an important role in bounding several expressions:

$$M_j=\frac{(\|\eta_j^*\|_{Q_j^{-1}}+\|Q_j^{-1}\|^{1/2}\epsilon)^3}{2\delta_j},\qquad j=1,\ldots,m+1. \tag{34}$$

Because $\eta^*+\beta d\in B$, we have from (31) and $\beta\|d\|\le\epsilon$ that $\|\eta_j^*\|\le\|\eta^*\|\le N+\epsilon$ for every $j=1,\ldots,m+1$, and, hence, by a simple algebraic manipulation, that $M_j\le N_j$ for every $j=1,\ldots,m+1$, where N_j is given by (30). Now we consider inequality (33) for the following cases.

- $j\in I$. In this case, $\|\eta_j^*\|_{Q_j^{-1}}<1$. Combining this with the fact that $\|d\|=\epsilon$ and (26), we obtain that for all $\alpha\in[0,1]$, the inequality $\|\eta_j^*+\alpha d_j\|_{Q_j^{-1}}<1$ is satisfied and as a result for every $\alpha\in[0,1]$, we have

$$z_j(\alpha)=\|\eta_j^*+\alpha d_j\|^2_{Q_j^{-1}}=\|\eta_j^*\|^2_{Q_j^{-1}}+2\alpha d_j^TQ_j^{-1}\eta_j^*+\alpha^2\|d_j\|^2_{Q_j^{-1}}.$$

Thus, $\gamma/\delta_j>(z_j)''_+(c)=2\|d_j\|^2_{Q_j^{-1}}$.

- $j\in J$. In this case, we have two subcases. Either $z_j(\alpha)=\|\eta_j^*+\alpha d_j\|^2_{Q_j^{-1}}$ for every α in some right neighborhood of c (i.e., $(c,c+\Delta)$ for some $\Delta>0$) or $z_j(\alpha)=2\|\eta_j^*+\alpha d_j\|_{Q_j^{-1}}-1$ for every α in some right neighborhood of c. In the first subcase, we have, similar to the case $j\in I$, that $\|d_j\|^2_{Q_j^{-1}}\le\gamma/2\delta_j$. In the second subcase, we have

$$\frac{\gamma}{\delta_j}>(z_j)''_+(c)\overset{\text{Lemma A.2}}{=}2\frac{\|d_j\|^2_{Q_j^{-1}}\|\eta_j^*\|^2_{Q_j^{-1}}-(d_j^TQ_j^{-1}\eta_j^*)^2}{\|\eta_j^*+cd_j\|^3_{Q_j^{-1}}} \tag{35}$$

$$=\frac{2\|\eta_j^*\|^2_{Q_j^{-1}}}{\|\eta_j^*+cd_j\|^3_{Q_j^{-1}}}\left(\|d_j\|^2_{Q_j^{-1}}-\frac{(d_j^TQ_j^{-1}\eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}}}\right)$$

$$>\frac{1}{N_j\delta_j}\left(\|d_j\|^2_{Q_j^{-1}}-\frac{(d_j^TQ_j^{-1}\eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}}}\right). \tag{36}$$

The last inequality is valid because $\|\eta_j^*\|_{Q_j^{-1}}>1$ for every $j\in J$ and

$$\begin{aligned}\|\eta_j^*+cd_j\|_{Q_j^{-1}}\le\|\eta_j^*\|_{Q_j^{-1}}+c\|d_j\|_{Q_j^{-1}}&\le\|\eta_j^*\|_{Q_j^{-1}}+c\|Q_j^{-1}\|^{1/2}\|d_j\|\\&\le\|\eta_j^*\|_{Q_j^{-1}}+c\|Q_j^{-1}\|^{1/2}\|d\|=\|\eta_j^*\|_{Q_j^{-1}}+c\|Q_j^{-1}\|^{1/2}\epsilon\\&\overset{c<1}{<}\|\eta_j^*\|_{Q_j^{-1}}+\|Q_j^{-1}\|^{1/2}\epsilon\overset{(34)}{=}\sqrt[3]{2M_j\delta_j}\le\sqrt[3]{2N_j\delta_j}.\end{aligned}$$

By Lemma A.3, there is $\theta_j\in\mathbb{R}$ such that

$$\|d_j-\theta_j\eta_j^*\|^2_{Q_j^{-1}}=\|d_j\|^2_{Q_j^{-1}}-\frac{(d_j^TQ_j^{-1}\eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}}}\overset{(36)}{<}N_j\gamma.$$

Combining the two subcases, we conclude that there exists $\theta_j\in\mathbb{R}$ (in the first subcase, take $\theta_j=0$) such that

$$\|d_j-\theta_j\eta_j^*\|^2_{Q_j^{-1}}\le\max\left\{N_j,\frac{1}{2\delta_j}\right\}\gamma.$$

- $j\in K_1^d$. In this case, $\|\eta_j^*+cd_j\|_{Q_j^{-1}}>1$ and thus $(z_j)''_+(c)$ has the same form as in (35) and so there exists $\theta_j\in\mathbb{R}$ such that

$$\|d_j-\theta_j\eta_j^*\|^2_{Q_j^{-1}}\le N_j\gamma.$$

We also have that $d_j^T Q_j^{-1} \eta_j^* > 0$, which implies that (see Lemma A.3)

$$\theta_j = \frac{d_j^T Q_j^{-1} \eta_j^*}{\|\eta_j^*\|^2_{Q_j^{-1}}} > 0.$$

- $j \in K_2^d$. Here we have two subcases: (1) $\|\eta_j^* + c d_j\|_{Q_j^{-1}} < 1$, and (2) $\|\eta_j^* + c d_j\|_{Q_j^{-1}} \geq 1$.

SUBCASE 1. Here, $z_j(\alpha) = \|\eta_j^* + \alpha d_j\|^2_{Q_j^{-1}}$ for α in a neighborhood around c, so $(z_j)''_+(c) = (z_j)''(c) = 2\|d_j\|^2_{Q_j^{-1}}$. Thus, as in the case $j \in I$, one has $\|d_j\|^2_{Q_j^{-1}} < \gamma/2\delta_j$.

SUBCASE 2. Because $q(\alpha) = \|\eta_j^* + \alpha d_j\|^2_{Q_j^{-1}}$ is a convex function of α and $q(0) = 1 \leq q(c)$, q is an increasing function on $[c, \infty)$, so that $q(\alpha) \geq q(c) \geq 1$ for all $\alpha \geq c$. This implies $z_j(\alpha) = 2\|\eta_j^* + \alpha d_j\|_{Q_j^{-1}} - 1$ for $\alpha \geq c$, and hence Lemma A.2 yields

$$(z_j)''_+(c) = 2\frac{\|d_j\|^2_{Q_j^{-1}} \|\eta_j^*\|^2_{Q_j^{-1}} - (d_j^T Q_j^{-1} \eta_j^*)^2}{\|\eta_j^* + c d_j\|^3_{Q_j^{-1}}}.$$

By (36), we have

$$\|d_j\|^2_{Q_j^{-1}} \left(1 - \frac{(d_j^T Q_j^{-1} \eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}} \cdot \|d_j\|^2_{Q_j^{-1}}}\right) < N_j \gamma.$$

As a result, at least one of the following two inequalities must be satisfied:

$$\|d_j\|^2_{Q_j^{-1}} < \sqrt{N_j \gamma},$$

$$1 - \frac{(d_j^T Q_j^{-1} \eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}} \cdot \|d_j\|^2_{Q_j^{-1}}} < \sqrt{N_j \gamma}.$$

We will show that the second inequality is impossible. Suppose otherwise that the second inequality is valid. By the definition of γ (cf. (27)), one has $\gamma < 1/4N_j \ \forall\, j$, and as a result we have $\sqrt{N_j \gamma} < 1/2$. Thus,

$$\frac{(d_j^T Q_j^{-1} \eta_j^*)^2}{\|\eta_j^*\|^2_{Q_j^{-1}} \cdot \|d_j\|^2_{Q_j^{-1}}} > 1 - \sqrt{N_j \gamma} > \frac{1}{2}. \tag{37}$$

Recall that for $j \in K_2^d$, $\|\eta_j^*\|_{Q_j^{-1}} = 1$ and $d_j^T Q_j^{-1} \eta_j^* \leq 0$, and so by substituting this in (37), we obtain

$$d_j^T Q_j^{-1} \eta_j^* < -\frac{\|d_j\|_{Q_j^{-1}}}{\sqrt{2}} < -\frac{\|d_j\|_{Q_j^{-1}}}{2}. \tag{38}$$

From this, it follows that for all $\alpha \in (0, 1]$ and $j \in K_2^d$,

$$\begin{aligned}
\|\eta_j^* + \alpha d_j\|^2_{Q_j^{-1}} &= \|\eta_j^*\|^2_{Q_j^{-1}} + 2\alpha d_j^T Q_j^{-1} \eta_j^* + \alpha^2 \|d_j\|^2_{Q_j^{-1}} \\
&\overset{(38),\, j\in K}{<} 1 - \alpha \|d_j\|_{Q_j^{-1}} + \alpha^2 \|d_j\|^2_{Q_j^{-1}} \\
&= 1 + \alpha \|d_j\|_{Q_j^{-1}} \left(-1 + \alpha \|d_j\|_{Q_j^{-1}}\right) \\
&\overset{0<\alpha\leq 1}{<} 1 + \alpha \|d_j\|_{Q_j^{-1}} \left(-1 + \|d_j\|_{Q_j^{-1}}\right) \\
&< 1 + \alpha \|d_j\|_{Q_j^{-1}} (-1 + \|Q_j^{-1}\|^{1/2} \epsilon) < 1,
\end{aligned}$$

where the last inequality follows from $\epsilon < 1/\|Q_j^{-1}\|^{1/2}$ (cf. (26)).

Therefore, we have a contradiction to the assumption that $\|\eta_j^* + c d_j\|_{Q_j^{-1}} \geq 1$. Thus, in this subcase, we have $\|d_j\|^2_{Q_j^{-1}} < \sqrt{N_j \gamma}$.

We summarize all our conclusions from the inequality $\gamma > \delta_j(z_j)''_+(c)$ for $j = 1, \dots, m+1$:

$$\|d_j\|_{Q_j^{-1}} < \sqrt{\frac{\gamma}{2\delta_j}}, \quad j \in I,$$

$$\|d_j - \theta_j \eta_j^*\|_{Q_j^{-1}} < \sqrt{\max\left\{N_j, \frac{1}{2\delta_j}\right\}\gamma}, \quad j \in J,$$

$$\|d_j - \theta_j \eta_j^*\|_{Q_j^{-1}} < \sqrt{N_j \gamma}, \quad j \in K_1^d,$$

$$\|d_j\|^2_{Q_j^{-1}} < \sqrt{N_j \gamma}, \quad j \in K_2^d,$$

where $\theta_j > 0$ for every $j \in K_1^d$. By Theorem 3.2 (see (19) and (20)), we obtain that there are numbers $\{\lambda_j\}_{j \in J \cup K_1^d}$ such that $\lambda_j \geq 0$ for all $j \in K_1^d$ and satisfy

$$\eta_j^* = \lambda_j v_j^p, \quad j \in J \cup K_1^d.$$

Defining $\tilde{\theta}_j = \lambda_j \theta_j$, the above four inequalities become

$$\|d_j\|_{Q_j^{-1}} < \sqrt{\frac{\gamma}{2\delta_j}}, \quad j \in I,$$

$$\|d_j - \tilde{\theta}_j v_j^p\|_{Q_j^{-1}} < \sqrt{\max\left\{N_j, \frac{1}{2\delta_j}\right\}\gamma}, \quad j \in J, \tag{39}$$

$$\|d_j - \tilde{\theta}_j v_j^p\|_{Q_j^{-1}} < \sqrt{N_j \gamma}, \quad j \in K_1^d, \tag{40}$$

$$\|d_j\|^2_{Q_j^{-1}} < \sqrt{N_j \gamma}, \quad j \in K_2^d.$$

Define a vector $u \in \mathbb{R}^{(m+1)n}$ by

$$u_j = \begin{cases} \tilde{\theta}_j v_j^p & \text{if } j \in J \cup K_1^d, \\ 0 & \text{otherwise.} \end{cases}$$

Now, define a norm on vectors of $\mathbb{R}^{(m+1)n}$ by $\|v\|_\alpha^2 := \sum_{j=1}^{m+1} \|v_j\|^2_{Q_j^{-1}}$, and denote by $|I|$ the cardinality of an index set I. Then,

$$\begin{aligned}
\|d-u\|_\alpha^2 &= \sum_{j=1}^{m+1} \|d_j - u_j\|^2_{Q_j^{-1}} \\
&= \sum_{j \in J \cup K_1^d} \|d_j - u_j\|^2_{Q_j^{-1}} + \sum_{j \in I \cup K_2^d} \|d_j - u_j\|^2_{Q_j^{-1}} \\
&= \sum_{j \in J \cup K_1^d} \|d_j - \tilde{\theta}_j v_j^p\|^2_{Q_j^{-1}} + \sum_{j \in I \cup K_2^d} \|d_j\|^2_{Q_j^{-1}} \\
&\overset{(39),(40)}{\leq} \sum_{j \in J \cup K_1^d} \max\left\{N_j, \frac{1}{2\delta_j}\right\}\gamma + \sum_{j \in I \cup K_2^d} \max\left\{\frac{\gamma}{2\delta_j}, \sqrt{N_j \gamma}\right\} \\
&\leq |J \cup K_1^d| \max_{j \in J \cup K_1^d} \left\{N_j \gamma, \frac{\gamma}{2\delta_j}\right\} + |I \cup K_2^d| \max_{j \in I \cup K_2^d} \left\{\frac{\gamma}{2\delta_j}, \sqrt{N_j \gamma}\right\} \\
&\leq (|J \cup K_1^d| + |I \cup K_2^d|) \max_{j=1,\dots,m+1} \left\{\sqrt{N_j \gamma}, N_j \gamma, \frac{\gamma}{2\delta_j}\right\} \\
&\overset{|J \cup K_1^d| + |I \cup K_2^d| = m+1,\, 0<\gamma<1}{\leq} \sqrt{\gamma}(m+1) \max_{j=1,\dots,m+1} \left\{\sqrt{N_j}, N_j, \frac{1}{2\delta_j}\right\}.
\end{aligned}$$

Note that $\|v\|^2 \leq \max_{j=1,\dots,m+1} \lambda_{\max}(Q_j)\|v\|_\alpha^2$ for every $v \in \mathbb{R}^{(m+1)n}$ and let C be given by (28). With these notations, we have

$$\|d - (\tilde{\theta}_j v_j^p)_{j=1}^{m+1}\| \leq C\sqrt[4]{\gamma},$$

where $d \in N_{X^*}(v^p) \cap B_\epsilon$, $\tilde{\theta}_j \geq 0$ for $j \in K_1^d$ and $\tilde{\theta}_j = 0$ for $j \notin J \cup K_1^d$. Hence,

$$d((\theta_j v_j^p)_{j=1}^{m+1}, N_{X^*}(v^p) \cap B_\epsilon) \leq C\sqrt[4]{\gamma} \overset{(27)}{<} \epsilon/4.$$

According to Theorem 3.4, to obtain the desired contradiction it is sufficient to prove that

$$\left\| \sum_{j \in J \cup K_1^d} \tilde{\theta}_j \alpha_j v_j^p \right\| < \xi,$$

and in fact

$$\begin{aligned}
\left\| \sum_{j \in J \cup K_1^d} \tilde{\theta}_j \alpha_j v_j^p \right\| &\overset{d \in W}{=} \left\| \sum_{j \in J \cup K_1^d} \tilde{\theta}_j \alpha_j v_j^p - \sum_{j=1}^{m+1} \alpha_j d_j \right\| \\
&= \left\| \sum_{j \in J \cup K_1^d} \alpha_j (\tilde{\theta}_j v_j^p - d_j) - \sum_{j \in I \cup K_2^d} \alpha_j d_j \right\| \\
&\leq \sum_{j \in J \cup K_1^d} |\alpha_j| \|\tilde{\theta}_j v_j^p - d_j\| + \sum_{j \in I \cup K_2^d} |\alpha_j| \|d_j\| \\
&\leq \sum_{j \in J \cup K_1^d} |\alpha_j| \sqrt{\lambda_{\max}(Q_j)} \|\tilde{\theta}_j v_j^p - d_j\|_{Q_j^{-1}} + \sum_{j \in I \cup K_2^d} |\alpha_j| \sqrt{\lambda_{\max}(Q_j)} \|d_j\|_{Q_j^{-1}} \\
&\overset{(39),(40)}{\leq} \sqrt[4]{\gamma} D,
\end{aligned}$$

where D is given by (29). But by the definition of γ, we have that $\sqrt[4]{\gamma} D < \xi$ and thus we have obtained the desired contradiction to Theorem 3.4 and the theorem is proved. □

Appendix. We collect here some simple technical results that are used throughout the paper.

Lemma A.1. *Let Q be a positive definite matrix, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$. If x satisfies the following quadratic inequality*

$$x^T Q x + 2b^T x \leq c, \tag{41}$$

then

$$\|x\|^2 \leq a,$$

where

$$a = \left(\frac{1}{\sqrt{\lambda_{\min}(Q)}} \sqrt{c + b^T Q^{-1} b} + \|Q^{-1} b\| \right)^2.$$

Proof. Rewrite (41) as $\|Q^{1/2}(x + Q^{-1}b)\|^2 \leq c + b^T Q^{-1} b$; then using $\|Q^{1/2}(x + Q^{-1}b)\|^2 \geq \lambda_{\min}(Q) \cdot \|x + Q^{-1}b\|^2$ and $\|x + Q^{-1}b\| \geq \|x\| - \|Q^{-1}b\|$, the result follows. □

Theorem A.1. *Let z: $\mathbb{R} \to \mathbb{R}$ be a continuous function over a closed interval $[a, b]$ with one-sided derivatives in (a, b). Then, there exists $c \in (a, b)$ such that the ratio $(z(b) - z(a))/(b - a)$ lies between the one-sided derivatives $z'_-(c)$ and $z'_+(c)$.*

Proof. Follows as an easy extension of the classical mean value theorem for differentiable functions. □

For any continuously differentiable function z: $\mathbb{R} \to \mathbb{R}$, we denote by z''_+ (z''_-) the right (left) derivative of its derivative z'.

Theorem A.2. *Let z: $\mathbb{R} \to \mathbb{R}$ be a continuously differentiable convex function. If z' has one-sided derivatives at a point $x \in \mathbb{R}$, then $z''_-(x), z''_+(x) \geq 0$.*

Proof. Because z is a differentiable convex function, it follows that z' is a nondecreasing function (see, e.g., Rockafellar [25, Theorem 24.1]). As a result, z''_-, z''_+ are nonnegative. □

Lemma A.2. *For any $d \in \mathbb{R}^{(m+1)n}$, define $z(\alpha) = 2\|\eta^* + \alpha d\|_{Q^{-1}} - 1$. Then,*

$$\frac{d^2 z}{d\alpha^2} := z''(\alpha) = 2 \frac{\|d\|_{Q^{-1}}^2 \|\eta^*\|_{Q^{-1}}^2 - (d^T Q^{-1} \eta^*)^2}{\|\eta^* + \alpha d\|_{Q^{-1}}^3}$$

for every α such that $\|\eta^* + \alpha d\|_{Q^{-1}} > 0$.

PROOF. By a straightforward calculation. □

LEMMA A.3. *Let Q be a positive definite matrix and let $u, v \in \mathbb{R}^n$. Then,*

$$\min_{\delta \in \mathbb{R}} \|u - \delta v\|_Q^2 = \|u\|_Q^2 - \frac{(u^T Q v)^2}{\|v\|_Q^2},$$

where the minimum is attained at $\delta^* = u^T Q v / \|v\|_Q^2$.

PROOF. One has $\|u - \delta v\|_Q^2 = \|u\|_Q^2 - 2(u^T Q v)\delta + \|v\|_Q^2 \delta^2$, and the result follows immediately by minimizing the resulting one-dimensional quadratic function. □

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