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A probabilistic result for the max-cut problem on random graphs

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Abstract

We consider the max-cut problem on a random graph G with n vertices and weights w_{ij} being independent bounded random variables with the same fixed positive expectation μ and variance σ^2. It is well known that the max-cut number $\mathrm{mc}(G)$ always exceeds $\frac{1}{2}\sum_{i<j} w_{ij}$. We prove that with probability greater than p_n the max-cut number satisfies

$$\frac{1}{2}\sum_{i<j} w_{ij} \leqslant \mathrm{mc}(G) \leqslant q_n \left(\frac{1}{2}\sum_{i<j} w_{ij}\right),$$

where p_n, q_n are explicitly expressed in terms of the problem's data and such that p_n, q_n approach 1 as $n \to \infty$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Given an undirected graph $G=(V,E)$ with nonnegative weights $w_{ij}=w_{ji}$ on the edges $(i,j)\in E$, the maximum cut problem is that of finding a subset $S\subseteq V$ such that $\sum_{i\in S, j\in V\setminus S} w_{ij}$ is maximized. The maximum cut of G will be denoted by $\mathrm{mc}(G)$.

The max-cut problem is of fundamental importance in combinatorial optimization and is known to be NP-complete [5]. The max-cut problem also arises in several practical applications. Recently, Goemans and Williamson [7] discovered an approximation algorithm for max-cut whose accuracy is significantly better than all previously known algorithms. This algorithm was based on a semi-definite programming relaxation which can be solved approximately in polynomial time. For relevant literature and recent results on the max-cut problem, we refer the reader to Goemans and Williamson [7] and the more recent survey of Goemans [6] and references therein.

It is well known that the max-cut number $\mathrm{mc}(G)$ always exceeds $\frac{1}{2}\sum_{i<j} w_{ij}$. The main result of this paper shows that if $(w_{ij})_{i<j}$ are independent bounded random variables with the same expectation $\mu > 0$ and

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variance σ^2 then

$$\frac{1}{2}\sum_{i<j} w_{ij} \leqslant \mathrm{mc}(G) \leqslant q_n \left(\frac{1}{2}\sum_{i<j} w_{ij}\right)$$

with probability greater than p_n, where p_n and q_n are explicitly given and such that $p_n, q_n \to 1$ as $n \to \infty$.

To prove our result we first show in the next section that on any graph G it holds that

$$\frac{1}{2}\sum_{i<j} w_{ij} \leqslant \mathrm{mc}(G) \leqslant \left(1 - \frac{n\lambda_{\min}(W)}{2\sum_{i<j} w_{ij}}\right)\frac{1}{2}\sum_{i<j} w_{ij},$$

where $\lambda_{\min}(W)$ denotes the minimum eigenvalue of the matrix $W = (w_{ij})$. The last inequality leads us to investigate the behavior of the ratio $r_n(W) := (1 - n\lambda_{\min}(W)/2\sum_{i<j} w_{ij})^{-1}$, which depends on n and on the minimum eigenvalue of the matrix W. In Section 3, we show by numerical experiments, that on different types of random graphs, the ratio $r_n(W)$ approaches 1 as n gets larger. This is formalized in Section 4 where our main result is presented and explicit expressions for p_n and q_n are given in terms of n and the parameter distribution of W.

2. Bounds for the max-cut

Let $G = (V, E)$ be an undirected graph with vertex set $V = 1, 2, \ldots, n$ and nonnegative weights $w_{ij} = w_{ji}$ on the edges $(i, j) \in E$ with $w_{ij} = 0$; $\forall (i, j) \notin E$. The maximum cut of G consists of finding the set $S \subset V$ to maximize the weight of the edges with one point in S and the other point in $\bar{S} := V \setminus S$. To each vertex i, assign the variable $x_i = 1$ if $i \in S$ and -1 otherwise. Then the problem of finding the weight of the maximum cut can be equivalently written as the following integer quadratic problem:

$$(\mathrm{M}) \quad \max \ \frac{1}{4}\sum_{i,j=1}^{n} w_{ij}(1 - x_i x_j)$$
$$\text{s.t. } x_i \in \{-1, 1\}^n, \quad i = 1, \ldots, n.$$

We will use the following notation. The value of the max-cut problem (M) will be denoted by $\mathrm{mc}(G)$. The matrix $W = (w_{ij})_{i,j=1}^{n}$ will stand for the weight matrix with $w_{ij} = w_{ji} \geqslant 0$ $\forall i \neq j$ and $w_{ii} = 0$ $\forall i \in V$. The feasible set of (M) will be denoted by $D := \{x \in R^n\colon x_i^2 = 1 \ i = 1, \ldots, n\} \equiv \{-1, 1\}^n$. Recall that for any symmetric matrix A, the minimum and maximum eigenvalues denoted, respectively, by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ satisfy the relations

$$\lambda_{\min}(A) \leqslant z^{\mathrm{T}} A z \leqslant \lambda_{\max}(A) \quad \forall \|z\| = 1.$$

The following result is the starting point of our analysis. For completeness we include here a self-contained and easy proof of this result (see also Remark 2.1).

Lemma 2.1. *Let $G = (V, E)$ be an undirected graph with a symmetric nonnegative weights matrix $W \in R^{n \times n}$. Then*

$$\frac{1}{2}\sum_{i<j} w_{ij} \leqslant \mathrm{mc}(G) \leqslant \left(1 - \frac{n\lambda_{\min}(W)}{2\sum_{i<j} w_{ij}}\right)\frac{1}{2}\sum_{i<j} w_{ij}.$$

Proof. Using our notation we have

$$\mathrm{mc}(G) = \max_{x \in D} \frac{1}{4}\sum_{i,j} w_{ij}(1 - x_i x_j)$$
$$= \max_{x \in D} \frac{1}{4}\sum_{i,j} w_{ij} - \frac{1}{4} x^{\mathrm{T}} W x.$$

Now, consider the sum

$$\sum_{x \in D} x^{\mathrm{T}} W x = \sum_{x \in D}\left(\sum_{i=1}^{n} w_{ii} x_i^2 + \sum_{i \neq j} w_{ij} x_i x_j\right)$$
$$= \left(\sum_{i=1}^{n}\sum_{x \in D} w_{ii} x_i^2 + \sum_{i \neq j}\sum_{x \in D} w_{ij} x_i x_j\right)$$
$$= 2^n \,\mathrm{tr}(W) + 0 = 0,$$

since here, $\mathrm{tr}(W) := \sum_{i=1}^{n} w_{ii} = 0$. Therefore, $\exists \bar{x} \in D$ which satisfies $\bar{x}^{\mathrm{T}} W \bar{x} \leqslant 0$, and the value of the cut at this point is

$$\frac{1}{4}\sum_{i,j} w_{ij} - \frac{1}{4}\bar{x}^{\mathrm{T}} W \bar{x} \geqslant \frac{1}{4}\sum_{i,j} w_{ij} = \frac{1}{2}\sum_{i<j} w_{ij}$$

proving the lower bound. To derive the upper bound we first write $\mathrm{mc}(G)$ as

$$\mathrm{mc}(G) = \frac{1}{4}\sum_{i,j} w_{ij} + \frac{1}{4}\max_{x \in D} -x^{\mathrm{T}} W x$$
$$= \frac{1}{2}\sum_{i<j} w_{ij} - \frac{1}{4}\min_{x \in D} x^{\mathrm{T}} W x. \tag{2.1}$$

Since $z^{\mathrm{T}} W z \geqslant \lambda_{\min}(W)$ $\forall \|z\|^2 = 1$, and $D \subset x\colon \|x\|^2 = n$ we obtain

$$x^{\mathrm{T}} W x \geqslant n\lambda_{\min}(W) \quad \forall x \in D.$$

Using this last inequality in (2.1) we thus have

$$\mathrm{mc}(G) \leqslant \frac{1}{2}\sum_{i<j} w_{ij} - \frac{n}{4}\lambda_{\min}(W) = \frac{1}{2}\sum_{i<j} w_{ij}\left(1 - \frac{n\lambda_{\min}(W)}{2\sum_{i<j} w_{ij}}\right). \qquad \square$$

The lower bound $\frac{1}{2}\sum_{i<j} w_{ij}$ is well known in the literature. The greedy algorithm and local search both find a cut with value greater than or equal to $\frac{1}{2}\sum_{i<j} w_{ij}$. Our interest here is on investigating the upper bound $(1 - n\lambda_{\min}(W)/2\sum_{i<j} w_{ij})1/2\sum_{i<j} w_{ij}$ on the max-cut mc(G). By denoting

$$r_n(W) := \left(1 - \frac{n\lambda_{\min}(W)}{2\sum_{i<j} w_{ij}}\right)^{-1},$$

we see from Lemma 4.1 that if $r_n(W)$ is close to 1 as n tends to infinity then $\frac{1}{2}\sum_{i<j} w_{ij}$ is a good approximation for the max-cut for large values of n. The next section describes numerical experiments showing that $r_n(W)$ is indeed getting close to 1 when n is getting larger.

Remark 2.1. Eigenvalue bounds and their use in approximating the optimal value of the max-cut mc(G) have been extensively studied in the literature, and we refer the reader to the work of Delorme and Poljak [2] and the survey by Mohar and Poljak [8]. In particular, a better upper bound than the one derived in Lemma 2.1, can be found in [2]. More precisely, it is shown in [2] that mc$(G) \leqslant \varphi(G)$ with

$$(\mathrm{DM}) \quad \varphi(G) = \min_{u \in R^n}\left\{\frac{n}{4}\lambda_{\max}(L + \mathrm{diag}(u)) : u^{\mathrm{T}}e = 0\right\},$$

where L denotes the Laplacian matrix of the graph G, e is the vector of ones, and diag(u) is the diagonal matrix with entries u_i. In fact, it can be easily shown that the bound $\varphi(G)$ is nothing else but the *dual* bound for problem (M). Moreover, it should be noted that our bound could also be derived from the dual bound $\varphi(G)$, by choosing the feasible point $u = -We + (1/n)\mathrm{tr}(L)e$, where tr$(L)$ is the trace of L.

Remark 2.2. We would like to emphasize that here the upper bound derived in Lemma 2.1 is for the sole purpose of deriving a probabilistic analysis of the bound by exploiting powerful properties of eigenvalues of random symmetric matrices, (see Section 4). Yet, the numerical experiments given below also show that the simple explicit bound given in Lemma 2.1 already provides for small values of n a good approximation on the random graphs under consideration. The bound $\varphi(G)$, while theoretically better, requires on the other hand the numerical solution of a difficult non-smooth optimization problem (DM).

3. Behavior of $r_n(W)$: numerical experiments

To understand the behavior of the ratio $r_n(W)$ which depends on the minimum eigenvalue of the matrix W and n we consider the following experiment on random graphs. We ran 1000 instances on graphs of n vertices with $n = 32, 64, 128, 256, 512$. We consider 3 types of random graphs when the weights of the matrix W are generated as follows:

Type A: each weight w_{ij} $(i < j)$ is 0 or 1 in probability $\frac{1}{2}$; and with $w_{ii} = 0$ and $w_{ij} := w_{ji}$ $(i > j)$.

Type B: each weight w_{ij} $(i < j)$ is randomly drawn from the set $\{0, 1, 2, \ldots, 10\}$ with uniform distribution, and with $w_{ii} = 0$ and $w_{ij} := w_{ji}$ $(i > j)$.

Type C: each weight w_{ij} $(i < j)$ is randomly drawn from the set $\{0, 1, 2, \ldots, 100\}$ with uniform distribution, and with $w_{ii} = 0$ and $w_{ij} := w_{ji}$ $(i > j)$.

We summarize the results of our experiment in Table 1, where we denote by $r(G)$ the ratio $r_n(W)$, and by $\hat{r}(G)$ its average over 1000 runs. Likewise,

Table 1
Values of $\hat{r}(G)$ on random graphs

n	Type	$\hat{r}(G)$	$r_{\min}(G)$	$r_{\max}(G)$
	A	0.73	0.69	0.77
32	B	0.80	0.77	0.83
	C	0.82	0.79	0.84
	A	0.79	0.77	0.81
64	B	0.85	0.84	0.87
	C	0.86	0.85	0.87
	A	0.84	0.83	0.85
128	B	0.89	0.88	0.9
	C	0.9	0.89	0.9
	A	0.88	0.88	0.89
256	B	0.92	0.92	0.92
	C	0.93	0.92	0.93
	A	0.91	0.91	0.92
512	B	0.94	0.94	0.94
	C	0.95	0.94	0.95

$r_{\min}(G)$ and $r_{\max}(G)$ denote, respectively, the minimum and maximum values of the ratio $r_n(W)$ over the 1000 runs, and n denotes the size of the graph.

From the results summarized in Table 1 it can be seen that the ratio $\hat{r}(G)$ tends to 1 as n gets larger. Another observation is that the rate of convergence depends on the way the weights are selected. The above numerical experiments motivate us to theoretically analyze the behavior of r_n.

4. Probabilistic analysis of the upper bound

It is known that for unweighted random graphs, the maximum eigenvalue satisfies $\lambda_{\max}(A(G))\geqslant k_{\min}$, where $k_{\min}$ is the smallest vertex degree in the graph and $A(G)$ is the adjacency matrix of a graph G, see e.g., [1]. Therefore, $\lambda_{\max}(A(G))$ is of magnitude $O(n)$. A surprising fact is that for a weighted random graph, $\lambda_{\min}(W)$ is of magnitude $O(n^{1/2+\varepsilon})$ for any $\varepsilon > 0$ under quite general assumptions made on the way W is randomly generated. This latest fact which was proven by Füredi and Komlós [4] will be a key ingredient in the proof of our main result. More precisely, we need a more specific result tailored to our needs which will be derived from the analysis developed in [4] and is given in Lemma 4.1 below. We first state our assumptions and introduce some new notation. The random graph G on n vertices with weights matrix W is assumed to satisfy the following conditions:

(C1) W is an $n\times n$ matrix where the entries $(w_{ij})_{i<j}$ are independent bounded random variables with the same expectation $\mu > 0$ and variance σ^2.

(C2) $w_{ij} = w_{ji}$ $\forall i > j$ and $w_{ii} = 0$ $\forall i$.

Lemma 4.1. *Let W be a random symmetric matrix satisfying* (C1) *and* (C2), *with eigenvalues* $lambda_1(W)\geqslant\lambda_2(W)\geqslant\cdots\geqslant\lambda_n(W)$. *Then for any* $v\geqslant 0$ *and for any* $k\leqslant(\sigma/K)^{1/3}n^{1/6}$ we have

$$\mathrm{Prob}\left\{\max_{2\leqslant i\leqslant n}|\lambda_i(W)| > 2\sigma\sqrt{n}+v\right\} < \sqrt{n}\left(1-\frac{v}{2\sigma\sqrt{n}+v}\right)^k,$$

where $K\in(0,+\infty)$ *is such that* $|w_{ij}-\mu|\leqslant K,\ \forall i<j$.

Proof. Let A be a random symmetric matrix with entries a_{ij} such that $(a_{ij})_{i\leqslant j}$ are independent bounded random variables with bound K, and with the same expectation $\mu=0$ and variance σ^2, and let $a_{ij}=a_{ji}$ for $i>j$. Then, from the claim 3.3 in [4, pp. 237–238, see also p. 236] we have for any $v\geqslant 0$ and $k\leqslant(\frac{\sigma}{K})^{1/3}n^{1/6}$:

$$\mathrm{Prob}\left\{\max_{1\leqslant i\leqslant n}|\lambda_i(A)| > 2\sigma\sqrt{n}+v\right\} < \sqrt{n}\left(1-\frac{v}{2\sigma\sqrt{n}+v}\right)^k.$$

Let E be the matrix with all entries 1. Applying the above inequality with the random symmetric matrix $A:=W-\mu E$, we thus have

$$\mathrm{Prob}\left\{\max_{1\leqslant i\leqslant n}|\lambda_i(W-\mu E)| > 2\sigma\sqrt{n}+v\right\} < \sqrt{n}\left(1-\frac{v}{2\sigma\sqrt{n}+v}\right)^k.$$

To complete the proof, it remains to show that

$$\max_{2\leqslant i\leqslant n}|\lambda_i(W)|\leqslant\max_{1\leqslant i\leqslant n}|\lambda_i(W-\mu E)|. \tag{4.2}$$

Noting that

$$\max_{2\leqslant i\leqslant n}|\lambda_i(W)|=\max\{|\lambda_2(W)|,|\lambda_{\min}(W)|\},$$

$$\max_{1\leqslant i\leqslant n}|\lambda_i(W-\mu E)|=\max\{|\lambda_{\max}(W-\mu E)|, |\lambda_{\min}(W-\mu E)|\}$$

and using the facts that $\lambda_2(W)\leqslant\lambda_{\max}(W-\mu E)$ and $\lambda_{\min}(W)\geqslant\lambda_{\min}(W-\mu E)$ (see e.g., Lemmas 1 and 2 in [4, pp. 237–238]), it can be verified that (4.2) holds, and hence the desired result follows. □

We now state and prove our main result.

Theorem 4.1. *Let G be a random graph with n vertices and weights matrix W satisfying* (C1) *and* (C2). *Then given $\eta\in(0,\frac{1}{2})$ and $\alpha\in(0,1)$ the following hold*:
$\forall n>1$ with probability greater than

$$1-\sqrt{n}\left(\frac{1}{n^\eta+1}\right)^{(\sigma/K)^{1/3}n^{1/6}}-\frac{2\sigma^2}{n(n-1)\mu^2\alpha^2},$$

we have

$$\frac{1}{2}\sum_{i<j}w_{ij}\leqslant \mathrm{mc}(G) \leqslant\frac{1}{2}\sum_{i<j}w_{ij}\left(1+\frac{2\sigma}{\mu(1-\alpha)}\frac{\sqrt{n}+n^{1/2+\eta}}{n-1}\right).$$

Proof. First, by applying Lemma 4.1 with $k := (\sigma/K)^{1/3} n^{1/6}$ and $v := 2\sigma n^{1/2+\eta}$ we obtain

$$\begin{aligned}\text{Prob}\{|\lambda_{\min}(W)| > 2\sigma\sqrt{n} + 2\sigma n^{1/2+\eta}\} &\leqslant \text{Prob}\left\{\max_{2\leqslant i\leqslant n} |\lambda_i(W)| > 2\sigma\sqrt{n} + 2\sigma n^{1/2+\eta}\right\} \\ &< \sqrt{n}\left(\frac{2\sigma\sqrt{n}}{2\sigma\sqrt{n} + 2\sigma n^{1/2+\eta}}\right)^{(\sigma/K)^{1/3} n^{1/6}} \\ &= \sqrt{n}\left(\frac{1}{n^{\eta}+1}\right)^{(\sigma/K)^{1/3} n^{1/6}}. \end{aligned} \tag{4.3}$$

Define $W_t := \sum_{i,j} w_{ij} = 2\sum_{i<j} w_{ij}$ (since here $w_{ii} = 0$) and recall that w_{ij} $(i<j)$ are independent random variables with the same expectation μ and variance σ^2. Then, using Tschebycheff inequality [3, p. 247], we obtain that

$$\forall \tau > 0 \quad \text{Prob}\left\{\left|\sum_{i<j} (w_{ij} - \mu)\right| > \tau\right\} \leqslant \frac{n(n-1)\sigma^2}{2\tau^2}.$$

Pick $\tau := n(n-1)/2\mu\alpha$ with $\alpha \in (0,1)$, then

$$\text{Prob}\left\{\left|\frac{W_t}{2} - \frac{n(n-1)}{2}\mu\right| > \frac{n(n-1)}{2}\mu\alpha\right\} \leqslant \frac{2\sigma^2}{n(n-1)\mu^2\alpha^2}.$$

Hence, in particular, we get

$$\begin{aligned}\text{Prob}\left\{\frac{W_t}{2} < \frac{n(n-1)}{2}(1-\alpha)\mu\right\} &\leqslant \text{Prob}\left\{\left|\frac{W_t}{2} - \frac{n(n-1)}{2}\mu\right| > \frac{n(n-1)}{2}\mu\alpha\right\}, \\ &\leqslant \frac{2\sigma^2}{n(n-1)\mu^2\alpha^2}. \end{aligned} \tag{4.4}$$

For convenience, we now define the following events:

$$A_n := \{|\lambda_{\min}(W)| > 2\sigma\sqrt{n} + 2\sigma n^{1/2+\eta}\},$$

$$B_n := \left\{\frac{W_t}{2} < \frac{n(n-1)}{2}(1-\alpha)\mu\right\}$$

and the probability

$$p_{n,\eta} := 1 - \sqrt{n}\left(\frac{1}{n^{\eta}+1}\right)^{(\sigma/K)^{1/3} n^{1/6}} - \frac{2\sigma^2}{n(n-1)\mu^2\alpha^2}.$$

Combining (4.3) and (4.4) we then obtain

$$\text{Prob}\{A_n \cup B_n\} \leqslant \text{Prob}(A_n) + \text{Prob}(B_n) < 1 - p_{n,\eta}$$

and hence:

$$\text{Prob}\{\bar{A}_n \cap \bar{B}_n\} > p_{n,\eta},$$

where

$$\bar{A}_n := \{|\lambda_{\min}(W)| \leqslant 2\sigma\sqrt{n} + 2\sigma n^{1/2+\eta}\} \quad \text{and}$$

$$\bar{B}_n := \left\{\frac{W_t}{2} \geqslant \frac{n(n-1)}{2}(1-\alpha)\mu\right\}$$

denote the complementary events of A_n and B_n, respectively. Therefore, with probability greater than $p_{n,\eta}$ we have obtained that

$$\begin{aligned} r_n(W) &= \frac{1}{1 - n\lambda_{\min}(W)/W_t} = \frac{1}{1 + n|\lambda_{\min}(W)|/W_t} \\ &\geqslant \frac{1}{1 + (2\sigma n^{3/2} + \sigma n^{3/2+\eta})/[n(n-1)(1-\alpha)\mu]} \\ &= \frac{1}{1 + [2/(1-\alpha)]\sigma/\mu(\sqrt{n} + n^{1/2+\eta})/(n-1)} \end{aligned}$$

where in the first equality we used the fact that $\lambda_{\min}(W) \leqslant 0$ (since here $\text{tr}(W) = 0 = \sum_{i=1}^{n} \lambda_i(W) \geqslant n\lambda_{\min}(W)$), and this completes the proof. □

An easy by-product of Theorem 4.1 is the following asymptotic result.

Corollary 4.1. *The value of the ratio* $\frac{1}{2}\sum_{i<j} w_{ij}/\text{mc}(G)$ *tends to* 1 *with probability approaching* 1 *as* $n \to \infty$. *More precisely*, $\forall \varepsilon > 0$, $\exists N$ *such that* $\forall n > N$ *with a probability greater than* $1-\varepsilon$ *we have*

$$1 - \varepsilon \leqslant \frac{\frac{1}{2}\sum_{i<j} w_{ij}}{\text{mc}(G)} \leqslant 1.$$

Proof. Note that $\forall \eta \in (0,1)$ and $\forall \alpha \in (0,1)$ we have that as

$$n \to \infty : p_{n,\eta} \to 1 \quad \text{and}$$

$$r_{n,\eta} := \frac{1}{1 + 2/(1-\alpha)\sigma/\mu(\sqrt{n} + n^{1/2+\eta})/(n-1)} \to 1.$$

Therefore $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n \geqslant N$ it holds that

$$1 - p_{n,\eta} < \varepsilon \quad \text{and} \quad r_{n,\eta} > 1 - \varepsilon$$

and hence with probability greater than $1-\varepsilon$ we have

$$\frac{\frac{1}{2}\sum_{i<j} w_{ij}}{\text{mc}(G)} \geqslant r_{n,\eta} > 1 - \varepsilon. \qquad \square$$

Table 2
Values of $p_{n,\eta}$ and $r_{n,\eta}$

n	$p_{n,1/4}$	$r_{n,1/4}$	$p_{n,1/8}$	$r_{n,1/8}$	$p_{n,1/16}$	$r_{n,1/16}$
100	0.330	0.535	−0.313	0.633	−0.818	0.672
200	0.623	0.588	−0.094	0.698	−0.766	0.740
300	0.729	0.618	0.001	0.733	−0.766	0.740
400	0.788	0.638	0.071	0.756	−0.761	0.797
1000	0.917	0.697	0.320	0.819	−0.661	0.857
5000	0.993	0.784	0.745	0.897	−0.158	0.926
10000	0.998	0.814	0.866	0.920	−0.128	0.945
20000	0.999	0.841	0.940	0.939	0.404	0.959
30000	0.999	0.855	0.966	0.947	0.545	0.966
40000	0.999	0.864	0.978	0.953	0.634	0.970
50000	0.999	0.871	0.984	0.956	0.695	0.973
100000	0.999	0.890	0.995	0.967	0.842	0.980

We note that an asymptotic result similar to the one given in Corollary 4.1 can be found in [2, Theorem 8] where it was established that $\varphi(G)/\mathrm{mc}(G) \to 1$ as $n \to \infty$.

The following example illustrates Theorem 4.1, by computing the explicit numbers $p_{n,\eta}, r_{n,\eta}$ for the choice $\alpha := \frac{1}{32}$. Consider matrices with entries either 0 or 1 in probability $\frac{1}{2}$. We then get $\sigma = \frac{1}{2}\ \mu = \frac{1}{2}$ and $K = \frac{1}{2}$. Applying Theorem 4.1 to this class of random matrices gives that with probability greater than $p_{n,\eta}$ we have

$$\frac{\frac{1}{2}\sum_{i<j} w_{ij}}{\mathrm{mc}(G)} \geqslant r_{n,\eta},$$

where

$$p_{n,\eta} := 1 - \sqrt{n}\left(\frac{1}{n^{\eta}+1}\right)^{n^{1/6}} - \frac{2048}{n(n-1)},$$

$$r_{n,\eta} := \frac{1}{1 + \frac{64}{31}(\sqrt{n} + n^{1/2+\eta})/(n-1)}.$$

Table 2 summarizes the values of the probabilities $p_{n,\eta}$ and the ratio $r_{n,\eta}$ for various choices of n and $\eta \in (0, \frac{1}{2})$. Note that negative values in the probability columns indicate that the theorem does not furnish any useful information.

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