SIAM J. OPTIM.
Vol. 11, No. 1, pp. 179–188

GLOBAL OPTIMALITY CONDITIONS FOR QUADRATIC OPTIMIZATION PROBLEMS WITH BINARY CONSTRAINTS*

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Abstract. We consider nonconvex quadratic optimization problems with binary constraints. Our main result identifies a class of quadratic problems for which a given feasible point is global optimal. We also establish a necessary global optimality condition. These conditions are expressed in a simple way in terms of the problem's data. We also study the relations between optimal solutions of the nonconvex binary quadratic problem versus the associated relaxed and convex problem defined over the l_∞ norm. Our approach uses elementary arguments based on convex duality.

Key words. quadratic programming, optimality conditions, nonconvex optimization, integer programming, convex duality, max-cut problem

AMS subject classifications. 90C20, 90C26, 90C09

PII. S1052623498336930

1. Introduction.

This work is concerned with quadratic optimization problems with binary constraints of the form

$$\text{(D)} \quad \min\{q(x) : \ x \in D := \{-1, 1\}^n\}, \tag{1.1}$$

where q is the quadratic function $q(x) = \frac{1}{2}x^tQx + b^tx$, where Q is an $n \times n$ symmetric matrix, and where $b \in \mathbb{R}^n$ are the given data. Problems of the above type arise naturally in several important combinatorial optimization problems, such as the max-cut problem. These problems are known to be NP hard; see, e.g., Garey and Johnson [2]. One typical approach to solve these problems is to construct lower bounds for approximating the optimal value. The classical technique to obtain bounds is either via a continuous relaxation or via the dual problem, which is usually followed by branch and bound type algorithms for refining it. This kind of approach was used, e.g., by Shor [5], and several variants of this technique, including various relaxations of the constraint set can be found in several works; see, e.g., the recent survey paper of [1] and references therein. More recently, semidefinite programming relaxations of (D) have been studied and proven to be quite powerful for finding approximate optimal solutions; see, e.g., [3] and references therein.

1.1. Motivation.

This paper is not concerned with computation of bounds for problem (D). Our main goal here is to exploit the peculiar structure of problem (D) in order to characterize global optimal solutions of problem (D), as well as to study the relations between the optimal solutions of (D) and the optimal solutions of its continuous relaxation (C) defined by

$$\text{(C)} \quad \min\{q(x) : \ x \in C := \{x : \ -1 \le x_i \le 1, \ i = 1, \dots, n\}\}.$$

We derive a sufficient optimality condition which guarantees that a given feasible point in D is a global optimal for problem (D) as well as a necessary global optimality

*Received by the editors April 16, 1998; accepted for publication (in revised form) February 13, 2000; published electronically July 25, 2000.
http://www.siam.org/journals/siopt/11-1/33693.html

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condition. An interesting fact about these conditions is that they are simply expressed in terms of the problem's data $[Q,b]$ involving only primal variables and do not involve any dual variables. To motivate the kind of conditions we are looking at, consider the following trivial example. Let Q be the diagonal matrix $Q = \mathrm{diag}(\lambda_j)_{j=1}^n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n > 0$, and let $b \in \mathbb{R}^n$ be the given data. We then ask under which conditions on the data $[Q, b]$ we can write

$$\min\{q(x) : x \in D\} = \min\{q(x) : x \in C\}.$$

In this example, the function q is separable and can be written as

$$q(x) = \frac{1}{2}x^t Qx + b^t x = \sum_{j=1}^{n} \frac{1}{2}\lambda_j x_j^2 + b_j x_j.$$

It is easy to verify that for any $a, b \in \mathbb{R}$ we have

$$\min\left\{\frac{1}{2}ax^2 + bx : -1 \leq x \leq 1\right\} = \begin{cases} -b^2(2a)^{-1} & \text{if } |ba^{-1}| < 1, \\ (2a)^{-1} + b & \text{if } |ba^{-1}| \geq 1, b \leq 0, \\ (2a)^{-1} - b & \text{if } |ba^{-1}| \geq 1, b \geq 0. \end{cases}$$

From the above computation, we thus have that a sufficient (and in this case necessary) condition to have the optimal value of the continuous minimum (C) equal to the optimal value of the discrete one (D) is simply

$$\lambda_j \leq |b_j| \quad \forall j = 1, \ldots, n. \tag{1.2}$$

This condition shows that for this particular example, we need to ask that the matrix Q is in the sense of inequality (1.2) *smaller* than the vector b. Another way to look at (1.2) is that when Q is in some sense smaller than b, then we can disregard the quadratic term and solve the trivial problem $\min_{x\in D} b^t x$.

In the next section, using simple convex duality arguments, we derive the sufficient global optimality condition for the general problem (D). This condition, like the condition derived for the trivial example above, also requires that Q is in some sense "smaller" than b. We also derive a necessary global optimality condition which is similar in form to the sufficient condition. Both conditions are simply expressed in terms of the problem's data $[Q, b]$ and do not involve any dual variables. In section 3 we treat the special case of (D), when the matrix Q is positive semidefinite. In that case, problem (D) remains nonconvex due to the constraints set; however, its continuous relaxation (C) becomes a *convex problem*. Applying the results of section 2, we then establish relations between the optimal solutions of (C) and (D). In particular, we find necessary and sufficient conditions for a vector $x \in D$ to be the solution of both (C) and (D). Furthermore, we characterize a global optimal solution of (D), whenever it is close enough to an optimal solution of the corresponding relaxed convex problem (C). We conclude the paper in section 4 with a simple application.

1.2. Notations and definitions. Throughout this paper we will use the following notations and definitions. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$, and $\mathbb{R}^n_+, \mathbb{R}^n_{++}$ stand for the nonnegative and positive orthant, respectively. For a vector $x \in \mathbb{R}^n$, the Euclidean norm (l_2-norm) and l_∞- norm are denoted, respectively, by $||x|| := (\sum_{i=1}^n x_i^2)^{1/2}$ and $||x||_\infty := \max_{1\leq i\leq n} |x_i|$. Let $\{e_j\}_{j=1}^n$ be the canonical basis of $\mathbb{R}^n$, and let the vector of all 1's be denoted by e, i.e., $e = (1, \ldots, 1)^T$.

Given an $n \times n$ matrix Q, $\mathrm{Diag}(Q)$ denotes the $n \times n$ diagonal matrix with entries q_{ii}. For $x \in \mathbb{R}^n$, the corresponding capital letter will define the *diagonal* $n \times n$ matrix $X := \mathrm{diag}(x)$ with ith diagonal element $x_i,\ i = 1, \ldots, n$, and thus we will also write $x = Xe$.

The feasible set $\{-1, 1\}^n$ of problem (D) can be written in a continuous form equivalently as:

$$D := \{x \in \mathbb{R}^n :\ x_i^2 = 1,\ i = 1, \ldots, n\}.$$

The following three equivalent formulations of the convex relaxation of D will be useful to us:

$$\begin{aligned} C &= \{x \in \mathbb{R}^n :\ ||x||_\infty \le 1\} \\ &= \{x \in \mathbb{R}^n :\ -1 \le x_i \le 1,\ i = 1, \ldots, n\} \\ &= \{x \in \mathbb{R}^n :\ x_i^2 \le 1,\ i = 1, \ldots, n\}. \end{aligned}$$

Clearly, the following relation holds: $D \subset C$.

We will denote the optimization problem of minimizing the quadratic function $q(x)$ over the set D by (D) and its global optimal value by $q_D(x)$. A similar notation is used when optimizing $q(x)$ over the set C.

For a symmetric $n \times n$ real matrix Q with elements $q_{ij} = q_{ji},\ i, j = 1, \ldots, n$, we denote by $\lambda_i(Q) \equiv \lambda_i, i = 1, \ldots, n$ its eigenvalues ordered as

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n.$$

We also use $\lambda_n \equiv \lambda_{\min}(Q) = \min\{x^T Q x,\ ||x|| = 1\}$. The matrix Q is positive semidefinite, denoted by $Q \succeq 0$ (positive definite, denoted by $Q \succ 0$) if and only if $\lambda_n \ge 0$ ($\lambda_n > 0$). The trace of Q is defined by $\mathrm{tr}(Q) = \sum_{i=1}^n q_{ii} = \sum_{i=1}^n \lambda_i$ and it holds that $n\lambda_{\min}(Q) \le \mathrm{tr}(Q)$.

2. Global optimality conditions. Consider the nonconvex quadratic problem

$$\text{(D)} \quad \min\{q(x) :\ x_i^2 = 1,\ i = 1, \ldots, n\}.$$

This section is divided in two parts in which we first derive the sufficient globally optimality conditions and then the necessary one.

Sufficient conditions. Let $y \in \mathbb{R}^n$ be the multiplier associated with the constraints of (D) and form the Lagrangian

$$L(x, y) = q(x) + \sum_{i=1}^n y_i(x_i^2 - 1).$$

Defining the diagonal matrix $Y = \mathrm{diag}(y)$, L can be written as

$$L(x, y) = \frac{1}{2} x^T (Q + Y) x + b^T x - \frac{e^T y}{2}. \tag{2.1}$$

The dual problem corresponding to (D) is then defined by the *concave* maximization problem

$$\text{(DD)} \quad \sup\{h(y) : y \in \mathbb{R}^n \cap \mathrm{dom} h\},$$

where here h is the dual functional

$$h(y) := \inf\{L(x,y) : \ x \in \mathbb{R}^n\}, \tag{2.2}$$

and $\text{dom}h = \{y \in \mathbb{R}^n : \ h(y) > -\infty\}$.

From standard duality we always have the *weak* duality relation

$$q(x) \geq h(y) \quad \forall x \in D, \ \forall y \in \mathbb{R}^n \cap \text{dom}h.$$

Strong duality here of course does not hold since problem (D) is nonconvex. However, we recall the following useful result, which follows from basic duality theory [4].

LEMMA 2.1. *If there exists $\bar{x} \in D$ and $\bar{y} \in \mathbb{R}^n \cap \text{dom}h$ such that $q(\bar{x}) = h(\bar{y}) = \inf_x L(x, \bar{y})$, then $\bar{x}$ is a global optimal solution of* (D).

Thus, if we are lucky enough to guess such a pair $(\bar{x}, \bar{y})$ satisfying the conditions of Lemma 2.1, we can conclude that $\bar{x}$ globally solves (D). The special structure of problem (D) precisely allows us to identify such a pair. First we need to recall an elementary result on quadratic functions which will be helpful to make explicit the feasible set of the dual problem (DD).

LEMMA 2.2. *Let A be an $n \times n$ symmetric matrix, and let $f : \mathbb{R}^n \to \mathbb{R}$ be the quadratic function $f(x) = \frac{1}{2}x^T Ax + b^T x$, where $b \in \mathbb{R}^n$. Then, $\inf\{f(x) : \ x \in \mathbb{R}^n\} > -\infty$ if and only if the following two conditions hold:*

(i) $\exists x \in \mathbb{R}^n : \ Ax + b = 0$.

(ii) *The matrix A is positive semidefinite.*

We can now establish the following sufficient global optimality condition.

THEOREM 2.3. *Consider problem* (D) *with the data* $[Q, b]$, *with Q a real symmetric matrix. Let $x = Xe \in D$. If*

$$[\text{SC}] \qquad \lambda_n(Q)e \geq XQXe + Xb,$$

then x is a global optimal solution for (D).

Proof. Applying Lemma 2.2 on the dual objective h defined via (2.1)–(2.2), we have $\inf\{L(x,y) : \ x \in \mathbb{R}^n\} > -\infty$ if and only if the following conditions hold:

$$\exists x \in \mathbb{R}^n : \ (Q + Y)x + b = 0, \tag{2.3}$$

$$Q + Y \succeq 0. \tag{2.4}$$

Let x be any feasible point of (D). Then, since $x = Xe$ with $X = \text{diag}(x)$, from $x_i^2 = 1, \ i = 1, \ldots, n$, we also have $X^2 = I$. Now, let

$$y := -(Xb + XQXe). \tag{2.5}$$

We first show that the pair (x, y) just defined above satisfies (2.3). Indeed with $x = Xe$ and y defined in (2.5),

$$\begin{aligned}(Q + Y)x + b &= QXe + YXe + b \\ &= QXe + Xy + b \\ &= QXe - X^2 b - X^2 QXe + b \\ &= 0 \qquad (\text{since } X^2 = I).\end{aligned}$$

Now using (2.3) we can rewrite the dual objective h as

$$\begin{aligned}h(y) &= \inf_{x \in \mathbb{R}^n} \left\{ \frac{1}{2}x^T(Q + Y)x + b^T x - e^T y \right\} \\ &= -\frac{1}{2}x^T(Q + Y)x - \frac{e^T y}{2},\end{aligned}$$

with x satisfying (2.3) and such that $Q + Y \succeq 0$. Using the above expression for h, we now compute for the pair $(x = Xe, y = -XQXe - Xb)$:

$$\begin{aligned} h(y) &= -\frac{1}{2}e^T X(Q+Y)Xe - \frac{1}{2}e^T y \\ &= -\frac{1}{2}e^T XQXe - e^T y \\ &= \frac{1}{2}e^T XQXe + b^T Xe = q(Xe) = q(x). \end{aligned}$$

To complete the proof it thus remains to show that y defined in (2.5) is feasible for (DD), i.e., that $Q+Y \succeq 0$, and the result will follow from Lemma 2.1. For that, note that we always have

$$\lambda_n(Q+Y) \geq \lambda_n(Q) + \lambda_n(Y),$$

and hence $Q+Y$ is positive semidefinite if $\lambda_n(Q) \geq -\lambda_n(Y)$. But since Y is diagonal, from (2.5) we have $-\lambda_n(Y) = \max_i(Xb + XQXe)_i$ and the later inequality can thus be written as $\lambda_n(Q)e \geq XQXe + Xb$, and the proof is completed. $\square$

Necessary conditions. We now derive global necessary optimality conditions which resemble the sufficient conditions derived in Theorem 2.3.

THEOREM 2.4. *Consider problem* (D) *with the data* $[Q, b]$, *where* Q *is a real symmetric matrix. If* $x \in D$ *is a global minimum for* (D), *then*

$$[\text{NC}] \qquad XQXe + Xb \leq \text{Diag}(Q)e.$$

Proof. If $x \in D$ is a global minimum for (D), then

$$q(x) \leq q(z) \quad \forall z \in D.$$

In particular, for $z = z_1 := -2x_1e_1 + x = (-x_1, x_2, \ldots, x_n)^T \in D$, where $e_1 = (1, 0, \ldots 0)^T$, we obtain

$$\begin{aligned} \frac{1}{2}x^T Qx + b^T x &\leq \frac{1}{2}(x - 2x_1e_1)^T Q(x - 2x_1e_1) + b^T(x - 2x_1e_1) \\ &= \frac{1}{2}x^T Qx + 2x_1^2 e_1^T Qe_1 - 2x_1 e_1^T Qx - 2x_1 b^T e_1 + b^T x. \end{aligned}$$

Since $x_i^2 = 1, e_1^t Qe_1 = q_{11}$, the later inequality reduces to

$$x_1 e_1^T Qx + x_1 b^T e_1 \leq q_{11}.$$

In a similar way we can show that for any $j = 1, \ldots, n$

$$x_j e_j^T Qx + x_j b^T e_j \leq q_{jj},$$

which proves the relation [NC]. $\square$

It is interesting to note that both the necessary and optimality conditions are expressed only in terms of the primal variables and do not involve any dual variables. Moreover, rewriting the optimality conditions in the form

$$\begin{aligned} &[\text{SC}] \quad -Xb \geq X(Q - \lambda_{\min}(Q)I)Xe, \\ &[\text{NC}] \quad -Xb \geq X(Q - \text{Diag}(Q)I)Xe, \end{aligned}$$

we can interpret these as mentioned in the introduction by saying that a global optimal solution of (D) can be identified when the matrix Q is "smaller" than the vector b in the sense of the inequalities above. Several remarks are now in order regarding the derived optimality conditions.

Remark 2.5. In the case of pure quadratic optimization problems, i.e., when $b \equiv 0$, then the sufficient condition [SC] becomes $\lambda_{\min}(Q)e \geq XQXe$, which forces Xe to be the minimum eigenvector of Q. Thus, in the case of pure quadratic optimization problems, the sufficient condition becomes less informative. However, this difficulty can be handled by converting the pure quadratic problem into an equivalent one with a nonzero linear term in the objective. A standard and simple way to do this is just to observe that when in problem (D) $q(x) := 1/2x^TQx$, then since $q(x) = q(-x)$, we can just fix the value of an arbitrary component of x, say $x_k = 1$, and immediately get a nonhomogeneous quadratic objective, which has the same objective function value (see section 4 for an application).

Remark 2.6. Recall that $q_{jj} \geq \lambda_{\min}(Q)\ \forall j = 1,\ldots,n$, i.e., $\mathrm{Diag}(Q)e \geq \lambda_{\min}(Q)e$. Thus, using the sufficient optimality condition [SC] derived in Theorem 2.3, we have the natural implication

$$\lambda_{\min}(Q)e \geq XQXe + Xb \Longrightarrow \mathrm{Diag}(Q)e \geq XQXe + Xb.$$

Remark 2.7. Let $\bar{x} := Xe \in D$. Then [NC] implies

$$\mathrm{tr}(Q) \geq \bar{x}^TQ\bar{x} + b^T\bar{x},$$

where $\mathrm{tr}(Q) = \sum_{i=1}^n q_{ii}$. On the other hand, [SC] implies

$$n\lambda_{\min}(Q) \geq \bar{x}^TQ\bar{x} + b^T\bar{x}.$$

Since $\mathrm{tr}(Q) \geq n\lambda_{\min}(Q)$, one could be tempted to conjecture that $n\lambda_{\min}(Q) \geq \bar{x}^TQ\bar{x} + b^T\bar{x}$ could be considered as a potentially "better" sufficient condition for $\bar{x} \in D$ to be a global minimum. This is, however, not true as illustrated by the following simple example.

EXAMPLE 2.8. *Consider problem* (D) *in* $\mathbb{R}^2$ *with* $q(x) := x_1^2 - \frac{1}{2}x_2^2 + 6x_1 + 2x_2$. *The optimal solution is obtained at* $x^* = (-1,-1)^T$. *Now, let* $\bar{x} = (-1,1)^T$. *Since here* $\lambda_{\min}(Q) = -1$ *and* $n = 2$, *one can easily verify that* $n\lambda_{\min}(Q) = -2 \geq \bar{x}^TQ\bar{x} + b^T\bar{x} = -3$, *yet* $\bar{x}$ *is not global optimal.*

Remark 2.9. Let $x = (\sigma(b_i))_{i=1}^n$, where $\sigma(b_i) = 1$ if $b_i \geq 0$ and -1 otherwise. Then [SC] reduces to $XQXe \leq |b| + \lambda_{\min}(Q)e$. Thus, if the later inequality holds with $X = \mathrm{diag}(\sigma(b))$, the optimal solution of problem (D) is given by $x = (\sigma(b_i))_{i=1}^n$, namely, as the solution of the trivial problem $\min\{b^Tx : x \in D\}$; i.e., problem (D) can be solved by removing the quadratic term from the objective function.

We end this section by mentioning that we can state global optimality conditions for more general quadratic problems (and in particular for $\{0,1\}$ quadratic programs) of the form

$$\min\{q(x) : x \in \{a,c\}^n\}, \tag{2.6}$$

where $a < c$ are given real numbers. Using the linear transformation

$$x = \frac{c-a}{2}y + \frac{c+a}{2}e,$$

the above problem is transformed to $\min\{q'(y) : y \in D\}$, where $q'(y)$ can be explicitly written in terms of Q, b, a, c. A straightforward computation shows that [SC] and [NC] become, respectively,

$$\frac{c-a}{2}\lambda_{\min}(Q)e \geq \frac{a+c}{2}YQYe + Yb + \frac{a+c}{2}YQe, \tag{2.7}$$

$$\frac{c-a}{2}\mathrm{Diag}(Q)e \geq \frac{a+c}{2}YQYe + Yb + \frac{a+c}{2}YQe, \tag{2.8}$$

and the optimal solution x of problem (2.6) can be recovered from the optimal solution y via the linear transformation given above.

3. The positive semidefinite case. Let Q be a positive semidefinite matrix. Then (D) is still nonconvex because of the constraints $x \in D = \{-1, 1\}^n = \{x \in \mathbb{R}^n : x_i^2 = 1,\ i = 1, \dots, n\}$. However, the corresponding relaxed problem (C) becomes the *convex* problem:

$$\text{(C)} \quad \min\{q(x) :\ x_i^2 \leq 1,\ i = 1, \dots, m\}.$$

Then the question of the relations between the solution of the "easy" convex problem (C) versus the "hard" nonconvex problem (D) arises. Our first result shows that there is a simple necessary and sufficient condition for a point in D to be the solution of both the convex problem (C) and the nonconvex problem (D).

THEOREM 3.1. *Consider the nonconvex problem* (D) *with data [Q,b], with Q a real symmetric positive semidefinite matrix. Let $x = Xe \in D$. Then x is a solution of both* (C) *and* (D) *if and only if*

$$XQXe + Xb \leq 0.$$

Proof. First, suppose that $XQXe + Xb \leq 0$. Since (C) is convex and satisfies Slater's condition, strong duality applies and we have $\min\{q(x) : x \in C\} = \max\{h(y) : y \geq 0\}$, where h is the dual objective function of (C), which is the same as the one given in (2.2), except that here $y \in \mathbb{R}^n_+$. As in the proof of Theorem 2.3 with $y = -(XQXe + Xb)$, which is nonnegative by our assumption, we obtain $h(y) = q_C(Xe) = q_C(x)$, showing that x is a solution of (C) and hence of (D). To prove the converse, suppose $x = Xe$ solves (C) and (D). From the KKT optimality conditions for (C) we have $(Q + Y)x + b = 0,\ Y \succeq 0$, where $y \in \mathbb{R}^n_+$ are the multipliers for the constraints of (C). Therefore,

$$\begin{aligned} XQXe + Xb &= X(Qx + b) \\ &= -XYx \\ &= -Y, \ \text{ since } \ x \in D, \end{aligned}$$

and hence the result follows since $y \in \mathbb{R}^n_+$. ◻

Our next result characterizes an optimal solution of (D) whenever it is "close enough" to an optimal solution of the relaxed convex problem (C).

THEOREM 3.2. *Consider the problem* (D) *with data [Q,b], with Q a real symmetric positive semidefinite matrix. Let x be an optimal solution of the convex problem* (C). *If $y \in D$ satisfies the conditions*

(i) $y_i = x_i$ *when* $x_i^2 = 1$,

(ii) $YQ(y - x) \leq \lambda_{min}(Q)e$, *then y is a global optimal solution for* (D).

Proof. Since (C) is a convex problem and Slater's condition holds, then x solves (C) if and only if the KKT conditions hold, i.e., there exists $\lambda \geq 0$ such that

$$(Q+\Lambda)x + b = 0, \tag{3.1}$$

$$\lambda_i(x_i^2 - 1) = 0, \ i = 1, \ldots, n, \tag{3.2}$$

where $\Lambda := \mathrm{diag}(\lambda)$. Set $\delta := y - x$, and $\Delta := \mathrm{diag}(\delta)$. Then

$$\begin{aligned} YQYe + Yb &= Y(Qy+b) \\ &= Y(Q(x+\delta)+b) \\ &= Y(-\Lambda x + Q\delta) \ \ (\text{using } \ (3.1)) \\ &= (X+\Delta)(-\Lambda x + Q\delta) \\ &= -X\Lambda x + (X+\Delta)Q\delta - \Delta\Lambda x \\ &= -\lambda + YQ\delta - \Delta\Lambda x, \end{aligned}$$

where in the last equality we use (3.2). Now, we claim that $\Delta\Lambda x = 0$. Indeed, if $\delta_i = 0$, then $\lambda_i\delta_i = 0$, and if $\delta_i \neq 0$, then from the assumption of the theorem, this means $x_i^2 \neq 1$, and hence from (3.2) this implies $\lambda_i = 0$. Therefore, $\lambda_i\delta_i = 0 \ \forall i$, and from the above computations, together with the fact that $\lambda \geq 0$, we have obtained

$$YQYe + Yb = -\lambda + YQ\delta \leq YQ\delta = YQ(y-x).$$

Invoking Theorem 2.3 then completes the proof. ∎

Note that when $x_i^2 \neq 1$ for some i, then the corresponding binary value y_i in the theorem above can be chosen as $y_i = \sigma(x_i)$, where $\sigma(x_i) = 1$ if $x_i \geq 0$ and -1 otherwise.

EXAMPLE 3.3. *Consider the problem* (D) *with data [Q,b], where*

$$Q = \begin{pmatrix} 4 & 2 & 0 & 2 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 4 & 2 \\ 2 & 2 & 2 & 4 \end{pmatrix}, \ b = \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix}.$$

Here, we have $\lambda_{\min}(Q) = 1.036$ so that Q is positive definite. The solution of the relaxed convex problem (C) is obtained at the point $x = (-0.875, -0.875, -1, 0.625)^T$ and thus we can take (by rounding as explained above) as a "closest" point $y \in D$ to x the vector $y = (-1, -1, -1, 1)^T$. Now we compute $YQ(y-x) = (0, 0, -0.75, 1)$ so that the inequality $YQ(y-x) \leq \lambda_{\min}(Q)e$ is satisfied, and therefore from Theorem 3.2, y is the minimizing vector of (D).

4. An application. We consider a simple application of our results to problems with pure quadratic objectives, originally motivated from the max-cut problem. Given an undirected weighted graph $G = (V, E), V = \{1, 2, \ldots, n\}$, with weights $w_{ij} = w_{ji} \geq 0$ on the edges $(i,j) \in E$ and with $w_{ij} = 0$ if $(i,j) \notin E$, the max-cut problem is to find the set of vertices $S \subset V$ that maximizes the weight of the edges with one end point in S and the other in its complement $\bar{S}$, i.e., to maximize the total weight across the cut $(S, \bar{S})$. The cut can be defined by the integer variables $x_i \in \{-1, 1\}$ assigned to each vertex i. Then, with $x_i = 1$ if $i \in S$ and -1 otherwise, the weight of the cut is $\sum_{i<j} w_{ij}(1 - x_i x_j)/2$, and the max-cut problem is equivalent to the quadratic optimization problem (see, e.g., [3]):

$$\text{(MC)} \quad \max\left\{\sum_{i<j} w_{ij}\frac{1 - x_i x_j}{2} :\ x_i^2 = 1,\ i = 1, \dots, n\right\}.$$

Problem (MC) can be reformulated equivalently as

$$\text{(MC)} \quad \min\left\{\sum_{i=j} w_{ij} x_i x_j :\ x_i^2 = 1,\ i = 1, \dots, n\right\},$$

with $w_{ii} = 0$. Defining the matrix $W = 2(w_{ij}), i, j = 1, \dots, n$, we then obtain the formulation of (MC) as a pure quadratic problem fitting our generic formulation (D) with data $[W, 0]$, namely,

$$\text{(MC)} \quad \min\left\{q(x) = \frac{1}{2}x^T W x :\ x \in D\right\}.$$

By elementary arguments we can obtain the following sufficient condition for a vertex to define a max-cut.

LEMMA 4.1. *Let $G = (V, E)$ be an undirected graph with $V = \{1, \dots, n\}$ and with weight matrix W. Let l be a vertex that satisfies the following condition:*

$$\forall k \in V \setminus l :\quad w_{kl} \geq \sum_{i \neq l} w_{ik}. \tag{4.1}$$

Then l defines a max-cut; i.e., the max-cut is $S = \{l\}$ and $\bar{S}$ is the complementary set with the remaining vertices.

In other words, Lemma 4.1 says that under a particular condition as given in (4.1) on the matrix W, the vector $(-1, \dots, -1, \underbrace{1}_{k}, -1, \dots, -1)^T$ (meaning $x_k = 1, x_i = -1\, \forall i \neq k$) is the minimizing vector of the problem (MC). This result relies on the fact that the matrix W in the max-cut problems satisfies the very special conditions $\operatorname{Diag}(W) = 0$ and $w_{ij} \geq 0$. This motivates us to ask if a similar type of result can be established for an *arbitrary* pure quadratic problem, namely, when W is an $n \times n$ arbitrary symmetric matrix. An application of Theorem 2.3 leads us to establish a similar result for a class of matrices satisfying a sort of "eigenvalue-row-dominance" condition akin to the concept of diagonally dominant matrices.

PROPOSITION 4.2. *Let W be an $n \times n$ symmetric matrix that satisfies the following condition:*

$$\forall k \neq l :\quad w_{kl} \geq \sum_{i \neq l} w_{ik} - \lambda_{\min}(W(k)),$$

where $W(k)$ is the $(n-1) \times (n-1)$ matrix obtained from W by removing the kth row and column. Then the vector $(-1, \dots, -1, \underbrace{1}_{k}, -1, \dots, -1)^T$ is the minimizing point of problem (D) *with data [W,0].*

Proof. Without loss of generality we prove the result only for $k = 1$. By Remark 2.5, we can substitute $x_1 = 1$ and obtain a nonhomogeneous equivalent problem with data $[W', b']$ defined by

$$w'_{ij} = w_{ij} \text{ if } i \neq 1, j \neq 1;\ w'_{ij} = 0 \text{ if } i = 1 \text{ or } j = 1,$$
$$b' = (w_{j1})_{j=1}^n.$$

The above transformation obviously reduces the dimension of the original problem with data $[W, 0]$ posed in $\mathbb{R}^n$ to a nonhomogeneous problem which can now be defined in $\mathbb{R}^{n-1}$, with data $[W(1), b(1)]$, where $W(1)$ is obtained by removing the first row and column of W and $b(1)$ the first row of b'. Then, letting $X := -I_{n-1\times n-1}$, in Theorem 2.3 it follows that if

$$\lambda_{\min}(W(1))e \geq W'e - b(1),$$

then $(-1, \ldots, -1)^T \in \mathbb{R}^{n-1}$ is the solution of problem (D) with data $[W(1), b(1)]$ and thus $(\underbrace{1}_{1}, -1, \ldots, -1)^T \in \mathbb{R}^n$ is the solution of (D) with data $[W, 0]$. Similarly, the above argument can be repeated for each k, and the proof is completed. $\square$

Acknowledgments. The authors thank the referees for the careful review and constructive comments and suggestions.

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