A SEQUENTIAL ASCENDING PARAMETER METHOD FOR SOLVING CONSTRAINED MINIMIZATION PROBLEMS[*]

AMIR BECK[†], AHARON BEN-TAL[†], AND LUBA TETRUASHVILI[†]

Abstract. In this paper, a method for solving constrained convex optimization problems is introduced. The problem is cast equivalently as a parametric unconstrained one, the (single) parameter being the optimal value of the original problem. At each stage of the algorithm the parameter is updated, and the resulting subproblem is only approximately solved. A linear rate of convergence of the parameter sequence is established. Using an optimal gradient method due to Nesterov [Dokl. Akad. Nauk SSSR, 269 (1983), pp. 543–547] to solve the arising subproblems, it is proved that the resulting gradient-based algorithm requires an overall of $O(\log(1/\varepsilon)/\sqrt{\varepsilon})$ inner iterations to obtain an ε-optimal and feasible solution. An image deblurring problem is solved, demonstrating the capability of the algorithm to solve large-scale problems within reasonable accuracy.

Key words. convex optimization, linear rate of convergence, optimal gradient methods, image deblurring

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1. Introduction. Consider the following general constrained minimization problem:

$$
\begin{aligned}
\text{(1.1)} \qquad \min \quad & f(\mathbf{x}) \\
\text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m, \\
& \mathbf{x} \in X.
\end{aligned}
$$

It is well known (see, e.g., [5, Lemma 2.3.4]) that the optimal value t^* of (1.1) is the smallest root of the single variable function

$$
\text{(1.2)} \qquad F^*(t) = \min_{\mathbf{x} \in X} \left[\max\{f(\mathbf{x}) - t, g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})\} \right].
$$

The optimal set of (1.1) is the set of minimizers in the above minimization problem when $t = t^*$.

In this paper we analyze the following simple scheme for finding the optimal value of (1.1):

$$
\text{(1.3)} \qquad t_{k+1} = t_k + F^*(t_k).
$$

The generated sequence monotonically ascends to t^*. The resulting method is called the SAP (sequential ascending parameter) method, and its basic properties are given in section 2.

SAP is in fact a theoretical algorithm since $F^*(t_k)$ cannot usually be computed *exactly*, as it is the value of an optimization problem. In section 3, we introduce an implementable version of the SAP algorithm, called ISAP. In this method, given an "accuracy parameter" $\varepsilon > 0$, it is assumed that an algorithm—let it be called $\mathcal{M}$— solves the corresponding subproblem with ε-accuracy. We prove that the resulting

ISAP algorithm finds an ε-optimal and feasible solution of problem (1.1), by which we mean a solution which is ε-optimal *and* ε-feasible. It is shown that the rate of convergence is *linear*, and based on this rate an upper bound on the computational effort needed to achieve an ε-optimal and feasible solution is given.

It should be noted that the SAP method (1.3) was analyzed in [3], and that an implementable version of a slightly different updating rule was considered in [2] in the context of bundle methods. However, in these papers, only a *sublinear* convergence rate was established.

In section 4, the method $\mathcal{M}$ is specified to be the algorithm of Nesterov from [4]. This method happens to be optimally adapted to the max-type objective function to be minimized in problem (1.2); we call this method OPTGRAD. When the objective and constraint functions in problem (1.1) are differentiable, with Lipschitz continuous gradients, OPTGRAD generates an ε-optimal solution in $O(1/\sqrt{\epsilon})$ iterations. Combined with the linear rate of ISAP, the method ISAP + OPTGARD needs an overall of $O(\log(1/\varepsilon)/\sqrt{\epsilon})$ OPTGRAD iterations to solve (1.1) with ε-accuracy.

The capability of ISAP in solving large-scale constrained convex problems, with reasonable accuracy, is demonstrated in section 5. There we solve an image deblurring problem with 196,602 variables within an accuracy of $\varepsilon = 10^{-3}$. The algorithm needed only three outer iterations. This low number of outer iterations was observed in many other numerical tests we performed, and was significantly smaller than the number of iterations needed when using a bisection method. Moreover, this behavior was essentially independent of how far the initial parameter t_1 was from the optimal one, t^*.

2. Problem formulation.

2.1. A parametric representation. Consider problem (1.1), where f and $g_1, \ldots, g_m$ are continuous functions over $\mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$ is a closed convex set. By denoting

$$G(\mathbf{x}) = \max_{i=1,2,\ldots,m} \{g_i(\mathbf{x})\},$$

problem (1.1) can be reformulated as

$$
(2.1) \qquad (P) \qquad
\begin{array}{ll}
\min & f(\mathbf{x}) \\
\text{s.t.} & G(\mathbf{x}) \leq 0, \\
& \mathbf{x} \in X.
\end{array}
$$

Throughout the paper we make the following assumptions on problem (P).

Assumption A1. The minimum of problem (P), denoted by t^*, is finite and attained. The optimal set of (P) is denoted by X^*.

Assumption A2. For each $t \in \mathbb{R}$, the minimum of the function $\max\{f(\mathbf{x}) - t, G(\mathbf{x})\}$ over the set X is attained.

Note that Assumptions A1 and A2 hold true, for instance, when one (or more) of the functions $f, g_1, \ldots, g_m$ is coercive or when X is a compact set. It is well known that the constrained problem (P) is equivalent to the following problem of minimizing a max-type function over X:

$$(2.2) \qquad \min_{\mathbf{x} \in X}\{\max[f(\mathbf{x}) - t^*, G(\mathbf{x})]\}.$$

The optimal set of (2.2) is X^*, and its optimal value is zero. This means that if the optimal value is known, then the constrained problem (P) can be recast as a

minimization problem over the set X, which is "simpler" than the feasible set of (P). Of course, in most cases the optimal value t^* is unknown, and therefore some kind of a one-dimensional search for finding it is required. This one-dimensional search is performed on the univariate function

$$(2.3) \qquad F^*(t) = \min_{\mathbf{x} \in X} \left[\max\{ f(\mathbf{x}) - t, G(\mathbf{x}) \} \right].$$

The function $F^*(t)$ is well defined since $F^*(t) > -\infty$ for all $t \in \mathbb{R}$ by Assumption A1. The next lemma spells out some elementary and well-known properties of F^*, which are recalled here for completeness and proven in Appendix A.

LEMMA 2.1. *Let t^* be the optimal value of problem* (P). *Then the following hold:*
(a) *F^* is nonincreasing over $\mathbb{R}$.*
(b) *$F^*(t) > 0$ for all $t < t^*$.*
(c) *$F^*(t) \leq 0$ for all $t \geq t^*$.*
(d) *F^* is Lipschitz continuous with constant 1. That is,*

$$(2.4) \qquad F^*(u) - F^*(v) \leq v - u \text{ for every } u \leq v.$$

2.2. The SAP method. Lemma 2.1 establishes the fact that t^* is the smallest root of a nonincreasing continuous univariate function over $\mathbb{R}$. To find this root, one can employ, for example, a bisection procedure. We will nonetheless consider a different method in which the parameter t increases at each iteration; it will consequently be called the *sequential ascending parameter* (SAP) algorithm and is given next.

Algorithm SAP.
- **Initialization:** Choose $t_1 < t^*$.
- **General step.**

$$(2.5) \qquad t_{k+1} = t_k + F^*(t_k), \quad k = 1, 2, \dots .$$

The SAP algorithm was analyzed in [3], where it was shown that the sequence $\{F^*(t_k)\}_{k \geq 1}$ converges to 0 in a sublinear rate and that the sequence $\{t_k\}_{k \geq 0}$ is either finite if $t_k = t^*$ for some k or strictly increasing if infinite and converges to the optimal value t^*. We recall for completeness these basic facts on the SAP method, which are summarized in the next theorem, whose simple proof is given in Appendix B.

THEOREM 2.2. *Let $\{t_k\}_{k \geq 1}$ be the sequence generated by the SAP method. Then*
(a) *$t_k \leq t^*$ for every $k \geq 1$;*
(b) *for every $k \geq 1$, $t_k \leq t_{k+1}$ and $t_k = t_{k+1}$ if and only if $t_k = t^*$;*
(c) *$F^*(t_k) \leq \frac{t^* - t_1}{k}$ for every $k \geq 1$;*
(d) *$t_k \to t^*$ and $F^*(t_k) \to 0$ as $k \to \infty$.*

Another variant of the algorithm in which the update step takes the form $t_{k+1} = t_k + \eta F^*(t_k)$ for some $\eta \in (0,1)$ was analyzed in [2] for the convex case where a sublinear rate of convergence was established.

In a sense, the above SAP algorithm is a "conceptual" algorithm, since it assumes that the subproblems associated with the evaluation of $F^*(t_k)$ are solved *exactly*, and in addition no stopping criteria is provided. A practical implementation of the algorithm should assume that the subproblems are solved only up to some tolerance and should offer some kind of a stopping criterion. In the next section we analyze this "implementable" SAP method in the convex case and show that its rate of convergence is in fact *linear*.

3. The implementable SAP method—The convex case.

3.1. The method. From this section on we assume that the functions $f, g_1, \ldots,$ g_m are convex over $\mathbb{R}^n$. Note that in this case F^* is also a convex function. We will also assume, in addition to Assumptions A1 and A2, the following Slater-type condition.

Assumption A3. The scalar $G_{\min}$ defined by

$$(3.1) \qquad G_{\min} \equiv \min\{G(\mathbf{x}) : \mathbf{x} \in X\}$$

is finite and negative, and the minimum in (3.1) is attained.

So far we have described the SAP method for the case where the exact values of the function $F^*(t_k)$ are computed. However, in a "real" implementation, the values of $F^*(t_k)$ are computed up to some tolerance (as their computation involves the solution of an optimization problem). Our ultimate objective is to find an ε-*optimal and feasible solution*, that is, a point $\tilde{\mathbf{x}} \in X$ satisfying $f(\tilde{\mathbf{x}}) - t^* \leq \varepsilon$, $G(\tilde{\mathbf{x}}) \leq \varepsilon$.

Suppose now that there exists a method $\mathcal{M}$ for finding an ε-optimal solution of problems of the type

$$H^* = \min_{\mathbf{x} \in X} \max\{h_1(\mathbf{x}), h_2(\mathbf{x})\}.$$

The input of the method $\mathcal{M}$ consists of the convex functions h_1, h_2, the closed convex set X, and a tolerance parameter $\varepsilon > 0$. The output of $\mathcal{M}$ is an $\mathbf{x} \in X$ satisfying

$$H^* \leq \max\{h_1(\mathbf{x}), h_2(\mathbf{x})\} \leq H^* + \varepsilon.$$

We are now ready to develop the *implementable SAP* (ISAP) method, which depends on a predescribed tolerance parameter $\varepsilon > 0$ and on the method $\mathcal{M}$. We will use the notation

$$(3.2) \qquad F_t(\mathbf{x}) \equiv \max\{f(\mathbf{x}) - t, G(\mathbf{x})\}.$$

In this notation we have $F^*(t) = \min_{\mathbf{x} \in X} F_t(\mathbf{x})$.

Algorithm ISAP.
- **Initialization:** Choose $t_1 < t^*$ and $\varepsilon > 0$.
- **Step k ($k \geq 1$).** Employ method $\mathcal{M}$ with input $\left(f(\cdot) - t_k, G(\cdot), X, \frac{\varepsilon}{3}\right)$, and obtain an $\mathbf{x}_k$ satisfying

$$F^*(t_k) \leq F_{t_k}(\mathbf{x}_k) \leq F^*(t_k) + \frac{\varepsilon}{3}.$$

If $F_{t_k}(\mathbf{x}_k) > \frac{2}{3}\varepsilon$, then update

$$(3.3) \qquad\qquad t_{k+1} := F_{t_k}(\mathbf{x}_k) + t_k$$

and set $k \leftarrow k + 1$. Otherwise, stop.

The output of the ISAP algorithm is $\mathbf{x}_K$, where K denotes the index of the last iteration.

3.2. Convergence analysis. The next theorem establishes the fact that the sequence $\{t_k\}_{k=1}^K$ is strictly increasing and—excluding the last iterate—upper bounded by t^*. It is also shown that the last iterate $\mathbf{x}_K$ is an ε-optimal and feasible solution of problem (P) given in (2.1).

THEOREM 3.1. *Let $\{t_k, \mathbf{x}_k\}_{k=1}^K$ be the sequence generated by the ISAP method. Then we have the following:*

(a) $t_1 < t_2 < \cdots < t_K$ *and $t_k < t^*$ for every $k = 1, \ldots, K-1$.*

(b) $t_K \leq t^* + \varepsilon/3$.

(c) *At the last iteration K,*

$$(3.4) \qquad f(\mathbf{x}_K) - t^* \leq \varepsilon, \quad G(\mathbf{x}_K) \leq \varepsilon.$$

Proof. (a) For $k < K$ the stopping criteria is not satisfied; i.e., we have $F_{t_k}(\mathbf{x}_k) > \frac{2}{3}\varepsilon$. Thus, for all $k < K$,

$$F^*(t_k) \geq F_{t_k}(\mathbf{x}_k) - \frac{\varepsilon}{3} > 0.$$

Hence, from Lemma 2.1(b) and (c), we obtain that $t_k < t^*$ for all $k < K$. The positivity of $F^*(t_k)$ for $1 \leq k \leq K-1$, along with the updating rule (3.3), implies that $t_k < t_{k+1}$ for all $1 \leq k < K$ and consequently $t_1 < t_2 < \cdots < t_{K-1} < t_K$.

(b) Let $\mathbf{x}^*$ be an optimal solution of problem (P). Then

$$t_K = F_{t_{K-1}}(\mathbf{x}_{K-1}) + t_{K-1} \leq F^*(t_{K-1}) + \frac{\varepsilon}{3} + t_{K-1}$$

$$\leq \max\{f(\mathbf{x}^*) - t_{K-1}, G(\mathbf{x}^*)\} + \frac{\varepsilon}{3} + t_{K-1}$$

$$= \max\{t^* - t_{K-1}, G(\mathbf{x}^*)\} + \frac{\varepsilon}{3} + t_{K-1} = t^* + \frac{\varepsilon}{3},$$

where in the last inequality we used the fact that $t_{K-1} < t^*$ and the feasibility of $\mathbf{x}^*$.

(c) Since $(t_K, \mathbf{x}_K)$ satisfies the stopping criteria we have

$$F_{t_K}(\mathbf{x}_K) \leq \frac{2}{3}\varepsilon.$$

By the definition of $F_t(\cdot)$ given in (3.2), it follows that

$$G(\mathbf{x}_K) \leq \frac{2}{3}\varepsilon < \varepsilon, \quad f(\mathbf{x}_K) - t_K \leq \frac{2}{3}\varepsilon,$$

which combined with the inequality $t_K \leq t^* + \varepsilon/3$, proven in part (b), implies that

$$f(\mathbf{x}_K) - t^* \leq f(\mathbf{x}_K) - t_K + \frac{\varepsilon}{3} \leq \varepsilon,$$

establishing the desired result.   □

Our main result is the linear rate of convergence of the method, whose proof heavily relies on the following key lemma.

LEMMA 3.2. *Let $\{t_k\}_{k=1}^K$ be the sequence generated by the ISAP method. Suppose that*[1] *$K > 2$. Then for all $1 < k \leq K$,*

$$(3.5) \qquad F^*(t_k) \leq (1 - |(F^*)'(t^*)|)F^*(t_{k-1}),$$

---

[1]Otherwise, either the first or second iterate already satisfies the stopping criteria and is therefore an ε-optimal and feasible solution.

where $(F^*)'(t^*)$, which is a subgradient of F^* at t^*, satisfies

$$-1 < (F^*)'(t^*) < 0.$$

Proof. Suppose that $K > 2$. By the subgradient inequality we have that for all $t \in \mathbb{R}$,

$$F^*(t) - F^*(t^*) \geq (F^*)'(t^*)(t - t^*),$$

which, combined with the fact that $F^*(t^*) = 0$, yields

(3.6) $$F^*(t) \geq -(F^*)'(t^*)(t^* - t).$$

Note that since $K > 2$, it follows by Theorem 3.1(a) that $t_2 = t_1 + F_{t_1}(\mathbf{x}_1) < t^*$, implying that

(3.7) $$t_1 + F^*(t_1) \leq t_1 + F_{t_1}(\mathbf{x}_1) < t^*.$$

Substituting $t = t_1$ into (3.6) and using (3.7), it follows that

$$-(F^*)'(t^*) < 1.$$

We will now show that $(F^*)'(t^*)$ is negative. Let $\tilde{\mathbf{x}} \in X$ be a point satisfying $G(\tilde{\mathbf{x}}) < 0$. The existence of such a point is guaranteed by Assumption A3. Set $\tilde{t} = f(\tilde{\mathbf{x}}) + 1$. Then

$$F^*(\tilde{t}) = \min_{\mathbf{x}} F_{\tilde{t}}(\mathbf{x}) \leq F_{\tilde{t}}(\tilde{\mathbf{x}}) = \max\{f(\tilde{\mathbf{x}}) - \tilde{t}, G(\tilde{\mathbf{x}})\} < \max\{-1, G(\tilde{\mathbf{x}})\} < 0.$$

The left-hand side of (3.6) after substituting $t = \tilde{t}$ is negative, implying that the value of $(F^*)'(t^*)(t^* - \tilde{t})$ is positive. Therefore, $(F^*)'(t^*)$ is negative, which means that (3.6) can be rewritten as

(3.8) $$F^*(t) \geq |(F^*)'(t^*)|(t^* - t),$$

where $(F^*)'(t^*) \in (-1, 0)$.

Let k be an integer satisfying $1 < k \leq K$. Then $\alpha \equiv \frac{t_k - t_{k-1}}{t^* - t_{k-1}} \in (0, 1)$. Simple algebra shows that $t_k = \alpha t^* + (1 - \alpha)t_{k-1}$, which, combined with the convexity of F^*, implies that

(3.9) $$F^*(t_k) \leq \alpha F^*(t^*) + (1 - \alpha)F^*(t_{k-1}) = (1 - \alpha)F^*(t_{k-1})$$
$$= \left(1 - \frac{t_k - t_{k-1}}{t^* - t_{k-1}}\right) F^*(t_{k-1}).$$

Using the updating rule $t_k := F_{t_{k-1}}(\mathbf{x}_{k-1}) + t_{k-1}$ and the fact that $F^*(t_{k-1}) \leq F_{t_{k-1}}(\mathbf{x}_{k-1})$, we obtain

(3.10) $$F^*(t_k) \leq \left(1 - \frac{F^*(t_{k-1})}{t^* - t_{k-1}}\right) F^*(t_{k-1}),$$

which, along with (3.8), establishes the desired result. $\quad\square$

The linear rate of convergence is established in the next theorem.

THEOREM 3.3 (linear rate of convergence of ISAP). *Let $\{(t_k, \mathbf{x}_k)\}_{k=1}^K$ be the sequence generated by the ISAP method, and suppose that $K > 2$. Let $\mathbf{x}_G$ be a minimizer of $\min_{\mathbf{x} \in X} G(\mathbf{x})$, and let $G_{\min} = G(\mathbf{x}_G)$. Then for all $1 < k \leq K$*

$$(3.11) \qquad F^*(t_k) \leq \rho^{k-1} F^*(t_1),$$

where

$$(3.12) \qquad \rho = \frac{f(\mathbf{x}_G) - t^*}{|G_{\min}| + f(\mathbf{x}_G) - t^*} \in [0, 1).$$

Proof. Let $\tilde{t} = f(\mathbf{x}_G) - G_{\min}$. Then

$$F^*(\tilde{t}) = \min_{\mathbf{x} \in X}\{\max[f(\mathbf{x}) - \tilde{t}, G(\mathbf{x})]\} = \min_{x \in X}\{\max[f(\mathbf{x}) - f(\mathbf{x}_G) + G_{\min}, G(\mathbf{x})]\}$$

$$(3.13) \qquad \leq \max\{f(\mathbf{x}_G) - f(\mathbf{x}_G) + G_{\min}, G(\mathbf{x}_G)\} = G_{\min}.$$

It follows by Assumption A3 that $G_{\min} < 0$ and thus $\tilde{t} > t^*$. Using the subgradient inequality together with the fact that $F^*(t^*) = 0$, we obtain that

$$(F^*)'(t^*) \leq \frac{F^*(\tilde{t})}{\tilde{t} - t^*},$$

which combined with (3.13) yields

$$(3.14) \qquad (F^*)'(t^*) \leq -\frac{|G_{\min}|}{|G_{\min}| + f(\mathbf{x}_G) - t^*}.$$

Recall (Lemma 3.2) that $(F^*)'(t^*) < 0$. Utilizing this fact, it follows that the latter inequality is equivalent to

$$1 - |(F^*)'(t^*)| \leq \frac{f(\mathbf{x}_G) - t^*}{|G_{\min}| + f(\mathbf{x}_G) - t^*}.$$

The required result (3.11) then follows from Lemma 3.2. $\qquad \square$

Remark 3.1. A proof almost identical to that given for Theorem 3.3 shows that inequality (3.11) can be replaced by

$$F^*(t_k) \leq \left(\frac{f(\tilde{\mathbf{x}}) - t^*}{|G(\tilde{\mathbf{x}})| + f(\tilde{\mathbf{x}}) - t^*}\right)^{k-1} F^*(t_1),$$

where $\tilde{\mathbf{x}} \in X$ is *any* point satisfying $G(\tilde{\mathbf{x}}) < 0$ and not necessarily a minimizer of $\min\{G(\mathbf{x}) : \mathbf{x} \in X\}$.

Note that the utopian SAP method coincides with the ISAP method when $\varepsilon = 0$ and when $\mathbf{x}_k$ is taken at each iteration to be the exact minimizer of $\min_{\mathbf{x} \in X} F_{t_k}(\mathbf{x})$. The stopping criteria is taken to be $t_k = t^*$. The analysis of the results so far does not depend on the positivity of ε, and therefore we can state a variant of Theorem 3.3 for the SAP method.

COROLLARY 3.4 (linear rate of convergence of SAP). *Let $\{t_k\}_{k=1}^K$ be the sequence generated by the SAP method with stopping criteria $F^*(t_k) = 0$. Here, $K = \infty$ corresponds to the case where the sequence is infinite. If $2 < K < \infty$, then for every $k = 2, \ldots, K$*

$$(3.15) \qquad F^*(t_k) \leq \rho^{k-1} F^*(t_1),$$

where ρ is defined in (3.12). *If $K = \infty$, then* (3.15) *holds for all $k \geq 1$.*

Returning to the more realistic ISAP method, it is important to note that we have established a linear rate of convergence for the outer process in ISAP, which updates the estimates t_k for the optimal function value. In addition, at each iteration, a method, denoted by $\mathcal{M}$, is employed. We will assume that this is also an iterative method whose iterations will be called "inner iterations." The complexity of the method can be estimated as the overall number of inner iterations which are performed. In order to estimate the complexity, it is imperative that some kind of an upper bound on the number of inner iterations at each evaluation of F^* be given. Given such a bound, the next corollary establishes an upper bound on the total number of inner iterations.

THEOREM 3.5. *Suppose that the number of computations which are required to employ the method $\mathcal{M}$ at each iteration of the ISAP method is upper bounded by $H(\varepsilon)$. Then after at most $(A + B\log(1/\varepsilon))H(\varepsilon)$ inner iterations, where*

$$A = \frac{\log(3F^*(t_1))}{\log(1/\rho)} + 2, \qquad B = \frac{1}{\log(1/\rho)},$$

and ρ is defined in (3.12), *the ISAP method produces an ε-optimal and feasible solution.*

Proof. First note that the inequality

$$(3.16) \qquad \rho^{k-1}F^*(t_1) \leq \varepsilon/3,$$

after taking log from both sides and after some rearrangement of terms, is equivalent to

$$(3.17) \qquad k \geq \frac{\log(3F^*(t_1)/\varepsilon)}{\log(1/\rho)} + 1.$$

Now, let us show that the number of outer iterations, K, of the ISAP method satisfies

$$(3.18) \qquad K \leq \frac{\log(3F^*(t_1)/\varepsilon)}{\log(1/\rho)} + 2.$$

Suppose by contradiction that $K > \frac{\log(3F^*(t_1)/\varepsilon)}{\log(1/\rho)} + 2$. Then this would mean that $K - 1 > \frac{\log(3F^*(t_1)/\varepsilon)}{\log(1/\rho)} + 1$. But then, by the equivalency of (3.16) and (3.17), it follows that

$$\rho^{(K-1)-1}F^*(t_1) \leq \varepsilon/3,$$

and hence, by Theorem 3.3, we have that $F^*(t_{K-1}) \leq \varepsilon/3$, which implies that

$$F_{t_{K-1}}(\mathbf{x}_{K-1}) \leq F^*(t_{K-1}) + \frac{\varepsilon}{3} \leq \frac{2}{3}\varepsilon.$$

Therefore, the stopping criterion is satisfied for iteration $K-1$, which is a contradiction to the definition of K as the first index for which the stopping criteria is satisfied. The result now follows by combining the bound (3.18) with the fact that each outer iteration of the ISAP method consists of at most $H(\varepsilon)$ inner iterations. ☐

3.3. Some examples.

Example 3.6. The linear convergence rate of the ISAP and SAP methods established in Theorem 3.3 and Corollary 3.4, respectively, rely on the validity of Assumption A3, which assumes the existence of a point $\tilde{\mathbf{x}} \in X$ such that $G(\tilde{\mathbf{x}}) < 0$. In this example we show that this assumption is in fact essential. Consider the problem

$$
\begin{aligned}
\min \quad & x \\
\text{s.t.} \quad & x^2 \leq 0.
\end{aligned}
$$

Here $X = \mathbb{R}$, $f(x) = x$, $G(x) = x^2$. Obviously Assumption A3 is not satisfied. The only feasible point is $x = 0$, and the optimal value is 0. Now, let us apply the SAP method with $t_1 = -1$:

$$
t_1 = -1, \quad t_{k+1} = t_k + F^*(t_k), \qquad k = 1, 2, \ldots .
$$

Here $F^*(t_k) = \min_x \max\{x - t_k, x^2\}$. It is easy to see that the optimal solution of the latter minimization problem is the smallest x satisfying $x - t_k = x^2$. Thus, $x_k = \frac{1 - \sqrt{1 - 4t_k}}{2}$ and therefore

$$
t_{k+1} = t_k + F^*(t_k) = t_k + x_k - t_k = x_k = \frac{1 - \sqrt{1 - 4t_k}}{2}.
$$

By Theorem 2.2(d), $F^*(t_k) \to 0$ as $k \to \infty$. We will show that the sequence $\{F^*(t_k)\}_{k \geq 1}$ does not converge to 0 in a linear rate. To prove that, we will first show by induction that $t_k \leq -\frac{1}{k}$. Of course $t_1 = -1 \leq -1$. Suppose that $t_k \leq -\frac{1}{k}$. We would like to show that $t_{k+1} \leq -\frac{1}{k+1}$. Indeed, since $t_k \leq -\frac{1}{k}$, we have

$$
t_{k+1} = \frac{1 - \sqrt{1 - 4t_k}}{2} \leq \frac{1 - \sqrt{1 + 4/k}}{2}.
$$

It is therefore sufficient to prove that

$$
\frac{1 - \sqrt{1 + 4/k}}{2} \leq -\frac{1}{k+1},
$$

which can be easily seen to hold for all $k \geq 1$. The inequality $t_k \leq -\frac{1}{k}$ implies that

$$
F^*(t_k) = x_k^2 = t_{k+1}^2 \geq \frac{1}{(k+1)^2},
$$

showing that $F^*(t_k)$ does not converge with a linear rate to 0.

Example 3.7. When Assumption A3 is satisfied, then it was shown in Corollary 3.4 that linear rate of convergence of $\{F^*(t_k)\}_{k \geq 1}$ is guaranteed. Specifically,

$$
F^*(t_k) \leq \rho^{k-1} F^*(t_1), \quad k = 1, 2, \ldots,
$$

where the linear factor ρ is given by

$$
q = \frac{f(x_G) - t^*}{|G_{\min}| + f(x_G) - t^*}
$$

and $x_G \in \operatorname{argmin}\{G(x) : x \in X\}$. Consider now the problem

$$
\begin{aligned}
\min \quad & x \\
\text{s.t.} \quad & x^2 - \epsilon \leq 0.
\end{aligned}
$$

Here $t^* = -\sqrt{\epsilon}$, $x_G = 0$, $f(x_G) = 0$, $G_{\min} = -\epsilon$, so that

$$\rho = \frac{\sqrt{\epsilon}}{\sqrt{\epsilon} + \epsilon} = \frac{1}{1 + \sqrt{\epsilon}}.$$

As a result,

$$F^*(t_k) \leq \left(\frac{1}{1 + \sqrt{\epsilon}}\right)^{k-1} F^*(t_1).$$

We will now show that the sequence indeed converges at a linear rate (but not faster). Note that here

$$t_{k+1} = \frac{1 - \sqrt{1 - 4t_k + 4\epsilon}}{2}.$$

We will chose the initial point as $t_1 = -2\sqrt{\epsilon}$. Invoking the mean value theorem on the function $g(\theta) \equiv \frac{1 - \sqrt{1 - 4\theta + 4\epsilon}}{2}$ over the interval $[t_k, -\sqrt{\epsilon}]$, we obtain that

$$|t_{k+1} + \sqrt{\epsilon}| = |g(t_k) - g(-\sqrt{\epsilon})| = |g'(\xi)(t_k + \sqrt{\epsilon})| = \frac{1}{\sqrt{1 - 4\xi + 4\epsilon}}|t_k + \sqrt{\epsilon}|,$$

where $\xi \in (t_k, -\sqrt{\epsilon})$. We will assume that $t_1 = -2\sqrt{\epsilon}$ so that $\xi \in (-2\sqrt{\epsilon}, -\sqrt{\epsilon})$. Consequently,

$$|t_{k+1} + \sqrt{\epsilon}| \leq \frac{1}{\sqrt{1 + 4\sqrt{\epsilon} + 4\epsilon}}|t_k + \sqrt{\epsilon}| = \frac{1}{1 + 2\sqrt{\epsilon}}|t_k + \sqrt{\epsilon}|,$$

$$|t_{k+1} + \sqrt{\epsilon}| \geq \frac{1}{\sqrt{1 + 8\sqrt{\epsilon} + 4\epsilon}}|t_k + \sqrt{\epsilon}| \geq \frac{1}{1 + 4\sqrt{\epsilon}}|t_k + \sqrt{\epsilon}|,$$

so that

$$\sqrt{\epsilon}\left(\frac{1}{1 + 4\sqrt{\epsilon}}\right)^{k-1} \leq |t_k + \sqrt{\epsilon}| \leq \sqrt{\epsilon}\left(\frac{1}{1 + 2\sqrt{\epsilon}}\right)^{k-1}.$$

As a result, since $F^*(t_k) \leq t^* - t_k$, we have

$$F^*(t_k) \leq \sqrt{\epsilon}\left(\frac{1}{1 + 2\sqrt{\epsilon}}\right)^{k-1}.$$

In addition, by (3.14) and (3.8), it follows that

$$F^*(t_k) \geq |(F^*)'(t^*)| \cdot |t_k + \sqrt{\epsilon}| \geq \frac{|G_{\min}|}{|G_{\min}| + f(x_G) + \sqrt{\epsilon}}|t_k + \sqrt{\epsilon}| \geq \frac{\varepsilon}{\varepsilon + \sqrt{\epsilon}}|t_k + \sqrt{\epsilon}|$$

so that

$$F^*(t_k) \geq \frac{\epsilon}{\varepsilon + \sqrt{\epsilon}}\left(\frac{1}{1 + 4\sqrt{\epsilon}}\right)^{k-1}.$$

To summarize,

$$\frac{\epsilon}{\epsilon + \sqrt{\epsilon}}\left(\frac{1}{1 + 4\sqrt{\epsilon}}\right)^{k-1} \leq F^*(t_k) \leq \sqrt{\epsilon}\left(\frac{1}{1 + 2\sqrt{\epsilon}}\right)^{k-1}.$$

We thus conclude that the sequence converges at a linear rate but not faster than a linear rate. Note that the sequence $\{F^*(t_k)\}_{k \geq 1}$ lies between two geometric sequences, one with a linear factor $\frac{1}{1+4\sqrt{\epsilon}}$ and the other with a linear factor $\frac{1}{1+2\sqrt{\epsilon}}$. Our theoretical result states that the sequence is upper bounded by a geometric sequence with a linear factor $\frac{1}{1+\sqrt{\epsilon}}$.

4. A gradient-based method for solving smooth convex optimization problems. Consider now problem (1.1) with the functions $f, g_1, \ldots, g_m$ assumed to be continuously differentiable on $\mathbb{R}^n$ (in addition to being convex), whose gradients are Lipschitz continuous with constants $L_f, L_{g_1}, \ldots, L_{g_m}$, respectively. Applying the ISAP method to problem (1.1) requires solving (up to some tolerance) problems of the form

$$(4.1) \qquad \min_{\mathbf{x} \in X} \left[\max \left\{ f(\mathbf{x}) - t, g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}) \right\} \right].$$

A possible approach for solving such a problem is to use a gradient-based method, that is, a method that exploits only zero- and first order information at each iteration (function values and gradients, respectively). One reason for implementing such an approach is that for large-scale problems this is sometimes the only viable alternative, since its dominant computational effort consists of relatively "cheap" matrix-vector multiplications, in contrast to second order methods, which often require the solution of huge linear systems.

In [4] Nesterov devised an "optimal" first order method for solving problems of the type

$$(4.2) \qquad \min_{\mathbf{x} \in X} F(\mathbf{h}(\mathbf{x})),$$

where $X \subseteq \mathbb{R}^n$ is a closed convex set, $\mathbf{h} = (h_1, \ldots, h_m)^T : \mathbb{R}^n \to \mathbb{R}^m$, and $h_1, \ldots, h_m$ are continuously differentiable convex functions over $\mathbb{R}^n$ whose gradients are Lipschitz with constants $L_{h_1}, \ldots, L_{h_m}$, respectively. The function $F : \mathbb{R}^m \to \mathbb{R}$ is a convex positively homogenous function of degree one satisfying that the set $\partial F(0)$ contains only vectors with nonnegative components. One example of such a function is the max function $(y_1, \ldots, y_m) \mapsto \max\{y_1, \ldots, y_m\}$. The algorithm of [4], which we call OPTGRAD, is described below.

OPTGRAD—An optimal gradient-based method for solving (4.2).
- **Input:** $h_1, \ldots, h_N$—Continuously differentiable convex functions over $\mathbb{R}^n$ whose gradients are Lipschitz continuous.
  $L_{h_1}, \ldots, L_{h_N}$—the Lipschitz constants of the gradients of $h_1, \ldots, h_N$, respectively.
- **Step 0.** Take $\mathbf{y}_1 = \mathbf{x}_0 \in \mathbb{R}^n$, $t_1 = 1$.
- **Step k ($k \geq 1$).** Compute

$$(4.3) \quad \mathbf{x}_k = \operatorname*{argmin}_{\mathbf{x} \in X} \left\{ F(\mathbf{h}(\mathbf{y}_k) + J_{\mathbf{h}}(\mathbf{y}_k)(\mathbf{x} - \mathbf{y}_k)) + \frac{F(\mathbf{L})}{2} \|\mathbf{x} - \mathbf{y}_k\|^2 \right\},$$

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2},$$

$$\mathbf{y}_{k+1} = \mathbf{x}_k + \left(\frac{t_k - 1}{t_{k+1}} \right) (\mathbf{x}_k - \mathbf{x}_{k-1}).$$

The description of the algorithm uses the following notation: the vector $\mathbf{L}$ is comprised of all Lipschitz constants, $\mathbf{L} = (L_{h_1}, \ldots, L_{h_N})^T$; for any $\mathbf{z} \in \mathbb{R}^n$ the matrix $J_{\mathbf{h}}(\mathbf{z}) \in \mathbb{R}^{N \times n}$ is the Jacobian matrix given by

$$J_{\mathbf{h}}(\mathbf{z}) = \begin{pmatrix} \nabla h_1(\mathbf{z})^T \\ \vdots \\ \nabla h_N(\mathbf{z})^T \end{pmatrix}.$$

The following result was shown in [4, Theorem 2].

THEOREM 4.1. *Let $\{\mathbf{x}_k\}_{k=0}^{\infty}$ be the sequence generated by the OPTGRAD method, and let $\mathbf{x}^*$ be an optimal solution of (4.2). Then we have the following:*

(a)

$$F(\mathbf{h}(\mathbf{x}_k)) - F(\mathbf{h}(\mathbf{x}^*)) \leq \frac{4F(\mathbf{L})\|\mathbf{x}_0 - \mathbf{x}^*\|^2}{(k+2)^2}.$$

(b) *The method finds an ε-optimal solution after at most $\frac{2\|\mathbf{x}_0 - \mathbf{x}^*\|\sqrt{F(\mathbf{L})}}{\sqrt{\varepsilon}}$ iterations.*

A straightforward conclusion of the above result is given in the following corollary.

COROLLARY 4.2. *Suppose that X is compact and that its diameter is given by*

(4.4) $$\Theta = \max_{\mathbf{x},\mathbf{y}\in X} \|\mathbf{x} - \mathbf{y}\|.$$

Then the OPTGRAD method finds an ε-optimal solution after at most $\frac{2\Theta\sqrt{F(\mathbf{L})}}{\sqrt{\varepsilon}}$ iterations.

Suppose now that the ISAP method is employed on problem (1.1) and that at each iteration the emerging subproblem (4.1) is solved using the OPTGRAD method with input functions $f(\cdot) - t_k$, $g_1(\cdot), \ldots, g_m(\cdot)$ (in particular $N = m+1$) and with F being the max function. Then by Corollary 4.2 it follows that if X is compact, then at most $\frac{2\sqrt{3}\Theta\sqrt{\max\{L_f, L_{g_1}, \ldots, L_{g_m}\}}}{\sqrt{\varepsilon}}$ iterations are required in order to find an $\frac{\varepsilon}{3}$-optimal solution for the subproblem (4.1) with an appropriate choice of t. We can then invoke Theorem 3.5 to obtain the following result on the bound of the number of overall inner iterations.

COROLLARY 4.3. *Suppose that the ISAP method is employed with the subproblems (1.1) solved by applying $\lfloor \frac{2\sqrt{3}\Theta\sqrt{\max\{L_f, L_{g_1}, \ldots, L_{g_m}\}}}{\sqrt{\varepsilon}} \rfloor$ ("inner") iterations of the OPTGRAD method. Then the ISAP method requires at most $\tilde{A}\frac{1}{\sqrt{\varepsilon}} + \tilde{B}\frac{\log(1/\varepsilon)}{\sqrt{\varepsilon}}$ inner iterations to obtain an ε-optimal solution where*

$$\tilde{A} = 2\sqrt{3}\Theta \left(\frac{\log(3F^*(t_1))}{\log(1/\rho)} + 2 \right) \sqrt{\max\{L_f, L_{g_1}, \ldots, L_{g_m}\}},$$

$$\tilde{B} = 2\sqrt{3}\Theta \frac{\sqrt{\max\{L_f, L_{g_1}, \ldots, L_{g_m}\}}}{\log(1/\rho)}$$

and where ρ is defined in (3.12).

Note that step (4.3) requires the solution of an optimization problem of the form

(4.5) $$\min_{\mathbf{x}\in X} \left\{ F(\mathbf{A}\mathbf{x} + \mathbf{b}) + \frac{\gamma}{2}\|\mathbf{x} - \mathbf{z}\|^2 \right\},$$

where $\mathbf{A} \in \mathbb{R}^{N\times n}$, $\mathbf{b} \in \mathbb{R}^n$, and $\mathbf{z} \in \mathbb{R}^n$. When F is the max function, then (4.5) can be rewritten as the following minmax problem:

(4.6) $$\min_{\mathbf{x}\in X} \max_{\mathbf{y}\in\Delta_N} \left\{ \mathbf{y}^T(\mathbf{A}\mathbf{x} + \mathbf{b}) + \frac{\gamma}{2}\|\mathbf{x} - \mathbf{z}\|^2 \right\},$$

where $\Delta_N = \left\{ \mathbf{v} \in \mathbb{R}^N : \sum_{i=1}^N v_i = 1, \mathbf{v} \geq \mathbf{0} \right\}$ is the N-dimensional unit-simplex. The objective in (4.6) is convex in $\mathbf{x}$ and concave in $\mathbf{y}$. As a result, the order of

maximization and minimization can be replaced [6, Corollary 37.3.2], resulting in the dual problem

$$\max_{\mathbf{y} \in \Delta_N} \min_{\mathbf{x} \in X} \left\{ \mathbf{y}^T (\mathbf{Ax} + \mathbf{b}) + \frac{\gamma}{2} \|\mathbf{x} - \mathbf{z}\|^2 \right\},$$

which after some rearrangement of terms takes the form

$$\max_{\mathbf{y} \in \Delta_N} \min_{\mathbf{x} \in X} \left\{ \frac{\gamma}{2} \left\| \mathbf{x} - \left(\mathbf{z} - \frac{1}{\gamma} \mathbf{A}^T \mathbf{y} \right) \right\|^2 + \mathbf{y}^T (\mathbf{Az} + \mathbf{b}) - \frac{1}{2\gamma} \|\mathbf{A}^T \mathbf{y}\|^2 \right\}.$$

Solving the inner minimization problem results in the following dual formulation of problem (4.5):

$$(4.7) \qquad \max_{\mathbf{y} \in \Delta_N} \left\{ \frac{\gamma}{2} \left\| H_X \left(\mathbf{z} - \frac{1}{\gamma} \mathbf{A}^T \mathbf{y} \right) \right\|^2 + \mathbf{y}^T (\mathbf{Az} + \mathbf{b}) - \frac{1}{2\gamma} \|\mathbf{A}^T \mathbf{y}\|^2 \right\},$$

where $H_X(\mathbf{w}) \equiv \mathbf{w} - P_X(\mathbf{w})$ and P_X is the orthogonal projection onto the set X. When $N \ll n$, it might be easier to solve the dual formulation (4.7) rather than the primal one (4.5) since the primal formulation has many more variables than the dual formulation.

5. Numerical examples.

5.1. ISAP versus bisection. Consider a class of constrained least squares problems of the form

$$(T) \qquad\qquad \min\{\|\mathbf{Ax} - \mathbf{b}\|^2 : \|\mathbf{Lx}\|^2 \le \eta_1, \mathbf{x} \in X\},$$

where $\mathbf{A} \in \mathbb{R}^{100 \times 100}$, $\mathbf{b} \in \mathbb{R}^{100}$, $\mathbf{L} \in \mathbb{R}^{100 \times 100}$, $\eta_1 \in \mathbb{R}_{++}$ and where

$$X = \{\mathbf{x} \in \mathbb{R}^{100} : \|\mathbf{x}\|^2 \le \eta_2\}.$$

Since only one constraint is present (disregarding the constraint defining the set X), problem (T) corresponds to the general model (1.1) with $m = 1$. At each iteration of the ISAP method we thus solve a problem of the form (4.1) with $m = 1$. Applying the OPTGRAD method on such a subproblem requires at each iteration the solution of a problem of the form (4.7) with $N = 2$. Since $N = 2$, the problem can be trivially reduced into a one-dimensional problem by the change of variables $y_1 = v$, $y_2 = 1 - v$, which results in the univariate problem

$$\max_{0 \le v \le 1} \left\{ \frac{\gamma}{2} \left\| H_X \left(\mathbf{z} - \frac{1}{\gamma} \mathbf{A}^T (v\mathbf{g}_1 + \mathbf{g}_2) \right) \right\|^2 \right.$$
$$\left. + (v\mathbf{g}_1 + \mathbf{g}_2)^T (\mathbf{Az} + \mathbf{b}) - \frac{1}{2\gamma} \|\mathbf{A}^T (v\mathbf{g}_1 + \mathbf{g}_2)\|^2 \right\},$$

where $\mathbf{g}_1 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$, $\mathbf{g}_2 = \begin{pmatrix} 0, 1 \end{pmatrix}^T$. The above problem was solved using a simple bisection procedure.

We compared the ISAP method to the following bisection procedure for finding the smallest root of $F^*(t)$. Both the bisection and ISAP methods use the OPTGRAD method for solving the corresponding subproblems with the number of inner iterations equal to the theoretical upper bound given in Corollary 4.3.

Algorithm Bisection.
- **Initialization:** Choose $a_1 < t^*$, $b_1 > t^*$ and $\varepsilon > 0$. Set $t_1 := \frac{a_1+b_1}{2}$.
- **Step k ($k \geq 1$).** Employ method $\mathcal{M}$ with input $(f(\cdot) - t_k, G(\cdot), X, \frac{\varepsilon}{3})$ and obtain an $\mathbf{x}_k$ satisfying

$$F^*(t_k) \leq F_{t_k}(\mathbf{x}_k) \leq F^*(t_k) + \frac{\varepsilon}{3}.$$

If $F_{t_k}(\mathbf{x}_k) > \frac{\varepsilon}{3}$, then update

$$a_{k+1} := t_k, \qquad b_{k+1} := b_k, \qquad t_{k+1} := \frac{a_{k+1} + b_{k+1}}{2}.$$

If $F_{t_k}(\mathbf{x}_k) \leq \frac{\varepsilon}{3}$, then set

$$a_{k+1} := a_k, \qquad b_{k+1} := t_k, \qquad t_{k+1} := \frac{a_{k+1} + b_{k+1}}{2}.$$

Stop if $b_{k+1} - a_{k+1} \leq \frac{\varepsilon}{3}$ and the output is an $\tilde{\mathbf{x}}$ satisfying

$$F^*(a_{k+1}) \leq F_{a_{k+1}}(\tilde{\mathbf{x}}) \leq F^*(a_{k+1}) + \frac{\varepsilon}{3}.$$

Otherwise, $k \leftarrow k + 1$.

It is easy to show that the output $\tilde{\mathbf{x}}$ is an ε-optimal and feasible solution.

Note that, in contrast to the ISAP method, the number of (outer) iterations required by the bisection method in order to reach a certain accuracy is fixed and is in fact equal to $\lceil \log_2 \left((b_1 - a_1)\frac{3}{\varepsilon} \right) \rceil$. A certain disadvantage of bisection in comparison to ISAP is that it requires the knowledge of an upper bound on the optimal function value t^*. In our experiment each of the components of $\mathbf{A}, \mathbf{b}, \mathbf{L}$ was randomly and independently generated from a standard normal distribution. 20 such realizations were considered, and each was solved for the following choices of parameters:

$$\varepsilon = 10^{-2}, 10^{-3}, \quad \eta_1 = 10, 100, 1000, \quad \eta_2 = 20.$$

The initial lower bound on the optimal function value was set to[2] $t_1 = -1000$. The average, minimum, maximum, and standard deviation of the number of iterations of the ISAP method over the 20 realization for each of the possible parameters is given in the following table. The last column—denoted N_{bisec}—contains the number of iterations required by the bisection algorithm to reach the desired accuracy given the initial interval $[-1000, 1000]$.

ε	η_1	Mean	Min	Max	Std	N_{bisec}
	10	14.45	12	17	1.6376	20
10^{-2}	100	5.3	5	6	0.4702	20
	1000	2.95	2	3	0.2236	20
	10	18.35	15	22	2.059	23
10^{-3}	100	6.25	5	7	0.5501	23
	1000	3.65	2	4	0.5871	23

Note that the number of iterations required by the ISAP method is always smaller than the number of iterations of the bisection procedure, and for the larger values of

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[2]The initial estimate of the optimal function value was chosen as a very loose lower bound to demonstrate the weak dependence of the method on t_1.

η_1 it is significantly smaller. The number of iterations of the ISAP method decreases as η_1 increases. This phenomenon can be explained by the fact that the linear factor ρ corresponding to the rate of convergence of the ISAP method and given in (3.12) is a decreasing function of $|G_{\min}|$, which, for problem (T), is equal to η_1. It is also worth noting that in all the runs we performed we observed that the method only weakly depends on t_1 in the sense that t_2 is rather close to the optimal function value despite the fact that t_1 might be far from it. As an illustration, we describe below one of the 20 runs corresponding to the parameters $\eta_1 = 1000$, $\varepsilon = 10^{-3}$, which required four iterations.

k	t_k	$F_{t_k}(\mathbf{x}_k)$
1	-1000	1000.2806
2	0.2806	0.7482
3	1.0288	0.001348
4	1.03024	$2.159 \cdot 10^{-6}$

Notice the huge jump from the initial value, $t_1 = -1000$, to the value after the first iteration, $t_2 = 0.2806$. This jump occurred in all the runs performed in this set of tests. In our experiments the number of inner iterations, that is, the iterations of the OPTGRAD method, was chosen as the theoretical upper bound given in Corollary 4.3 and is equal to a few thousands of iterations. In practice, this is somewhat of an overkill, and far fewer iterations are needed in order to reach the required accuracy. In fact, we repeated the runs when the number of inner iterations performed at each iteration of the ISAP method was 200, and the results were almost identical. The only difference in the sequence $\{t_k\}_{k=1}^K$ was in the sixth or larger digits.

In the next section we present a large-scale image deblurring example in which a more practical stopping criteria for the OPTGRAD method is used.

5.2. An image deblurring example. We conclude this section with an image deblurring example illustrating the ability of the ISAP to handle large-scale problems.
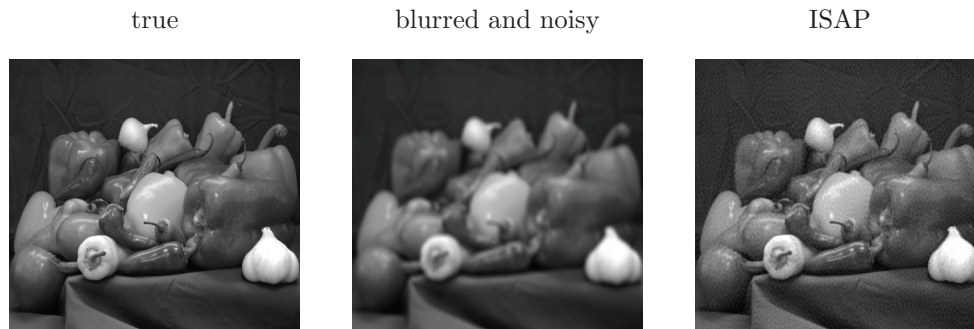

FIG. 1. *Deblurring of "peppers." Left: true image. Middle: blurred and noisy image. Right: solution of* (T) *by the ISAP method.*

Consider the 384×512 peppers gray image (left panel of Figure 1) scaled so that all the pixels are in the interval $[0, 1]$. The overall number of pixels is of course $196608 = 384 \cdot 512$. We blur the image with a Gaussian point spread function of dimension 9×9 with standard deviation 4 implemented by the MATLAB built-in function fspecial. Zero-mean normally distributed noise with standard deviation 0.002 is added to each of the components of the blurred image, and the result is the middle panel of Figure 1.

The blurring is performed under the assumption of reflexive boundary conditions. In mathematical terms the process can be described as

$$\mathbf{b} = \mathbf{A}\mathbf{x}_t + \mathbf{w},$$

where $\mathbf{x}_t \in \mathbb{R}^{196608}$ is the "vectorized" true image (that is, a vector obtained by stacking the column of the true image), $\mathbf{b}$ is the vectorized blurred and noisy image, $\mathbf{w} \in \mathbb{R}^{196608}$ is the noise vector whose components were independently and randomly generated from a zero-mean normal distribution with standard deviation 0.002, and $\mathbf{A} \in \mathbb{R}^{196608 \times 196608}$ is diagonalizable by the two-dimensional discrete cosine transform matrix; see [1] for further details.

The following constrained least squares problem was solved:

$$\begin{aligned} \min \quad & \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 \\ \text{s.t.} \quad & \|\mathbf{L}\mathbf{x}\|^2 \le \rho, \\ & \mathbf{x} \in X \equiv [0,1]^n. \end{aligned}$$

The parameter ρ was chosen to be $1.1 \cdot \|\mathbf{L}\mathbf{x}_t\|^2$ (that is, 10% more than the true value); the matrix $\mathbf{L}$ represents a discretization of the difference operator. Specifically, it is the matrix representing a two-dimensional convolution with the point spread function

$$\frac{1}{8} \begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}.$$

At each iteration of the ISAP method, the OPTGRAD method was invoked with the following stopping criterion. Let $\varphi_1, \varphi_2, \ldots$ be the sequence of function values generated by the OPTGRAD method. Denote the minimum function value over the first k iterations by $\varphi_{[k]} = \min_{i=1,\ldots,k} \varphi_i$. Since the OPTGRAD method is nonmonotone, $\varphi_{[k]}$ is not necessarily equal φ_k. The stopping criterion is "stop at the first iteration $k \ge 101$ in which $\varphi_{[k]} - \varphi_{[k-100]} < 10^{-4}$"; that is, the algorithm stops if a cumulative decrease of at least 10^{-4} was not gained in 100 consecutive iterations.

The right panel of Figure 1 is the focused reconstruction generated by the ISAP method employed with the parameter $\varepsilon = 10^{-3}$. The ISAP method required only three outer iterations, and the total CPU time was 783 seconds (Pentium 4, 2.34 GHz).

Appendix A. Proof of Lemma 2.1. Part (a) is straightforward from the definition of $F^*(t)$.

(b) Suppose that $t < t^*$, and assume in contradiction that $F^*(t) \le 0$. By Assumption A1 it follows that there exists $\mathbf{x} \in X$ such that $\max\{f(\mathbf{x}) - t, G(\mathbf{x})\} \le 0$, meaning that $f(\mathbf{x}) \le t < t^*, G(\mathbf{x}) \le 0$, which is a contradiction to the optimality of t^*.

(c) Let $t \ge t^*$, and let $\mathbf{x}^*$ be an optimal solution of problem (2.1). Then

$$F^*(t) = \min_{x \in X} \max\{f(\mathbf{x}) - t, G(\mathbf{x})\} \le \max\{f(\mathbf{x}^*) - t, G(\mathbf{x}^*)\} = \max\{t^* - t, G(\mathbf{x}^*)\} \le 0,$$

where the last inequality follows from the fact that x^* is feasible and $t^* \le t$.

(d) For any $u \le v$

$$\begin{aligned} F^*(v) &= \min_{\mathbf{x} \in X} \max\{f(\mathbf{x}) - u, G(\mathbf{x}) + v - u\} - v + u \\ &\ge \min_{\mathbf{x} \in X} \{\max\{f(\mathbf{x}) - u, G(\mathbf{x})\} - v + u = F^*(u) - v + u. \end{aligned}$$

Appendix B. Proof of Theorem 2.2. (a) The proof is by induction on k. By the definition of the SAP method, $t_1 \leq t^*$. Assume that $t_k \leq t^*$. Substituting $u = t_k, v = t^*$ in (2.4), we obtain that $F^*(t_k) \leq t^* - t_k$, implying that $t_{k+1} = t_k + F^*(t_k) \leq t^*$, thus proving the claim.

(b) Since $t_k \leq t^*$, then by Lemma 2.1(b) it follows that $F^*(t_k) \geq 0$, implying that $t_{k+1} = t_k + F^*(t_k) \geq t_k$. Furthermore, $t_{k+1} = t_k$ if and only if $F^*(t_k) = 0$, which, by the fact that $t_k \leq t^*$, is equivalent to $t_k = t^*$.

(c) Summing the update rule of the SAP method $t_{n+1} = F^*(t_n) + t_n$ for $n = 1, \ldots, k$, we obtain that

$$t_{k+1} - t_1 = \sum_{n=1}^{k} F^*(t_n),$$

which, combined with the inequalities $t_{k+1} \leq t^*$ and $\sum_{n=1}^{k} F^*(t_n) \geq k F^*(t_k)$, implies the result.

(d) The sequence $\{F^*(t_k)\}_{k \geq 1}$ converges to 0 since it is nonnegative and upper bounded by $\frac{t^* - t_1}{k}$. The sequence $\{t_k\}$ is nondecreasing and upper bounded and thus has a limit which we will denote by t_∞. Since $t_k \leq t^*$ for every $k \geq 1$ it follows that $t_\infty \leq t^*$. Taking $k \to \infty$ in the update formula (2.5) and utilizing the continuity of F^*, we obtain that $t_\infty = F^*(t_\infty) + t_\infty$, so that $F^*(t_\infty) = 0$, which, combined with the fact that $t_\infty \leq t^*$, implies that $t_\infty = t^*$. $\square$

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