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A FAST METHOD FOR FINDING THE GLOBAL SOLUTION OF THE REGULARIZED STRUCTURED TOTAL LEAST SQUARES PROBLEM FOR IMAGE DEBLURRING*

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Abstract. Given a linear system $\mathbf{Ax} \approx \mathbf{b}$ over the real or complex field, where both $\mathbf{A}$ and $\mathbf{b}$ are subject to noise, the total least squares (TLS) problem seeks to find a correction matrix and a correction right-hand side vector of minimal norm which makes the linear system feasible. To avoid ill posedness, a regularization term is added to the objective function; this leads to the so-called regularized TLS problem. A further complication arises when the matrix $\mathbf{A}$ and correspondingly the correction matrix must have a specific structure. This is modeled by the regularized structured TLS (RSTLS) problem. In general this problem is nonconvex and hence difficult to solve. However, the RSTLS problem arising from image deblurring applications under reflexive or periodic boundary conditions possesses a special structure where all relevant matrices are simultaneously diagonalizable (SD). In this paper we introduce an algorithm for finding the *global* optimum of the RSTLS problem with this SD structure. The devised method is based on decomposing the problem into single variable problems and then transforming them into one-dimensional unimodal real-valued minimization problems which can be solved globally. Based on the uniqueness and attainment properties of the RSTLS solution we show that a constrained version of the problem possesses a strong duality result and can thus be solved via a sequence of RSTLS problems.

Key words. structured total least squares, nonconvex optimization, image deblurring, unimodal functions, simultaneously diagonalizable matrices

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1. Introduction.

Given a linear system $\mathbf{Ax} \approx \mathbf{b}$ over the real or complex field, where both the matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$ are subjected to noise, the total least squares (TLS) problem seeks to minimize the sum of squared norms of the perturbations to both the model matrix and vector $\|\mathbf{E}\|^2 + \|\mathbf{w}\|^2$ subject to the condition that the perturbed system holds: $(\mathbf{A} + \mathbf{E})\mathbf{x} = \mathbf{b} + \mathbf{w}$. Although this problem is nonconvex, it can be solved efficiently and globally by using a spectral decomposition of the augmented matrix $(\mathbf{A}, \mathbf{b})$; see [14, 20].

In many applications, the matrix $\mathbf{A}$ has a specific linear structure, e.g., Toeplitz or Hankel, which imposes a requirement on the perturbation matrix $\mathbf{E}$ to possess a corresponding special structure. The TLS solution does not take into account this requirement, and consequently the structured TLS (STLS)[1] attracted intensive research; see, e.g., [1, 29, 34, 28, 25, 22]. The formulation of the STLS problem is

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[1]In some papers the STLS problem is also called *constrained total least squares*.

$$\text{(STLS):}\quad \begin{array}{ll} \min\limits_{\mathbf{E},\mathbf{x},\mathbf{w}} & \|\mathbf{E}\|^2 + \|\mathbf{w}\|^2 \\ \text{s.t.} & (\mathbf{A}+\mathbf{E})\mathbf{x} = \mathbf{b}+\mathbf{w}, \\ & \mathbf{E} \in \mathcal{L}, \end{array}$$

where $\mathcal{L}$ is a linear subspace. We remark that there are several generalizations of the above STLS formulation that are able to deal with multiple right-hand sides (that is, $\mathbf{b}$ and $\mathbf{x}$ are matrices) [23], structure of the right-hand side noise vector $\mathbf{w}$ [23], and other norms such as l_1, l_∞ [34], and weighted l_2 norms [24].

The STLS problem is a nonconvex problem, and thus finding its global solution is in general a difficult task. There are only a few exceptions to this state of affairs. For block circulant structures with unstructured blocks the corresponding STLS problem can be solved by decomposing the problem into several smaller TLS problems using the discrete Fourier transform [6]. Another tractable case arises when some of the columns of $\mathbf{A}$ are error-free while the others are subjected to noise. This problem is called the *generalized TLS* problem or *mixed LS-TLS* problem, and its solution can be obtained by computing a QR factorization of $\mathbf{A}$ and then solving a TLS problem of reduced dimension [19]. A more general problem is the *restricted TLS* problem introduced in [21]. There it is assumed that $(\mathbf{E},\mathbf{w}) = \mathbf{D}_1\widetilde{\mathbf{E}}\mathbf{C}_1$, where $\mathbf{D}_1$ and $\mathbf{C}_1$ are known matrices and $\widetilde{\mathbf{E}}$ is unknown. As was shown in [21], by choosing the matrices $\mathbf{D}_1$ and $\mathbf{C}_1$ appropriately, the restricted TLS problem contains as special cases any weighted least squares (LS), generalized LS, TLS, and generalized TLS problems. The restricted TLS problem can be solved by using the restricted singular value decomposition [37].

In this paper we consider yet another tractable class of STLS problems in which the global solution can efficiently be found. We deal with structures in which all of the matrices in $\mathcal{L}$ are square and can be diagonalized by a certain fixed orthogonal (or unitary in the complex case) matrix. These structures are called *simultaneously diagonalizable* (SD) structures. The motivation for considering such structures stems from image deblurring problems with spatially invariant point spread functions (PSF). For two-dimensional image deblurring problems it is well known that the matrix describing the blur operator can be diagonalized by a two-dimensional discrete Fourier transform matrix when *periodic boundary conditions* are assumed. For *reflexive boundary conditions* with symmetric PSF the corresponding matrix can be diagonalized by a two-dimensional discrete cosine transform matrix. Similar structures can be found in one-dimensional deconvolution problems. Section 2 contains a brief review of these structures.

A characteristic feature of image deblurring problems is that the matrix $\mathbf{A}$ is ill-conditioned, and as a result the STLS solution usually has a huge norm and as such is meaningless. Regularization is required in order to stabilize the solution. For the unstructured TLS problem several regularization methods are well known. Among them are truncation methods [11, 17] and Tikhonov regularization [13, 7], in which a quadratic penalty is added to the objective function or a quadratic constraint bounding the size of the solution norm is added to the problem [36, 33, 13, 8, 5].

For the STLS problem, Tikhonov regularization seems to be the most popular method. The resulting problem is called the regularized STLS problem (RSTLS) and is given by

$$\text{(RSTLS):}\quad \begin{array}{ll} \min\limits_{\mathbf{E},\mathbf{x},\mathbf{w}} & \|\mathbf{E}\|^2 + \|\mathbf{w}\|^2 + \rho\|\mathbf{L}\mathbf{x}\|^2 \\ \text{s.t.} & (\mathbf{A}+\mathbf{E})\mathbf{x} = \mathbf{b}+\mathbf{w}, \\ & \mathbf{E} \in \mathcal{L}. \end{array}$$

Common choices for $\mathbf{L}$ are the identity or a matrix approximating the first or second order derivative operator [16, 13, 18].

The RSTLS problem for structures arising in image deblurring was studied in several works. In [27] periodic boundary conditions are considered. By using the discrete Fourier transform the problem is decomposed into many complex-valued single-variable problems. The complex univariate problems are solved as two-variable nonconvex problems over the real domain by using the Davidon–Fletcher–Powell optimization algorithm.

In [31] an iterative algorithm of quasi-Newton form is applied for the RSTLS problem for reflexive boundary conditions that exploits the diagonalization properties of the associated matrices. The work [32] extends the structured total least norm algorithm [34] to include regularization, and image deblurring examples are discussed. This approach was also advocated in [12] for image deblurring problems with separable PSFs and in [26] for problems with zero boundary conditions.

In all of the above-mentioned works the optimization problems that need to be solved are nonconvex, and consequently the devised algorithms are not guaranteed to converge to a global optimum but rather to a stationary point. The main contribution of the present paper is the introduction of a method capable of obtaining the *global minimum* of the RSTLS problem for SD structures.

The paper is organized as follows. In section 2 we present a precise problem formulation followed by a brief review of the essential ingredients from image deblurring. The decomposition of the RSTLS problem into single-variable real- or complex-valued problems is discussed in section 3. These univariate problems are not necessarily unimodal, but we show in section 4 that they can be transformed into single-variable real-valued unimodal problems. Attainment and uniqueness conditions are also obtained. In section 5 we concentrate on circulant structures and show that, when the data are real-valued, there exists at least one real-valued optimal solution (although the corresponding single-variable problems are complex-valued). In section 6 we tackle the constrained version of the RSTLS problem, called CSTLS, and show that, based on the derived uniqueness properties and on a strong duality result, the constrained problem can be solved by a sequence of RSTLS problems. The paper ends in section 7 with detailed descriptions of the numerical algorithms and a demonstration of our method as applied to an image deblurring problem. A MATLAB implementation and documentation of the RSTLS and CSTLS methods for image deblurring problems with either periodic or reflexive boundary conditions can be found in [38].

1.1. Notation. A vector or matrix is called real-valued (complex-valued) if all of its entries are real (complex). For a complex scalar a, the complex conjugate is denoted by $\bar{a}$. Given a matrix $\mathbf{A}$ (a vector $\mathbf{v}$), the complex conjugate is denoted by $\mathbf{A}^*$ ($\mathbf{v}^*$). For a real-valued matrix $\mathbf{Q}$, the complex conjugate $\mathbf{Q}^*$ translates to the usual transpose $\mathbf{Q}^T$, and unitarity translates to orthogonality: $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. The root of -1 is denoted by $\mathbf{i} = \sqrt{-1}$. For a given vector $\mathbf{v}$, $\|\mathbf{v}\|$ denotes the Euclidean norm of $\mathbf{v}$, and, for a matrix $\mathbf{A}$, $\|\mathbf{A}\|$ denotes the Frobenius norm of the matrix. The Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$.

2. RSTLS for simultaneously diagonalizable structures.

2.1. Problem formulation. The RSTLS problem can be written as follows:

$$\text{(RSTLS):}\quad \begin{array}{ll} \min & \|\mathbf{E}\|^2 + \|\mathbf{w}\|^2 + \rho\|\mathbf{L}\mathbf{x}\|^2 \\ \text{s.t.} & (\mathbf{A}+\mathbf{E})\mathbf{x} = \mathbf{b} + \mathbf{w}, \\ & \mathbf{E} \in \mathcal{L}, \\ & \mathbf{x} \in \mathbb{F}^n, \mathbf{w} \in \mathbb{F}^m, \end{array} \tag{2.1}$$

where $\mathbf{A} \in \mathbb{F}^{m\times n}$ and $\mathbf{b} \in \mathbb{F}^m$, with $\mathbb{F}$ being either the real or the complex number field ($\mathbb{R}$ or $\mathbb{C}$, respectively). The parameter ρ is a positive real number, and the set $\mathcal{L}$ is a linear subspace of the set of all $m \times n$ matrices $\mathbb{F}^{m\times n}$. As was discussed in the introduction, this formulation was considered in several papers; see, e.g., [27, 31, 32, 12, 26].

In this paper we consider the case in which $m = n$ and $\mathcal{L}$ is a linear subspace of the set of all $n \times n$ matrices diagonalizable by a given unitary matrix. That is, $\mathcal{L} = \mathcal{L}_{\mathbf{Q}}$, where

$$\mathcal{L}_{\mathbf{Q}} = \{\mathbf{Q}^*\mathrm{diag}(\boldsymbol{\lambda})\mathbf{Q} : \boldsymbol{\lambda} \in \mathbb{F}^n\}, \tag{2.2}$$

with $\mathbf{Q}$ being a given unitary matrix (i.e., $\mathbf{Q}^*\mathbf{Q} = \mathbf{I}$). Such a structure is called a SD structure, with unitary transform. In our derivations we also assume that $\mathbf{A}, \mathbf{L} \in \mathcal{L}_{\mathbf{Q}}$. This particular structure is also discussed in, e.g., [27, 31].

In section 3 we will show that, as opposed to most structures, the RSTLS problem with an SD structure can be solved globally and efficiently. Before doing so, we will describe some image deblurring examples in which SD structures appear.

2.2. SD structures associated with image deblurring. We will now present four classes of SD structures that arise naturally in image deblurring problems. In addition to two-dimensional images, we will also consider one-dimensional signals and refer to them as "one-dimensional images." Before examining the four classes, we briefly review some essential facts and notation from image processing.

Many image deblurring problems can be modeled as $\mathbf{g} = \mathbf{Sf}$, where $\mathbf{g} \in \mathbb{R}^n$ is the blurred image and $\mathbf{f} \in \mathbb{R}^n$ is the unknown true image, whose size is assumed to be the same as the one of $\mathbf{g}$. The matrix $\mathbf{S}$ describes the blur operator. In the case of spatially invariant blurs, $\mathbf{Sf}$ is usually a convolution of a corresponding PSF and the true image $\mathbf{f}$.

The structure of the matrix $\mathbf{S}$ depends on the choice of boundary conditions, that is, the underlying assumptions on the image outside the field of view. Three very popular boundary conditions are (i) *zero boundary conditions*, in which all pixels outside the borders are assumed to be zero, (ii) *periodic boundary conditions*, in which it is assumed that the image repeats itself in all directions, (iii) *reflexive (Neumann) boundary conditions*, in which it is assumed that the scene outside of the boundaries is an image mirror of the image boundaries.

Let us illustrate the three types of boundary conditions. First, in the one-dimensional case consider the image

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

and then for zero, periodic, and reflexive boundary conditions the larger image looks like

$$\left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 1 \\ 2 \\ 3 \\ \hline 0 \\ 0 \\ 0 \end{array}\right), \quad \left(\begin{array}{c} 1 \\ 2 \\ 3 \\ \hline 1 \\ 2 \\ 3 \\ \hline 1 \\ 2 \\ 3 \end{array}\right), \quad \left(\begin{array}{c} 3 \\ 2 \\ 1 \\ \hline 1 \\ 2 \\ 3 \\ \hline 3 \\ 2 \\ 1 \end{array}\right),$$

respectively. In the two-dimensional case if we consider the image

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

then for zero, periodic, and reflexive boundary conditions the larger image looks like

$$\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 5 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 & 9 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 \\ \hline 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 \\ \hline 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 & 4 & 5 & 6 \\ 7 & 8 & 9 & 7 & 8 & 9 & 7 & 8 & 9 \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc|ccc} 9 & 8 & 7 & 7 & 8 & 9 & 9 & 8 & 7 \\ 6 & 5 & 4 & 4 & 5 & 6 & 6 & 5 & 4 \\ 3 & 2 & 1 & 1 & 2 & 3 & 3 & 2 & 1 \\ \hline 3 & 2 & 1 & 1 & 2 & 3 & 3 & 2 & 1 \\ 6 & 5 & 4 & 4 & 5 & 6 & 6 & 5 & 4 \\ 9 & 8 & 7 & 7 & 8 & 9 & 9 & 8 & 7 \\ \hline 9 & 8 & 7 & 7 & 8 & 9 & 9 & 8 & 7 \\ 6 & 5 & 4 & 4 & 5 & 6 & 6 & 5 & 4 \\ 3 & 2 & 1 & 1 & 2 & 3 & 3 & 2 & 1 \end{array}\right),$$

respectively. The structure of the matrix $\mathbf{S}$ depends on the underlying boundary conditions. Here we consider spatially invariant blurs which, as was already mentioned, imply that the blur is a convolution of given a PSF with the true (larger) image. For one-dimensional problems the PSF is just a vector $\mathbf{p} \in \mathbb{R}^d$ with an associated center $c \in \{1, 2, \dots, d\}$. The convolution operation is then:

$$g_i = \sum_{j=1}^{d} p_j f_{i+c-j}, \quad i = 1, \dots, n,$$

where $\mathbf{f} \in \mathbb{R}^d$ is the true image. Notice that the above formula uses values of $\mathbf{f}$ beyond the boundaries (indices smaller than 1 and larger than n), but these values are determined by the boundary conditions. For example, consider a one-dimensional image of length three: $\mathbf{f} = (f_1, f_2, f_3)^T$, and let the PSF array be $\mathbf{p} = (p_1, p_2, p_3)^T$ with $c = 2$. Then the blurred image $\mathbf{g}$ depends on the true image $\mathbf{f}$ via the relation $\mathbf{g} = \mathbf{S}\mathbf{f}$, where

$$\mathbf{S} = \begin{pmatrix} p_2 & p_1 & 0 \\ p_3 & p_2 & p_1 \\ 0 & p_3 & p_2 \end{pmatrix}, \begin{pmatrix} p_2 & p_1 & p_3 \\ p_3 & p_2 & p_1 \\ p_1 & p_3 & p_2 \end{pmatrix}, \begin{pmatrix} p_2 + p_3 & p_1 & 0 \\ p_3 & p_2 & p_1 \\ 0 & p_3 & p_2 + p_1 \end{pmatrix}$$

for zero, periodic, and reflexive boundary conditions, respectively. Note that the above three matrices have different structures (Toeplitz, circulant, and Toeplitz-plus-

Hankel). We now discuss four SD structures arising from one- and two-dimensional problems with either periodic or reflexive boundary conditions:[2]

1. *Circulant* [10]. For one-dimensional images with periodic boundary conditions, the structure of the model matrix is circulant, i.e., has the form
$$\mathbf{S} = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ s_n & s_1 & \cdots & s_{n-1} \\ \vdots & \vdots & & \vdots \\ s_2 & s_3 & \cdots & s_1 \end{pmatrix}.$$
All $n \times n$ circulant matrices are diagonalizable by the unitary discrete Fourier transform (DFT) matrix $\mathbf{F}_n$ given by
$$\mathbf{F}_n = \left(\frac{1}{\sqrt{n}} \omega^{(j-1)(k-1)} \right)_{j,k=1}^{n},$$
where $\omega = e^{\frac{2\pi \mathbf{i}}{n}}$. Multiplications of the DFT matrix $\mathbf{F}_n$ with vectors, as well as eigenvalue computation of circulant matrices, can be done very efficiently by using the fast Fourier transform (FFT) with a complexity of $O(n \log n)$.
2. *Block circulant with circulant blocks* [2]. For two-dimensional images of size $m \times n$ with periodic boundary conditions, the model matrix has a block circulant matrix with circulant blocks (BCCB) structure:
$$\mathbf{S} = \begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 & \dots & \mathbf{C}_n \\ \mathbf{C}_n & \mathbf{C}_1 & \dots & \mathbf{C}_{n-1} \\ \vdots & \vdots & & \vdots \\ \mathbf{C}_2 & \mathbf{C}_3 & \dots & \mathbf{C}_1 \end{pmatrix},$$
where $\mathbf{C}_1, \dots, \mathbf{C}_n$ are $m \times m$ circulant matrices. All BCCB matrices of the above size are diagonalizable by the unitary two-dimensional DFT matrix $\mathbf{F}_n \otimes \mathbf{F}_m$. As in the circulant case, computations with BCCB matrices can be performed by using the FFT.
3. *Toeplitz-plus-Hankel* [30]. For one-dimensional images with reflexive boundary conditions and symmetric PSF, the matrix $\mathbf{S}$ has a Toeplitz-plus-Hankel structure of the form [30]
$$T(\mathbf{s}) + H(\mathbf{s}),$$
where, for a given vector $\mathbf{s} = (s_1, \dots, s_n)^T \in \mathbb{R}^n$, $T(\mathbf{s})$ is the symmetric Toeplitz matrix whose first column is $\mathbf{s}$ and $H(\mathbf{s})$ is the Hankel matrix whose first and last columns are $(s_1, s_2, \dots, s_n, 0)^T$ and $(0, s_n, \dots, s_2, s_1)^T$, respectively. All Toeplitz-plus-Hankel matrices of the above form are diagonalizable by the orthogonal discrete cosine transform (DCT) matrix $\mathbf{C}_n$ given by
$$\mathbf{C}_n = \left(\sqrt{(2-\delta_{k1})/n} \cos \frac{\pi(2j-1)(k-1)}{2n} \right)_{j,k=1}^{n},$$
where, for two indices i and j, δ_{ij} denotes the Kronecker sign. Multiplications of the DCT matrix $\mathbf{C}_n$ with vectors, as well as eigenvalue computation

[2] We do not consider in this paper the zero boundary condition as it does not lead to an SD structure.

of circulant matrices, can be done very efficiently by using the fast cosine transform (FCT) with a complexity of $O(n \log n)$.

4. BTTB+BTHB+BHTB+BHHB *structure* [15, 30]. For two-dimensional images of size $m \times n$ with reflexive boundary conditions and a symmetric PSF, the matrix $\mathbf{S}$ is a sum of a BTTB (block Toeplitz with Toeplitz blocks), BTHB (block Toeplitz with Hankel blocks), BHTB (block Hankel with Toeplitz blocks), and BHHB (block Hankel with Hankel blocks) matrices. All matrices of this form are diagonalizable by the orthogonal two-dimensional DCT matrix $\mathbf{C}_n \otimes \mathbf{C}_m$. We note that the symmetry condition does occur in practice, for example, the Gaussian model for atmospheric turbulence blur, out-of-focus blurs, and certain classes of Moffat blurs [15].

We have thus described four SD structures arising from one- and two-dimensional deblurring problems. The first two classes correspond to $\mathbb{F} = \mathbb{C}$ (since the DFT matrix is complex-valued), and the last two classes correspond to $\mathbb{F} = \mathbb{R}$. Coming back to the RSTLS problem, we note that it is very natural to assume that the boundary conditions also apply to the regularization operator, and we can thus assume that $\mathbf{L} \in \mathcal{L}_{\mathbf{Q}}$.

3. Decomposition of the RSTLS problem for SD structures.

We begin by showing that the RSTLS problem (2.1) with an SD structure can be decomposed into n one-dimensional minimization problems.

THEOREM 3.1. *Consider the RSTLS problem* (2.1) *with* $m = n$ *and* $\mathcal{L} = \mathcal{L}_{\mathbf{Q}}$ *(see* (2.2)*), where* $\mathbf{Q} \in \mathbb{F}^{n\times n}$ *is a given unitary matrix. Suppose that* $\mathbf{A}, \mathbf{L} \in \mathcal{L}_{\mathbf{Q}}$*, and let* $\boldsymbol{\alpha}, \mathbf{l}$ *be the eigenvalues of* $\mathbf{A}$ *and* $\mathbf{L}$ *defined by the relations*

$$\mathbf{QAQ}^* = \mathrm{diag}(\boldsymbol{\alpha}), \quad \mathbf{QLQ}^* = \mathrm{diag}(\mathbf{l}). \tag{3.1}$$

Then any solution to the RSTLS problem is given by $\mathbf{x} = \mathbf{Q}^*\hat{\mathbf{x}}$*, where, for every* $i = 1, \dots, n$*, the* i*th component of* $\hat{\mathbf{x}}$*,* $\hat{x}_i$*, is an optimal solution to the one-dimensional problem*

$$\min_{\hat{x}_i} \left\{ \frac{|\alpha_i \hat{x}_i - \hat{b}_i|^2}{1 + |\hat{x}_i|^2} + \rho |l_i|^2 |\hat{x}_i|^2 \right\}, \tag{3.2}$$

where $\hat{\mathbf{b}} = \mathbf{Qb}$*. The optimal matrix* $\mathbf{E}$ *is given by*

$$\mathbf{E} = \mathbf{Q}^* \mathrm{diag}(\mathbf{r}) \mathbf{Q}, \tag{3.3}$$

where

$$r_i = -\frac{\overline{\hat{x}_i}(\alpha_i \hat{x}_i - \hat{b}_i)}{1 + |\hat{x}_i|^2}. \tag{3.4}$$

Proof. By using the relation $\mathbf{w} = (\mathbf{A} + \mathbf{E})\mathbf{x} - \mathbf{b}$, we can rewrite (2.1) as the following problem in the variables $\mathbf{E}$ and $\mathbf{x}$:

$$\min_{\mathbf{E},\mathbf{x}} \{ \|\mathbf{E}\|^2 + \|(\mathbf{A} + \mathbf{E})\mathbf{x} - \mathbf{b}\|^2 + \rho \|\mathbf{Lx}\|^2 : \mathbf{E} \in \mathcal{L}_{\mathbf{Q}}, \mathbf{x} \in \mathbb{F}^n \},$$

which, by the unitarity property of $\mathbf{Q}$, is the same as

$$\min_{\mathbf{E},\mathbf{x}} \{ \|\mathbf{QEQ}^*\|^2 + \|\mathbf{Q}(\mathbf{A} + \mathbf{E})\mathbf{Q}^*\mathbf{Qx} - \mathbf{Qb}\|^2 + \rho \|\mathbf{QLQ}^*\mathbf{Qx}\|^2 : \mathbf{E} \in \mathcal{L}_{\mathbf{Q}}, \mathbf{x} \in \mathbb{F}^n \}. \tag{3.5}$$

Since $\mathbf{E} \in \mathcal{L}_{\mathbf{Q}}$, we can make the change of variables $\mathbf{QEQ}^* = \mathrm{diag}(\mathbf{r})$, where $\mathbf{r} \in \mathbb{F}^n$ is an unknown variables vector. By combining this with (3.1) we conclude that (3.5) can be reformulated as

$$\min_{\mathbf{r},\hat{\mathbf{x}}}\{\|\mathrm{diag}(\mathbf{r})\|^2 + \|\mathrm{diag}(\boldsymbol{\alpha}+\mathbf{r})\hat{\mathbf{x}} - \hat{\mathbf{b}}\|^2 + \rho\|\mathrm{diag}(\mathbf{l})\hat{\mathbf{x}}\|^2 : \mathbf{r},\hat{\mathbf{x}} \in \mathbb{F}^n\},$$

where $\hat{\mathbf{x}} = \mathbf{Qx}$, and more explicitly as

$$\min_{\mathbf{r},\hat{\mathbf{x}}} \left\{ \sum_{i=1}^n (|r_i|^2 + |(\alpha_i + r_i)\hat{x}_i - \hat{b}_i|^2 + \rho|l_i|^2|\hat{x}_i|^2) : \mathbf{r},\hat{\mathbf{x}} \in \mathbb{F}^n \right\}.$$

The above optimization problem is separable with respect to the pairs of variables

$$(r_1,\hat{x}_1),(r_2,\hat{x}_2),\ldots,(r_n,\hat{x}_n),$$

implying that, for every i, the optimal $(r_i,\hat{x}_i)$ is the solution to the two-dimensional problem

$$\min_{r_i,\hat{x}_i} \left\{ |r_i|^2 + |(\alpha_i + r_i)\hat{x}_i - \hat{b}_i|^2 + \rho|l_i|^2|\hat{x}_i|^2 : r_i,\hat{x}_i \in \mathbb{F} \right\}. \tag{3.6}$$

Next, we fix $\hat{x}_i$ and minimize with respect to r_i. The result is

$$r_i = -\frac{\overline{\hat{x}_i}(\alpha_i\hat{x}_i - \hat{b}_i)}{1+|\hat{x}_i|^2}.$$

By substituting the above expression back into the objective function of (3.6) with some simple algebraic manipulations, we arrive at the following equivalent problem in the single variable $\hat{x}_i$:

$$\min_{\hat{x}_i} \left\{ \frac{|\alpha_i\hat{x}_i - \hat{b}_i|^2}{1+|\hat{x}_i|^2} + \rho|l_i|^2|\hat{x}_i|^2 \right\},$$

establishing the result. ◻

4. Solution and analysis of the RSTLS problem for SD structures.

In this section we study the one-dimensional (1D) problems (3.2) arising in the decomposition of the RSTLS problem. We show in section 4.1 that, although these problems are not unimodal,[3] they can be transformed into (strictly) unimodal problems and consequently solved efficiently and globally. This is especially crucial in image deblurring applications in which there are hundreds of thousands or even millions of 1D problems to be solved. Based on the uniqueness and attainment properties of the 1D problems, corresponding conditions for the RSTLS problem are established in section 4.2.

4.1. Solution of the single-variable problem.

Our goal in this section is to analyze the one-dimensional problem (3.2) and to devise an efficient solution method for solving it. Consider the problem

$$\min_{x\in\mathbb{F}} \left\{ f(x) = \frac{|ax-b|^2}{1+|x|^2} + |c|^2|x|^2 \right\}, \tag{4.1}$$

[3]A function $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ being a closed interval, is (strictly) unimodal if it has a unique local minimizer on I and is (strictly) decreasing from the left boundary of the interval to this unique minimum and (strictly) increasing from the minimum to the right boundary of the interval.

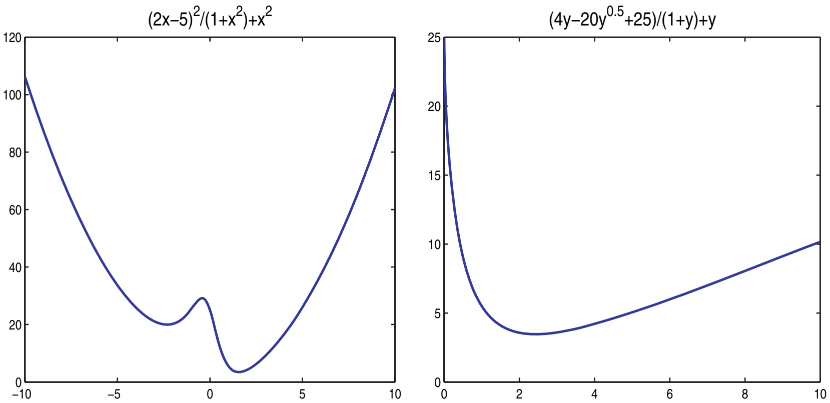

FIG. 1. *The objective function of problems* (4.5) *(left) and* (4.6) *(right).*

where $a, b, c \in \mathbb{F}$. If $c \neq 0$, then the objective function is coercive, and consequently its minimum is attained. The objective function of (4.1) is not unimodal (cf. Figure 1) and thus finding its global minimum efficiently is in principle a hard task. We will show in the next result that it can be solved via the minimization problem

$$\min_{y \geq 0} \left\{ g(y) \equiv \frac{|a|^2 y - 2|ab|\sqrt{y} + |b|^2}{1+y} + |c|^2 y \right\} \tag{4.2}$$

in the real nonnegative variable y. Before stating the result we briefly recall that for a real number $x \in \mathbb{R}$ the sign function is defined by

$$\operatorname{sgn}(x) \equiv \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0, \end{cases}$$

and for a complex number $z \in \mathbb{C}$ the sign function is given by

$$\operatorname{sgn}(z) \equiv \begin{cases} \frac{z}{|z|} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

LEMMA 4.1 (equivalence of problems (4.1) and (4.2)). *Consider problem* (4.1) *with* $a, b, c \in \mathbb{F}$. *Then*

(i) *If* $ab \neq 0$, *then* $\tilde{y}$ *is an optimal solution of* (4.2) *if and only if* $\tilde{x} = sgn(\bar{a}b)\sqrt{\tilde{y}}$ *is an optimal solution of* (4.1).

(ii) *If* $ab = 0$, *then* $\tilde{y}$ *is an optimal solution of* (4.2) *if and only if* $\tilde{x} = z\sqrt{\tilde{y}}$ *is an optimal solution of* (4.1) *for every* $z \in \mathbb{F}$ *satisfying* $|z| = 1$.

Proof. Let $\tilde{x}$ be an optimal solution of (4.1). Then by the optimality of $\tilde{x}$ we have

$$f(\tilde{x}) \leq f(z\tilde{x}) \text{ for every } z \in \mathbb{F} \text{ satisfying } |z| = 1,$$

which is the same as

$$\frac{|a\tilde{x} - b|^2}{1 + |\tilde{x}|^2} + |c|^2|\tilde{x}|^2 \leq \frac{|a(z\tilde{x}) - b|^2}{1 + |z\tilde{x}|^2} + |c|^2|z\tilde{x}|^2.$$

The latter inequality reduces to

$$\Re((1-z)a\bar{b}\tilde{x}) \geq 0. \tag{4.3}$$

We will now show that $a\bar{b}\tilde{x}$ is a nonnegative real number. This is obviously true if $\tilde{x} = 0$. Otherwise, we split the analysis into two cases:

Case I. If $ab \neq 0$, then substituting

$$z = \frac{\overline{a\bar{b}\tilde{x}}}{|a\bar{b}\tilde{x}|}$$

into (4.3) yields

$$\Re(a\bar{b}\tilde{x}) \geq |a\bar{b}\tilde{x}|,$$

implying that $a\bar{b}\tilde{x}$ is a nonnegative real number and, in particular, that $\operatorname{sgn}(\tilde{x}) = \operatorname{sgn}(\bar{a}b)$.

Case II. If $ab = 0$, the function f satisfies $f(zx) = f(x)$ for every $x, z \in \mathbb{F}$ such that $|z| = 1$ and thus $z\tilde{x}$ is also an optimal solution for every z satisfying $|z| = 1$.

A conclusion from the above two cases is that if the minimum of (4.1) is attained at a nonzero solution, then there must be at least one optimal solution $\tilde{x}$ for which $\operatorname{sgn}(\tilde{x}) = \operatorname{sgn}(\bar{a}b)$; consequently, we can make the change of variables $x = \operatorname{sgn}(\bar{a}b)\sqrt{y}$ which transforms problem (4.1) into (4.2). $\square$

Remark 4.1. Consider problem (4.1) with $\mathbb{F} = \mathbb{C}$ but with real data, i.e., $a, b, c \in \mathbb{R}$. Then a direct consequence of Lemma 4.1 is that if the optimal set of (4.1) is nonempty, then there must exist at least one real-valued optimal solution.

The following simple lemma establishes some key properties of problem (4.2). In particular, it is shown that problem (4.2) is strictly unimodal (in all interesting cases) and thus can be solved efficiently. This is in fact the main motivation for transforming problem (4.1) into (4.2).

LEMMA 4.2 (properties of problem (4.2)). *Consider problem* (4.2) *with* $a, b, c \in \mathbb{F}$. *Then*

(i) *the objective function* $g(y)$ *of* (4.2) *is quasi-convex*[4] *over* $[0, \infty)$*;*

(ii) *if* $c \neq 0$ *and* $\tilde{y}$ *is an optimal solution of* (4.2)*; then* $\tilde{y} \leq \frac{|b|^2}{|c|^2}$

(iii) *the solution of* (4.2) *is attained and unique if and only if* $(a, c) \neq (0, 0)$*;*

(iv) *if* $(a, c) \neq (0, 0)$, *then the objective function* $g(y)$ *of* (4.2) *is strictly unimodal over* $[0, \infty)$.

Proof. (i) We need to show that the level set $\{y : g(y) \leq \alpha\}$ is convex. Indeed,

$$\{y \geq 0 : g(y) \leq \alpha\} = \{y \geq 0 : (|a|^2 + |c|^2 - \alpha)y - 2|ab|\sqrt{y} + |c|^2y^2 + |b|^2 - \alpha \leq 0\}.$$

The latter is the zero level set of a convex function and hence convex.

(ii) Note that for $y \geq 0$

$$g(y) = \frac{(|a|\sqrt{y} - |b|)^2}{1 + y} + |c|^2y \geq |c|^2y.$$

Therefore, for $y > \frac{|b|^2}{|c|^2}$ we have

$$g(y) \geq |c|^2y > |b|^2 = g(0),$$

showing that there are no optimal solutions for (4.2) larger than $\frac{|b|^2}{|c|^2}$.

[4]A function $f : I \to \mathbb{R}$ ($I \subseteq \mathbb{R}$ being an interval) is quasi-convex if all of its level sets $\{x \in I : f(x) \leq \alpha\}$ are convex.

(iii) First consider the case $(a,c)=(0,0)$. Then $g(y)=|b|^2/(1+y)$. Hence it follows either that g does not attain a minimum (if $b\neq 0$) or that the minimum (namely, all $y\geq 0$) is nonunique (if $b=0$). Now consider the case $(a,c)\neq(0,0)$. We split the analysis into two subcases.

Subcase I. If $c\neq 0$, then $\lim_{y\to\infty} g(y)=\infty$, implying the attainment of the minimum. To show the uniqueness of the minimum in this case, assume in contradiction that the optimal solution of (4.2) is not unique. Then since the optimal set is convex (by quasi convexity) we conclude that the optimal set is an interval $I\subseteq[0,\infty)$ with a nonempty interior. Denote the optimal value by f^*. Then

$$g(y)=f^* \text{ for every } y\in I,$$

which can be explicitly written as

$$(|a|^2+|c|^2-f^*)y-2|ab|\sqrt{y}+|c|^2y^2+|b|^2-f^*=0 \text{ for every } y\in I.$$

By making the change of variables $z=\sqrt{y}$, we obtain

$$(|a|^2+|c|^2-f^*)z^2-2|ab|z+|c|^2z^4+|b|^2-f^*=0 \text{ for every } z\in J, \tag{4.4}$$

where $J=\{z : z^2\in I\}$ is an interval with a nonempty interior. However, (4.4) is impossible since an univariate quartic equation has at most four roots and thus cannot have an infinite number of roots.

Subcase II. Suppose that $c=0$. Then $a\neq 0$, and it is easy to see that g attains a unique minimum at $|b|^2/|a|^2$.

(iv) Since $(a,c)\neq(0,0)$, we know from part (iii) that g attains a unique global minimum on the interval $[0,\infty)$. Hence it remains to show that it is strictly decreasing from the origin to this minimum and strictly increasing when we go from this minimum to plus infinity. Suppose that this is not true. Then the function g must have a stationary point in $(0,\infty)$ which is different from the unique minimum. However, we will show that, for any $y>0$ such that $g'(y)=0$, we automatically have $g''(y)>0$; hence this stationary point y is at least a local minimum and, therefore, must be equal to the unique minimum of g on the interval $[0,\infty)$. Elementary differentiation gives

$$g'(y)=\frac{(|a|\sqrt{y}-|b|)(|a|\frac{1}{\sqrt{y}}+|b|)}{(1+y)^2}+|c|^2.$$

Now let $\tilde{y}>0$ be such that $g'(\tilde{y})=0$. Then

$$\begin{aligned} g''(\tilde{y}) &= -\frac{2}{(1+\tilde{y})^3}(|a|\sqrt{\tilde{y}}-|b|)\left(|a|\frac{1}{\sqrt{\tilde{y}}}+|b|\right)+\frac{|a||b|}{2(1+\tilde{y})^2\sqrt{\tilde{y}}}\left(1+\frac{1}{\tilde{y}}\right) \\ &\overset{g'(\tilde{y})=0}{=} \frac{2|c|^2}{1+\tilde{y}}+\frac{|a||b|}{2(1+\tilde{y})^2\sqrt{\tilde{y}}}\left(1+\frac{1}{\tilde{y}}\right)>0, \end{aligned}$$

where the positivity of the last expression comes from the fact that $(a,c)\neq(0,0)$; hence this last expression can be equal to zero only if both $c=0$ and $b=0$, but then, by taking into account that $\tilde{y}$ is also a stationary point, we would obtain $a=0$ as well, in contrast to $(a,c)\neq(0,0)$. □

The most important property of the function g is its strict unimodality (as stated in Lemma 4.2 (iv)). The strict unimodality property implies that there are no nonglobal local minima and thus enables us to invoke efficient one-dimensional solvers for

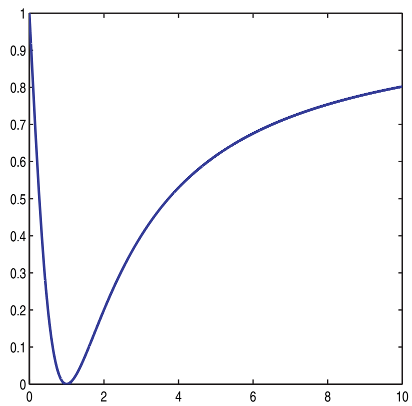

FIG. 2. *The function* $\frac{(\sqrt{y}-1)^2}{1+y}$ *from Remark* 4.2.

(strictly) unimodal functions that are guaranteed to converge to the global minimum. The following example illustrates this property.

Example 1. Consider problem (4.1) with $\mathbb{F} = \mathbb{R}$, $a = 2, b = 5$, and $c = 1$. In this case, problems (4.1) and (4.2) are given by

$$\min_{x} \left\{ \frac{(2x-5)^2}{1+x^2} + x^2 \right\} \tag{4.5}$$

and

$$\min_{y \geq 0} \left\{ \frac{4y - 20\sqrt{y} + 25}{1+y} + y \right\}, \tag{4.6}$$

respectively. The plots of the two functions are given in Figure 1.

Clearly, the objective function in (4.5) is not unimodal and indeed possesses a nonglobal local minimizer. The global solution of (4.5) is $\tilde{x} = 1.5606$ (given in four-digit accuracy). The objective function in (4.6) is, as guaranteed by Lemma 4.1, an unimodal function. The global minimum is $\tilde{y} = 2.4354$, and the relation $\tilde{x} = \sqrt{\tilde{y}}$ holds.

Remark 4.2. A natural question here is whether g is even more than quasi-convex, namely, convex. The answer to this question is negative. For example, for $a = b = 1$ and $c = 0$, the function g is clearly nonconvex as can be seen from Figure 2. Note, however, that this figure also illustrates the quasi convexity of g.

Combining Lemmas 4.1 and 4.2, we are now able to state the basic properties of problem (4.1).

LEMMA 4.3 (uniqueness for problem (4.1)). *The optimal solution of problem* (4.1) *is uniquely attained if and only if one of the following two conditions holds:*

(i) $a \neq 0$.

(ii) $a = 0, c \neq 0$, *and* $|b| \leq |c|$.

Proof. We will split the analysis into four cases:

Case I. $a \neq 0$ and $b \neq 0$. By Lemma 4.2(iii), since $a \neq 0$, the optimal solution of (4.2) is uniquely attained. Moreover, since $ab \neq 0$, then by Lemma 4.1, there is a one-to-one correspondence between optimal solutions of (4.1) and (4.2) (via the relation $\tilde{x} = \operatorname{sgn}(\bar{a}b)\sqrt{\tilde{y}}$), implying the uniqueness and attainment of the optimal solution of (4.1).

Case II. $a \neq 0$ and $b = 0$. The objective function of (4.2) in this case is strictly increasing, implying that the unique optimal solution of (4.2) is $\tilde{y} = 0$ and hence that the unique optimal solution of (4.1) is $\tilde{x} = 0$.

Case III. $a = 0$ and $b \neq 0$. By Lemma 4.2(iii), to guarantee the uniqueness and attainment of the optimal solution of (4.2) we must further assume that $c \neq 0$. The solution of (4.1) is unique if and only if the optimal solution $\tilde{y}$ of (4.2) is zero (otherwise, $z\sqrt{\tilde{y}}$ will be an optimal solution of (4.1) for every z satisfying $|z| = 1$). By the unimodality of g, the optimal solution is 0 if and only if $g'(0) \geq 0$, which is equivalent to $|b| \leq |c|$.

Case IV. $a = 0$ and $b = 0$. Again, as in the previous case, we further assume that $c \neq 0$. Here it is evident that the unique optimal solution is $\tilde{x} = 0$.

By combining the four cases we obtain the result. ∎

4.2. Uniqueness and attainment of the RSTLS solution. The result in section 4.1 collectively can be summed up in the following result.

THEOREM 4.1. *Consider the RSTLS problem* (2.1) *with* $m = n$ *and* $\mathcal{L} = \mathcal{L}_{\mathbf{Q}}$ *(see* (2.2)*), where* $\mathbf{Q} \in \mathbb{F}^{n \times n}$ *is a given unitary matrix. Let* $\hat{\mathbf{b}} = \mathbf{Q}^*\mathbf{b}$. *Suppose further that* $\mathbf{A}, \mathbf{L} \in \mathcal{L}_{\mathbf{Q}}$, *and let* $\boldsymbol{\alpha}, \mathbf{l}$ *be the eigenvalues of* $\mathbf{A}$ *and* $\mathbf{L}$ *given by the relations:*

$$\mathbf{Q}^*\mathbf{A}\mathbf{Q} = \operatorname{diag}(\boldsymbol{\alpha}), \quad \mathbf{Q}^*\mathbf{L}\mathbf{Q} = \operatorname{diag}(\mathbf{l}). \tag{4.7}$$

Then the solution to the RSTLS problem is uniquely attained if and only if for each $i = 1, \ldots, n$ *one of the following two conditions is satisfied:*

(i) $\alpha_i \neq 0$.

(ii) $\alpha_i = 0, l_i \neq 0$, *and* $|\hat{b}_i| \leq \sqrt{\rho}|l_i|$.

Proof. Note that the optimal $\mathbf{E}$ is uniquely defined via the optimal $\mathbf{x}$ by (3.3) and (3.4). Therefore, the uniqueness and/or attainment properties of the optimal solution of (2.1) amount to uniqueness and/or attainment of the single-variable problems (3.2), which combined with Lemma 4.3 establishes the result. ∎

Theorem 4.1 provides conditions for the optimal solution of the RSTLS problem to be uniquely attained. Based on this, we can derive a simpler condition:

$$\operatorname{Null}(\mathbf{A}) \cap \operatorname{Null}(\mathbf{L}) = \{\mathbf{0}\}, \tag{4.8}$$

which is sufficient for attainment of the optimal solution and necessary for the unique attainment of the optimal solution, as shown in the following theorem.

THEOREM 4.2. *Consider the setting of Theorem* 4.1. *Then*

(i) *if the optimal solution of* (2.1) *is uniquely attained, then condition* (4.8) *is satisfied;*

(ii) *if condition* (4.8) *is satisfied, then the optimal solution set of* (2.1) *is nonempty;*

(iii) *if* $\mathbf{A}$ *is nonsingular, then the solution of* (2.1) *is uniquely attained.*

Proof. (i) Note that by Theorem 4.1 a necessary condition for the optimal solution of (2.1) to be uniquely attained is that $|\alpha_i|^2 + |l_i|^2 \neq 0$ for every i, that is, α_i and l_i are not both zero for any given i. The eigenvalues of the matrix $\mathbf{A}^*\mathbf{A} + \mathbf{L}^*\mathbf{L}$ are exactly $|\alpha_i|^2 + |l_i|^2$, implying that $\mathbf{A}^*\mathbf{A} + \mathbf{L}^*\mathbf{L}$ is nonsingular; therefore,

$$\operatorname{Null}(\mathbf{A}) \cap \operatorname{Null}(\mathbf{L}) = \operatorname{Null}(\mathbf{A}^*\mathbf{A} + \mathbf{L}^*\mathbf{L}) = \{\mathbf{0}\}.$$

(ii) Assume that condition (4.8) holds. By Theorem 3.1, it is enough to show that for every $i = 1, \ldots, n$ the one-dimensional problem (3.2) has at least one optimal solution. Now by Lemma 4.1 it is sufficient to establish the attainment of the solution of

$$\min_{y \geq 0} \left\{ \frac{|\alpha_i|^2 y - 2|\alpha_i \hat{b}_i|\sqrt{y} + |\hat{b}_i|^2}{1+y} + \rho |l_i|^2 y \right\} \tag{4.9}$$

for every $i = 1, \ldots, n$, where $\alpha_i, \hat{b}_i$, and l_i are defined in the premise of Theorem 3.1. By Lemma 4.2(iii), this is guaranteed if $(\alpha_i, l_i) \neq (0,0)$ for every i, which, as shown in the proof of (i), is equivalent to condition (4.8).

(iii) It follows from the nonsingularity of $\mathbf{A}$ that all of its eigenvalues are nonzero, which, by Theorem 4.1, implies that the solution of (2.1) is uniquely attained. ∎

The following example shows by suitable counterexamples that the assumptions used in Theorem 4.2 are sufficient, but not necessary, for the corresponding statements to be true.

Example 2. (i) Consider problem (2.1) with $n = m = 1, A = (0), L = (1), b = (2), \rho = 1$, and $\mathbb{F} = \mathbb{R}$. Then condition (4.8) holds, but problem (2.1) has the two solutions $(E, x) = (1, 1)$ and $(E, x) = (-1, -1)$. This shows that the unique attainment of a solution of problem (2.1) is sufficient for condition (4.8) to hold but not necessary.

(ii) Consider problem (2.1) with $n = m = 1, A = (0), L = (0), b = (0), \rho = 1$, and $\mathbb{F} = \mathbb{R}$. Then every vector (E, x), with $E = 0$ and $x \in \mathbb{R}$ arbitrary, is a solution of problem (2.1), although condition (4.8) does not hold. Hence this condition is sufficient for problem (2.1) to have a nonempty solution set but not necessary.

It is interesting to compare the above conditions to the corresponding attainment/uniqueness conditions for the regularized least squares problem:

$$\text{(RLS):} \quad \min \|\mathbf{Ax} - \mathbf{b}\|^2 + \rho\|\mathbf{Lx}\|^2.$$

The optimal solution of (RLS), as opposed to the solution of the RSTLS problem, is always attained; it is unique if and only if condition (4.8) holds. This is in contrast to the RSTLS problem where condition (4.8) is only a necessary condition for unique attainment of the solution.

5. The RSTLS problem with circulant structure.

The RSTLS problem (2.1) with $\mathcal{L} = \mathcal{L}_{\mathbf{F}_n}$ ($\mathbf{F}_n$ being the $n \times n$ DFT matrix) corresponds to problems with circulant-structured matrices. Here the underlying number field is $\mathbb{F} = \mathbb{C}$ since the matrix $\mathbf{F}_n$ is complex-valued. However, in many applications the data $\mathbf{A}, \mathbf{b}$, and $\mathbf{L}$ are real-valued. The main result in this section is that if the optimal set of the RSTLS problem is nonempty, then there exists at least one real-valued optimal solution. Therefore, there is no drawback in analyzing the RSTLS problem over the complex field even when the data are real-valued.

THEOREM 5.1. *Consider the RSTLS problem with* $\mathbb{F} = \mathbb{C}, \mathcal{L} = \mathcal{L}_{\mathbf{F}_n}$, *with* $\mathbf{F}_n$ *being the* $n \times n$ *DFT matrix. Assume that* $\mathbf{A}$, $\mathbf{b}$, *and* $\mathbf{L}$ *are real-valued, that is,* $\mathbf{A} \in \mathbb{R}^{n \times n}, \mathbf{b} \in \mathbb{R}^n$, *and* $\mathbf{L} \in \mathbb{R}^{n \times n}$. *If the optimal set of* (RSTLS) *is nonempty, then there exists at least one optimal real-valued solution.*

Proof. We will require the following notation:

$$\mathcal{A} = \{\mathbf{z} \in \mathbb{C}^n : z_1 \in \mathbb{R}, z_{j+1} = \overline{z_{n+1-j}} \text{ for every } j = 1, \ldots, n-1\}.$$

To simplify the notation we omit the subscript in the $n \times n$ DFT matrix and denote it by $\mathbf{F}$ rather than by $\mathbf{F}_n$. The proof is based on the following three claims:

(i) Let $\mathbf{w} = \mathbf{Fv}$ for some $\mathbf{v} \in \mathbb{R}^n$. Then $\mathbf{w} \in \mathcal{A}$.

(ii) Let $\boldsymbol{\alpha}$ be the vector of eigenvalues of a real-valued circulant matrix $\mathbf{A}$. Then $\boldsymbol{\alpha} \in \mathcal{A}$.

(iii) Let $\mathbf{z} \in \mathcal{A}$. Then $\mathbf{F}^*\mathbf{z} \in \mathbb{R}^n$.

Proof of (i). First,

$$w_1 = (\mathbf{F}\mathbf{v})_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} v_i,$$

proving that $w_1 \in \mathbb{R}$. Next, for every $j = 1, \dots, n-1$ we have

$$w_{j+1} = (\mathbf{F}\mathbf{v})_{j+1} = \sum_{i=1}^{n} F_{j+1,i} v_i = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega^{j(i-1)} v_i. \tag{5.1}$$

On the other hand,

$$\begin{aligned}
\overline{w_{n+1-j}} = \overline{(\mathbf{F}\mathbf{v})_{n+1-j}} &= \sum_{i=1}^{n} \overline{F_{n+1-j,i}} v_i = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \overline{\omega^{(n-j)(i-1)}} v_i \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \omega^{j(i-1)} v_i \overset{(5.1)}{=} w_{j+1}.
\end{aligned}$$

Proof of (ii). Let $(s_1, s_2, \dots, s_n)$ be the first row of $\mathbf{A}$. The jth eigenvalue of the circulant matrix $\mathbf{A}$ is given by $\alpha_j = \sum_{i=1}^{n} \omega^{(i-1)(j-1)} s_i$. Then

$$\alpha_1 = \sum_{i=1}^{n} s_i \in \mathbb{R}$$

and

$$\overline{\alpha_{n+1-j}} = \overline{\sum_{i=1}^{n} \omega^{(i-1)(n-j)} s_i} = \sum_{i=1}^{n} \omega^{(i-1)j} s_i = \alpha_{j+1}$$

for every $j = 1, \dots, n-1$. Thus, $\boldsymbol{\alpha} \in \mathcal{A}$.

Proof of (iii). For every $i = 1, 2, \dots, n$:

$$\begin{aligned}
\sqrt{n}(\mathbf{F}^*\mathbf{w})_i &= \sqrt{n} \sum_{j=1}^{n} \overline{F_{j,i}} w_j = \sum_{j=1}^{n} \omega^{-(i-1)(j-1)} w_j \\
&\overset{\mathbf{w} \in \mathcal{A}}{=} w_1 + \sum_{j=2}^{n} \omega^{-(i-1)(j-1)} \overline{w_{n+2-j}} = w_1 + \sum_{j=2}^{n} \omega^{(i-1)(n+1-j)} \overline{w_{n+2-j}} \\
&\overset{k \leftarrow n+2-j}{=} w_1 + \sum_{k=2}^{n} \omega^{(i-1)(k-1)} \overline{w_k} = \sqrt{n} \sum_{k=1}^{n} F_{k,i} \overline{w_k} \\
&= \sqrt{n} \overline{\sum_{k=1}^{n} \overline{F_{k,i}} w_k} = \sqrt{n}\, \overline{(\mathbf{F}^*\mathbf{w})_i}.
\end{aligned}$$

By Theorem 3.1, an optimal solution of the RSTLS problem is given by $\mathbf{x} = \mathbf{F}^*\hat{\mathbf{x}}$, where $\hat{x}_i$, the ith component of $\hat{\mathbf{x}}$, is an optimal solution of (3.2). Recall that $\hat{\mathbf{b}} = \mathbf{F}\mathbf{b}$ for real-valued $\mathbf{b}$ and that $\boldsymbol{\alpha}$ and $\mathbf{l}$ are the eigenvalues vectors of the real-valued circulant matrices $\mathbf{A}$ and $\mathbf{L}$, respectively. Therefore, by properties (i) and (ii), $\hat{\mathbf{b}}, \boldsymbol{\alpha}, \mathbf{l} \in$

$\mathcal{A}$. Hence, $\hat{x}_1$ is the solution of (3.2) with $i = 1$ and with real data, which by Remark 4.1 implies that $\hat{x}_1$ is real. Moreover, for every $j = 1, \ldots, n-1$, $\hat{x}_j$ and $\hat{x}_{n+1-j}$ are the optimal solutions of

$$\min_{\hat{x}_{j+1}} \left\{ \frac{|\alpha_{j+1}\hat{x}_{j+1} - \hat{b}_{j+1}|^2}{1 + |\hat{x}_{j+1}|^2} + \rho |l_j|^2 |\hat{x}_j|^2 \right\},$$

$$\min_{\hat{x}_{n+1-j}} \left\{ \frac{|\alpha_{j+1}\overline{\hat{x}_{n+1-j}} - \hat{b}_{j+1}|^2}{1 + |\overline{\hat{x}_{n+1-j}}|^2} + \rho |l_{j+1}|^2 |\overline{\hat{x}_{n+1-j}}|^2 \right\},$$

respectively. Therefore, we can always choose the optimal solutions of these problems to satisfy $\hat{x}_{n+1-j} = \overline{\hat{x}_{j+1}}$. Thus, for the mentioned choice $\hat{\mathbf{x}} \in \mathcal{A}$ and by property (iii) this proves that $\mathbf{x} = \mathbf{F}^*\hat{\mathbf{x}}$ is real-valued. ∎

Remark 5.1. It can be shown by using the same methodology employed in the proof of Theorem 5.1 that there always exists a real-valued solution for the RSTLS problem with $\mathbf{Q} = \mathbf{F}_n \otimes \mathbf{F}_m$ (BCCB structure) whenever $\mathbf{A}, \mathbf{L}$, and $\mathbf{b}$ are real-valued.

The following two examples demonstrate the validity of Theorem 5.1.

Example 3. Let $\mathbf{Q} = \mathbf{F}_3$ (3×3 circulant matrices) and

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad \rho = 1.$$

Then

$$\boldsymbol{\alpha} = \mathrm{diag}(\mathbf{F}_3 \mathbf{A} \mathbf{F}_3^*) = \begin{pmatrix} 6 \\ -1.5 - 0.866025\mathbf{i} \\ -1.5 + 0.866025\mathbf{i} \end{pmatrix},$$

$$\hat{\mathbf{b}} = \mathbf{F}_3 \mathbf{b} = \begin{pmatrix} 8.660254 \\ -0.866025 + 0.5\mathbf{i} \\ -0.866025 - 0.5\mathbf{i} \end{pmatrix}, \qquad \mathbf{l} = \begin{pmatrix} 0 \\ 1.5 - 0.866025\mathbf{i} \\ 1.5 + 0.866025\mathbf{i} \end{pmatrix}.$$

The vector $\hat{\mathbf{x}}$ consisting of the optimal solutions the three arising optimization problems is

$$\hat{\mathbf{x}} = \begin{pmatrix} 1.443375 \\ 0.143941 - 0.249314\mathbf{i} \\ 0.143941 + 0.249314\mathbf{i} \end{pmatrix},$$

and the optimal solution

$$\mathbf{x} = \mathbf{F}_3^* \hat{\mathbf{x}} = \begin{pmatrix} 0.999543 \\ 0.999543 \\ 0.500913 \end{pmatrix}$$

is indeed real.

Example 4. Consider the RSTLS problem with $\mathbf{Q} = \mathbf{F}_3$ (3×3 circulant matrices) and

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \qquad \rho = 1.$$

Then

$$\boldsymbol{\alpha} = \mathrm{diag}(\mathbf{F}_3\mathbf{A}\mathbf{F}_3^*) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\mathbf{b}} = \mathbf{F}_3\mathbf{b} = \begin{pmatrix} 6.928203 \\ -1.732050 + \mathbf{i} \\ -1.732050 - \mathbf{i} \end{pmatrix}.$$

In this example the optimal solutions of the arising one-dimensional problems are not unique, and they consist of the collection of vectors $\hat{\mathbf{x}}$ of the form:

$$\hat{\mathbf{x}} = \begin{pmatrix} 2.309401 \\ 0.393319z_1 \\ 0.393319z_2 \end{pmatrix},$$

where z_1 and z_2 are complex numbers satisfying $|z_1| = |z_2| = 1$. Correspondingly, the set of optimal solutions of (RSTLS) consists of all vectors $\mathbf{F}_3^*\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is of the above form and is thus equal to

$$\{(a + z_1 b + z_2 c, a + bz_1\bar{\omega} + cz_2\omega, a + bz_1\omega + cz_2\bar{\omega})^T : |z_1| = |z_2| = 1\},$$

where $a = 2.309401, b = 0.393319$, and $\omega = e^{\frac{2\pi\mathbf{i}}{3}}$. The above set certainly contains complex-valued optimal solutions, but, if we choose $z_1 = \bar{z}_2$, we obtain a subset of real-valued optimal solutions:

$$\left\{\frac{1}{\sqrt{3}}(a + 2\cos(\theta)c, a + 2\cos(\theta + 2\pi/3)c, a + 2\cos(\theta - 2\pi/3)b)^T : 0 \le \theta \le 2\pi\right\}.$$

6. Solution of the CSTLS problem with SD structure. When the regularization is made by adding a constraint rather than by penalization, the problem becomes

$$\text{(CSTLS):}\quad \begin{array}{ll} \min\limits_{\mathbf{E},\mathbf{x}} & \|\mathbf{E}\|^2 + \|(\mathbf{A}+\mathbf{E})\mathbf{x} - \mathbf{b}\|^2 \\ \text{s.t.} & \|\mathbf{L}\mathbf{x}\|^2 \le \alpha, \\ & \mathbf{E} \in \mathcal{L}_{\mathbf{Q}}, \\ & \mathbf{x} \in \mathbb{F}^n, \end{array} \tag{6.1}$$

where $\alpha > 0$. We will show that the CSTLS problem can be solved by a sequence of RSTLS problems using a dual approach. We assume throughout this section that $\mathbf{A}$ is nonsingular. This assumption prevails in many image deblurring problems, although the matrix is often extremely ill conditioned.

The Lagrangian dual problem of (6.1) is given by

$$\max_{\lambda \ge 0} q(\lambda), \tag{6.2}$$

where

$$q(\lambda) = \begin{array}{ll} \min\limits_{\mathbf{E},\mathbf{x}} & \|\mathbf{E}\|^2 + \|(\mathbf{A}+\mathbf{E})\mathbf{x} - \mathbf{b}\|^2 + \lambda(\|\mathbf{L}\mathbf{x}\|^2 - \alpha) \\ \text{s.t.} & \mathbf{E} \in \mathcal{L}_{\mathbf{Q}}, \mathbf{x} \in \mathbb{F}^n. \end{array} \tag{6.3}$$

Therefore, evaluating a value of the dual objective function amounts to solving a single RSTLS problem which can be solved efficiently as shown in the previous sections. Since $\mathbf{A}$ is nonsingular, then by Theorem 4.2 (iii), the optimal solution of (6.3) is uniquely attained for all $\lambda \ge 0$, and we denote it by $(\mathbf{x}_\lambda, \mathbf{E}_\lambda)$. The function q has several important properties which are summarized in Lemma 6.1 below. The differentiability property of q (part (ii) of Lemma 6.1), relies on the uniqueness property and on the following well known result [9, Proposition 6.1.1].

THEOREM 6.1. *Let f and g be continuous functions defined on a compact set X. Let*

$$h(\lambda) \equiv \min_{\mathbf{x} \in X} \left\{ f(\mathbf{x}) + \lambda g(\mathbf{x}) \right\}, \quad \lambda \in [\lambda_1, \lambda_2],$$

and assume that there exists a unique minimizer $\mathbf{x}_\lambda$ to the above optimization problem for every $\lambda \in [\lambda_1, \lambda_2]$ denoted by $\mathbf{x}_\lambda$. Then h is differentiable for every $\lambda \in (\lambda_1, \lambda_2)$ and $h'(\lambda) = g(\mathbf{x}_\lambda)$.

In our case the compactness assumption is not satisfied; however, this difficulty can be avoided. We will use the following notation:

$$\begin{aligned} s(\mathbf{x}, \mathbf{E}) &= \|\mathbf{E}\|^2 + \|(\mathbf{A} + \mathbf{E})\mathbf{x} - \mathbf{b}\|^2, \\ t(\mathbf{x}, \mathbf{E}) &= \|\mathbf{L}\mathbf{x}\|^2 - \alpha, \\ Y &= \{(\mathbf{x}, \mathbf{E}) : \mathbf{x} \in \mathbb{F}^n, \mathbf{E} \in \mathcal{L}_{\mathbf{Q}}\}. \end{aligned}$$

Then, in this notation, the CSTLS problem can be written as

$$\min_{\mathbf{x}, \mathbf{E}} \{s(\mathbf{x}, \mathbf{E}) : t(\mathbf{x}, \mathbf{E}) \leq 0, (\mathbf{x}, \mathbf{E}) \in Y\}. \tag{6.4}$$

LEMMA 6.1. *Consider the function q given by* (6.3)*. Then*
(i) *q is concave over $[0, \infty)$;*
(ii) *$q(\lambda)$ is differentiable for every $\lambda > 0$ and $q'(\lambda) = \|\mathbf{L}\mathbf{x}_\lambda\|^2 - \alpha$;*
(iii) *$\lim_{\lambda \to \infty} q(\lambda) = -\infty$.*

Proof. (i) $q(\lambda)$ is the pointwise minimum of functions which are linear in λ and hence concave.

(ii) Let $\tilde{\lambda} > 0$, and let $\lambda_2 > \lambda_1 > 0$ be two positive numbers for which $\tilde{\lambda} \in (\lambda_1, \lambda_2)$. The dual objective can be written as

$$q(\lambda) = \min\{s(\mathbf{x}, \mathbf{E}) + \lambda t(\mathbf{x}, \mathbf{E}) : (\mathbf{x}, \mathbf{E}) \in Y\}. \tag{6.5}$$

From the nonsingularity of $\mathbf{A}$ and Theorem 4.2(iii) it follows that there exists a unique minimizer to the above problem which we denote by $(\mathbf{x}_\lambda, \mathbf{E}_\lambda)$. By Theorem 3.1 it follows that $\mathbf{x}_\lambda = \mathbf{Q}^* \mathbf{y}^\lambda$, where the ith component of $\mathbf{y}^\lambda$, y_i^λ, is the solution to

$$\min_{y_i} \left\{ \frac{|\alpha_i y_i - \hat{b}_i|^2}{1 + |y_i|^2} + \rho\lambda |l_i|^2 |y_i|^2 \right\}.$$

If $l_i = 0$, then $y_i^\lambda = \frac{\hat{b}_i}{\alpha_i}$ ($\alpha_i \neq 0$ for every i as an eigenvalue of a nonsingular matrix). Otherwise,

$$\rho\lambda |l_i|^2 |y_i^\lambda|^2 \leq \frac{|\alpha_i y_i^\lambda - \hat{b}_i|^2}{1 + |y_i^\lambda|^2} + \rho\lambda |l_i|^2 |y_i^\lambda|^2 \leq \frac{|\alpha_i 0 - \hat{b}_i|^2}{1 + 0^2} + \rho\lambda |l_i|^2 0^2 = |\hat{b}_i|^2,$$

so that $|y_i^\lambda|^2 \leq \frac{|\hat{b}_i|^2}{\rho\lambda |l_i|^2}$. Consequently, for every $\lambda \in [\lambda_1, \lambda_2]$,

$$|y_i^\lambda|^2 \leq \begin{cases} \left| \frac{\hat{b}_i}{\alpha_i} \right|^2 & l_i = 0, \\ \frac{|\hat{b}_i|^2}{\rho\lambda_1 |l_i|^2} & l_i \neq 0. \end{cases}$$

Hence, $\mathbf{y}^\lambda$ is bounded for every $\lambda \in [\lambda_1, \lambda_2]$ showing that $\mathbf{x}_\lambda = \mathbf{Q}^*\mathbf{y}^\lambda$ is also bounded over $[\lambda_1, \lambda_2]$; that is, there exists $\beta > 0$ for which $\|\mathbf{x}_\lambda\| \leq \beta, \lambda \in [\lambda_1, \lambda_2]$. Moreover, by the relation between the optimal $\mathbf{E}$ and the optimal $\mathbf{x}$ given by (3.3) and (3.4), it follows that $\mathbf{E}_\lambda$ is also bounded over $[\lambda_1, \lambda_2]$; namely, there exists $\gamma > 0$ for which $\|\mathbf{E}_\lambda\| \leq \gamma$. The dual objective function can thus be written as

$$q(\lambda) = \min\{s(\mathbf{x}, \mathbf{E}) + \lambda t(\mathbf{x}, \mathbf{E}) : (\mathbf{x}, \mathbf{E}) \in \tilde{Y}\},$$

where

$$\tilde{Y} = \{(\mathbf{x}, \mathbf{E}) : \mathbf{x} \in \mathbb{F}^n, \mathbf{E} \in \mathcal{L}_{\mathbf{Q}}, \|\mathbf{x}\| \leq \beta, \|\mathbf{E}\| \leq \gamma\}$$

is a compact set. Therefore, by Theorem 6.1, q is differentiable over (λ_1, λ_2) and in particular at $\tilde{\lambda}$ and $q'(\tilde{\lambda}) = t(\mathbf{x}_{\tilde{\lambda}}, \mathbf{E}_{\tilde{\lambda}}) = \|\mathbf{L}\mathbf{x}_{\tilde{\lambda}}\|^2 - \alpha$.

(iii) Since $\mathbf{E} = \mathbf{0}, \mathbf{x} = \mathbf{0}$ is feasible for (6.3), we obtain

$$q(\lambda) \leq \|\mathbf{b}\|^2 - \lambda\alpha,$$

establishing that $q(\lambda) \to -\infty$ as $\lambda \to \infty$. □

We will now show that, despite the nonconvexity of the CSTLS problem, strong duality holds.

THEOREM 6.2 (strong duality for CSTLS). *Let $\lambda^* > 0$ be a maximizer of* (6.2). *Then $q(\lambda^*)$ is equal to the optimal value of the primal problem* (6.1)*, and $(\mathbf{x}_{\lambda^*}, \mathbf{E}_{\lambda^*})$ is the optimal solution of* (6.1).

Proof. Since $\lambda^* > 0$ is the optimal solution of (6.2) and q is differentiable by Lemma 4.2(ii), we have $\|\mathbf{L}\mathbf{x}_{\lambda^*}\|^2 - \alpha = q'(\lambda^*) = 0$. Therefore, $\mathbf{x}_{\lambda^*}$ is a feasible solution of the primal problem (6.1), and

$$q(\lambda^*) = s(\mathbf{x}_{\lambda^*}, \mathbf{E}_{\lambda^*}) + \lambda^*(\|\mathbf{L}\mathbf{x}_{\lambda^*}\|^2 - \alpha) = s(\mathbf{x}_{\lambda^*}, \mathbf{E}_{\lambda^*}),$$

which, from basic duality theory, implies that λ^* and $(\mathbf{x}_{\lambda^*}, \mathbf{E}_{\lambda^*})$ are the dual and primal optimal solutions, respectively. □

The optimal λ^* is a root of the nondecreasing function $q'(\lambda)$ and can thus be found via a simple bisection procedure.

7. Implementation and a numerical example.

7.1. Implementation. The core of the numerical method for solving the RSTLS problem is the solution of n single-variable problems of the form (4.1). Since the number of these 1D problems might be huge (for example, for a two-dimensional 1024×1024 image, there are more than one million problems), it is imperative to find the *global* solution of each of them. The method will produce an erroneous solution even if one of the 1D problems is not solved correctly.

From numerical considerations the algorithm is split into two phases. In the first phase, we find the optimal solution of (4.2) up to a moderate tolerance ε (in our experiments $\varepsilon = 10^{-4}$). That is, the output of the first phase is an interval $[\ell, u]$, with $u - \ell < \varepsilon$, in which the optimal solution of (4.2) is guaranteed to reside. The goal of the first phase is to find a "small enough" interval in which the global solution is guaranteed to reside. Since in the course of the change of variables $x = \operatorname{sgn}(\bar{a}b)\sqrt{y}$ the accuracy of the solution might be reduced from ε to $\sqrt{\varepsilon}$, a second phase is invoked in which we seek the global minimizer x^* of the problem

$$\min_x \left\{ \frac{|a|^2x^2 - 2|ab|x + |b|^2}{1 + x^2} + |c|^2x^2 \right\} \tag{7.1}$$

in the interval $[\sqrt{\ell}, \sqrt{u}]$ up to a tolerance ε^2. The interval $[\sqrt{\ell}, \sqrt{u}]$ is small enough so that for all practical purposes the function in (7.1) is unimodal over $[\sqrt{\ell}, \sqrt{u}]$ and the global optimal solution given by $\text{sign}(\bar{a}b)x^*$ is obtained. A detailed description of the algorithm follows.

ALGORITHM SOLVE1D(a, b, c).

input	$a, b, c \in \mathbb{C}$
output	x - an optimal solution of (4.1).
comments	1. It is assumed that a and c are not both zero.
	2. The functions f_1 and f_2 called in the solver are given by
	$f_1(x; a, b, c) = \frac{(\lvert a\rvert\sqrt{x}-\lvert b\rvert)^2}{1+x} + \lvert c\rvert^2 x,$
	$f_2(x; a, b, c) = \frac{(\lvert a\rvert x-\lvert b\rvert)^2}{1+x^2} + \lvert c\rvert^2 x^2.$

If c is equal to zero up to some tolerance, then the output of the algorithm is b/a; otherwise, the upper bound is chosen.

if $c < 10^{-8}$
  $x = \frac{b}{a}$
  stop
else
  $u = \left|\frac{b}{c}\right|^2$
end if
$\ell = 0$
$s = \text{sgn}\,(\bar{a}b)$
Phase I. Activating an unimodal solver on the function f_1
while $(u - \ell) > \varepsilon$
    $x^- = \frac{2}{3}\ell + \frac{1}{3}u$
    $x^+ = \frac{1}{3}\ell + \frac{2}{3}u$
    $f^+ = f_1(x^+; a, b, c)$
    $f^- = f_1(x^-; a, b, c)$
    if $f^- \leq f^+$
      $u = x^+$
    else
      $\ell = x^-$
    end if
end while
Updating the lower and upper bounds.
$\ell = \sqrt{\ell}$
$u = \sqrt{u}$
Phase II. Activating an unimodal solver on the function f_2.
while $(u - \ell) > \varepsilon^2$
    $x^- = \frac{2}{3}\ell + \frac{1}{3}u$
    $x^+ = \frac{1}{3}\ell + \frac{2}{3}u$
    $f^+ = f_2(x^+; a, b, c)$
    $f^- = f_2(x^-; a, b, c)$
    if $f^- \leq f^+$
      $u = x^+$
    else
      $\ell = x^-$
    end if

end while
$x = s\frac{x^+ + x^-}{2}$
stop

We note that in the MATLAB implementation the minimization of the n 1D problems is done simultaneously using MATLAB's vector operations. For the exact implementation please see the (small) RSTLS MATLAB package available at [38]. Given the 1D solver, the solution of the RSTLS problem (2.1) is obtained via the following procedure.

ALGORITHM RSTLS $(\mathbf{Q}, \mathbf{A}, \mathbf{L}, \rho)$.

input	$\mathbf{Q} \in \mathbb{F}^{n\times n}$ - a unitary matrix.
	$\mathbf{A}, \mathbf{L} \in \mathcal{L}_{\mathbf{Q}}, \mathbf{b} \in \mathbb{F}^n$.
	$\rho \in \mathbb{R}_{++}$.
output	The $\mathbf{x}$-part of the optimal solution of (2.1).

Step 1. $\hat{\mathbf{b}} = \mathbf{Q}\mathbf{b}$.
Step 2. Compute the eigenvalues vectors $\boldsymbol{\alpha}, \mathbf{l}$ of $\mathbf{A}$ and $\mathbf{L}$ defined by the relations (3.1).
Step 3. For each $i = 1, \ldots, n$ call algorithm SOLVE1D with input $\alpha_i, \hat{b}_i, c_i$ and obtain an output $\hat{x}_i$.
Step 4. $\mathbf{x} = \mathbf{Q}^*\hat{\mathbf{x}}$, where $\hat{\mathbf{x}} = (\hat{x}_i)_{i=1}^n$.

Based on the RSTLS algorithm, the constrained version, problem (CSTLS), is solved via a simple bisection algorithm applied to $q'(\lambda)$, where q is the dual function defined by (6.3). The bisection is over the logarithm of base 10 of the dual variable λ.

ALGORITHM CSTLS$(\mathbf{Q}, \mathbf{A}, \mathbf{L}, \alpha)$.

input	$\mathbf{Q} \in \mathbb{F}^{n\times n}$ - a unitary matrix.
	$\mathbf{A}, \mathbf{L} \in \mathcal{L}_{\mathbf{Q}}, \mathbf{b} \in \mathbb{F}^n$.
	$\alpha \in \mathbb{R}_{++}$.
output	The $\mathbf{x}$-part of the optimal solution of (6.1).

Step 1. $u = 2, \ell = -4$.
Step 2. **while** $(u - \ell) > 0.1$
  $h = \frac{u+\ell}{2}$
  call Algorithm RSTLS with input $\mathbf{Q}, \mathbf{A}, \mathbf{L}, 10^h$ and obtain an output $\tilde{\mathbf{x}}$
  if $\|\mathbf{L}\tilde{\mathbf{x}}\|^2 < \alpha$
    $u = h$
  else
    $\ell = h$
  end if
end while
Step 3. $\mathbf{x} = \tilde{\mathbf{x}}$.

Note that the RSTLS and CSTLS algorithms use matrix-vector multiplications with the matrices $\mathbf{Q}$ and $\mathbf{Q}^*$ and require the computation of the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{L}$. When $\mathcal{L}_{\mathbf{Q}}$ is one of the four SD structures described in section 2.2 in the context of image deblurring, these operations can be efficiently performed by utilizing fast transforms: one- or two-dimensional FFT for periodic boundary conditions and one- or two-dimensional FCT for reflexive boundary conditions.

7.2. A numerical example. To demonstrate our approach we consider an image deblurring example. We start with the 512×512 Lena gray image (top left image of Figure 3) scaled so that all of the pixels are in the interval $[0, 1]$ and blur it with a Gaussian PSF of dimension 9×9 with standard deviation 6 implemented in the command psfGauss([9,9],6) from [15]; the values in the PSF range between 0.0095 and 0.0148. We assume that the blurring is not exactly known and that the observed PSF is a Gaussian PSF of dimension 9×9 with standard deviation 8. We then cut the margins by 20 rows and columns resulting in 492×492 and add a Gaussian white noise with standard deviation 10^{-3} (top right image of Figure 3). By assuming reflexive boundary conditions, the poor naive solution construction (i.e., $\mathbf{A}^{-1}\mathbf{b}$) is given in the left middle image of Figure 3. This poor quality of the naive solution is not surprising since the problem is extremely ill conditioned. In our experiments, the regularization matrix $\mathbf{L}$ represents a discretization of a differential operator corresponding to the PSF

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}.$$

The constrained least squares solution, that is, the solution of the problem

$$\min\{\|\mathbf{Ax} - \mathbf{b}\|^2 : \|\mathbf{Lx}\|^2 \leq \alpha\},$$

is presented in the right middle image. The CSTLS reconstructions under periodic and reflexive boundary conditions are the left and right bottom images, respectively. The parameter α is chosen as $1.2\|\mathbf{Lx}_{\text{true}}\|^2$. Clearly, the best reconstruction is provided by the CSTLS algorithm with reflexive boundary conditions. The artifacts in the CSTLS reconstruction with periodic boundary conditions are much more prominent. The relative error of the CSTLS reconstruction with reflexive boundary conditions, $\frac{\|\mathbf{x}_{\text{true}} - \mathbf{x}_{\text{CSTLS-R}}\|}{\|\mathbf{x}_{\text{true}}\|}$, is 0.0961, while for periodic boundary conditions the relative error is 0.1393. The constrained least squares solution gave the worst relative error: 0.15.

8. Conclusion and discussion. In this paper we have shown that the RSTLS problem for structures involving matrices which are simultaneously diagonalizable by a given unitary matrix can be efficiently and globally solved (as opposed to general structures). These SD structures appear in image deblurring problems with either reflexive or periodic boundary conditions. The solution method consists of first decomposing the problem into several real or complex one-dimensional problems which are not necessarily unimodal. In the described image deblurring examples, the decomposition is performed by using the FFT or the FCT. The one-dimensional problems are then globally solved by invoking an unimodal solver on a transformation of the problems. Numerical results demonstrate the effectiveness of the proposed approach.

Another type of boundary conditions are antireflective boundary conditions introduced in [35].[5] As stated in [35], antireflective boundary conditions further reduce the boundary artifacts. The reason is that zero Dirichlet and periodic boundary conditions introduce an artificial discontinuity at the border of the field of view; reflexive boundary conditions impose that the reflected image is globally continuous but introduce an artificial discontinuity of the first derivative, while antireflective boundary conditions using a central symmetry are able to maintain C^1 continuity in the case of signals and C^0 with normal derivative continuity for images.

[5]We thank an anonymous reviewer for referring us to the literature on this type of boundary conditions.

FIG. 3. *Deblurring of Lena.*

In analogy with the reflexive boundary conditions, matrix-vector operations, solution of linear systems, and eigenvalue computations in the antireflective setting can be done in $O(n \log n)$ real operations [4] (using the fast sine transform). It is also known that for these types of boundary conditions and with symmetric PSFs the set of all possible matrices is simultaneously diagonalizable [3]. However, the diagonalizing matrix is not unitary. The unitary property is essential to the analysis introduced in the current paper. Specifically, the decomposition of the RSTLS problem described in Theorem 3.1 will not be valid if the diagonalization is via a nonunitary matrix. Therefore, it does not seem possible to analyze the RSTLS problem with antireflective boundary conditions within the setting of the paper. It is an open question whether it is possible to exploit the special properties of antireflective boundary conditions in order to construct an efficient method for solving the corresponding RSTLS problem.

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