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Convexity Properties Associated with Nonconvex Quadratic Matrix Functions and Applications to Quadratic Programming

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Abstract We establish several convexity results which are concerned with nonconvex quadratic matrix (QM) functions: strong duality of quadratic matrix programming problems, convexity of the image of mappings comprised of several QM functions and existence of a corresponding S-lemma. As a consequence of our results, we prove that a class of quadratic problems involving several functions with similar matrix terms has a zero duality gap. We present applications to robust optimization, to solution of linear systems immune to implementation errors and to the problem of computing the Chebyshev center of an intersection of balls.

Keywords Quadratic matrix functions · Strong duality · Extended S-lemma · Semidefinite relaxation · Convexity of quadratic maps

1 Introduction

We consider convexity-type results related to quadratic functions $f:\mathbb{F}^{n\times r}\to\mathbb{R}$ of the form

$$f(Z)=\mathrm{Tr}(Z^*AZ)+2\Re(\mathrm{Tr}(B^*Z))+c,\quad Z\in\mathbb{F}^{n\times r},\tag{1}$$

where $A=A^*\in\mathbb{F}^{n\times n}$, $B\in\mathbb{F}^{n\times r}$ and $c\in\mathbb{R}$. Here $\mathbb{F}$ denotes either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. A function of the form (1) is called a *quadratic matrix function* of order r. In [1], this type of functions was analyzed over the real domain in relation to quadratic matrix programming (QMP) problems, namely problems

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involving QM functions. In particular, it was shown that nonconvex QMP problems with at most r constraints admit a tight semidefinite relaxation (SDR) and that strong duality holds. This result is a generalization of the well-known strong duality result for the class of quadratically constrained quadratic programming (QCQP) problems with a single quadratic constraint; see for example [7–10] for theoretical and algorithmic analysis. The tightness of the SDR result for QMP problems (or in particular for QCQP problems with a single constraint) can be interpreted as a type of hidden convexity property, since it also implies that the solution of the original *nonconvex* problem can be extracted from its *convex* SDR reformulation.

In this paper, we continue to study convexity results of (possibly nonconvex) QM functions. An overview of the literature reveals that there are three major convexity-type results with respect to quadratic functions and problems:

(i) *Zero duality gap*—the value of a QCQP problem is equal to the value of its dual problem.
(ii) *S-lemma-type result*—The statement "a quadratic inequality constraint is implied by a set of quadratic inequalities" is equivalent to a certain linear matrix inequality (LMI).
(iii) *Convexity of the image of quadratic mappings*—the image of a mapping comprised of several quadratic functions is convex.

We note that the first property is generally satisfied if and only if the corresponding SDR is tight. The reason for this is that the dual of a QCQP and its SDR are convex problems which are dual to each other and hence, under some regularity conditions, have the same value.

It is well known that the three categories—although not equivalent—are very closely related. Examples of derivations from one category to the other can be found throughout the literature. The earliest results connecting the different categories can be found in the works of Yakubovich and Fradkov and Yakubovich [2, 3] where it was shown that, by using separation theorems for convex sets, an appropriate S-lemma can be deduced from corresponding results on the convexity of the image of quadratic mappings. Years later, Polyak [4] proved a strong duality result for homogenous nonconvex quadratic problems involving two quadratic constraints by using a convexity property on the image of three homogenous quadratic forms (under the assumption that there exists a positive-definite linear combination of the corresponding matrices); this result was also recovered by Ye and Zhang [5]. Polyak's work [4] also provided an alternative proof of the well-known strong duality result for (generalized) trust region subproblems [6–10], which is based on the convexity property of mappings comprised of two *nonhomogenous* quadratic functions. In [11], Beck and Eldar use a complex version of the S-lemma in order to show strong duality property for nonconvex quadratic problems with two quadratic mappings over the complex domain. This result was also independently derived by Huang and Zhang [12]. In [11] it is also shown that, by comparing real and complex-valued images of quadratic mappings, one can establish a sufficient condition for strong duality of nonconvex quadratic problems with two quadratic constraints over the *real domain*. More interesting results concerning various convexity results of quadratic mappings and their relation to optimization problems can be found in the comprehensive survey of Polik and Terlaky [13], the book of Ben-Tal and Nemirovski [14] and in the paper of Hiriart-Urruty and Torki [15].

In this paper, we present convexity results of all three categories in connection to QM functions and problems. In Sect. 2, we review the essential facts from [1] about QMP problems and extend the analysis to the complex domain. Our motivation for considering the complex setting is twofold. First, optimization problems in many engineering applications naturally have complex-valued variables. Second, results over the complex and real domains are not the same. In particular, we show that QMP problems with r constraints over the real domain, or $2r$ constraints over the complex domain, have a tight SDR and that strong duality holds. In Sect. 3, we consider the image of quadratic mappings comprised of several QM functions. We show that a mapping comprised of at most r QM functions of order r in the real domain, or at most $2r$ QM functions in the complex domain, is always convex. Under some assumptions, we prove that an additional QM function can be added without damaging the convexity result. The latter result exploits a result of Au-Yeung and Poon [16] on the convexity of the numerical range of a certain class of homogenous quadratic functions. An extended S-lemma on QM functions is established in Sect. 4; applications to robust quadratic optimization and the solution of linear systems immune to implementation errors are presented. Finally, in Sect. 5 we present the special class of *uniform quadratic problems* in which strong duality is shown to follow from the strong duality result of QMP problems. This result is an extension and improvement of a result which was derived for the real case [17]. For convenience, some technical results are gathered in the appendices.

Notation The discussion throughout the paper is presented over the number field $\mathbb{F}$ which stands for either $\mathbb{R}$ or $\mathbb{C}$. The identity matrix of order r is denoted by I_r, $(\cdot)^*$ and $(\cdot)^T$ denote the Hermitian conjugate and the transpose of the corresponding matrices respectively. In order to be able to analyze the complex and real domains at the same time, we define the following field-dependent functions:

$$\theta(\mathbb{F}) \equiv \begin{cases} 1, & \mathbb{F}=\mathbb{R}, \\ 2, & \mathbb{F}=\mathbb{C}, \end{cases}$$

$$\varphi(n;\mathbb{F}) \equiv \begin{cases} n^2, & \mathbb{F}=\mathbb{C}, \\ \binom{n+1}{2}, & \mathbb{F}=\mathbb{R}. \end{cases}$$

It can be readily seen that $\varphi(n;\mathbb{F}) = \theta(\mathbb{F})\frac{n(n-1)}{2} + n$ and that $\varphi(n;\mathbb{F})$ is the dimension of $\mathcal{H}^n(\mathbb{F})$ over the real number field $\mathbb{R}$. The space $\mathcal{H}^n(\mathbb{F})$ denotes the space of matrices A over the field $\mathbb{F}$ that satisfy $A^* = A$. Therefore, $\mathcal{H}^n(\mathbb{R}) = \mathcal{S}^n$ is the set of real symmetric matrices and $\mathcal{H}^n(\mathbb{C}) = \mathcal{H}^n$ is the set of complex Hermitian matrices. Similarly, $\mathcal{H}^n_{++}(\mathbb{F})$ ($\mathcal{H}^n_{+}(\mathbb{F})$) is the set of all positive (semi)definite matrices over $\mathbb{F}$. For two matrices A and B, $A \succ B$ ($A \succeq B$) means that $A - B$ is positive definite (semidefinite). E^r_{ij} is the $r \times r$ matrix with one at the (i,j)th component and zero elsewhere. For a given square matrix U, $[U]_r$ denotes the southeast $r \times r$ submatrix of U, i.e., if $U = (u_{ij})^{n+r}_{i,j=1}$, then $[U]_r = (u_{ij})^{n+r}_{i,j=n+1}$. For simplicity, instead of inf/sup we use min/max; however, this does not mean that we assume that the optimum is attained and/or finite. The value of the optimal objective function of an optimization problem

$$(\mathrm{P}) \quad \min\{f(x) : x \in C\}$$

is denoted by val(P). The optimization problem (P) is called *bounded below* if the minimum is finite and termed *solvable* in the case where the minimum is finite and attained (similar definitions for maximum problems). We follow the MATLAB convention and use ";" for adjoining scalars, vectors or matrices in a column.

2 Review and Extension: QMP Problems in the Real and Complex Domains

QMP problems in the real domain were presented and analyzed in [1]. In this section, we extend the results of [1] to include both the complex and real domains. This section can also be regarded as a review of the results on QMP problems which will be the key ingredient for many of the results in this paper.

We begin by defining in Sect. 2.1 the concepts of QM functions and QMP problems. We then present in Sect. 2.2 a specially-devised SDR and dual of QMP problems. Finally, based on the constructed SDR and an extension of the rank reduction algorithm of Pataki [18, 19], we are able to show in Sect. 2.3 that QMP problems with r constraints over the real domain and QMP problems with $2r$ constraints over the complex field have a tight semidefinite relaxation.

2.1 Quadratic Matrix Functions and Problems

A *quadratic matrix (QM) function* of order r is a function $f : \mathbb{F}^{n\times r} \to \mathbb{R}$ of the form

$$f(Z) = \mathrm{Tr}(Z^* A Z) + 2\Re(\mathrm{Tr}(B^* Z)) + c, \quad Z \in \mathbb{F}^{n\times r}, \tag{2}$$

where $A \in \mathcal{H}^n(\mathbb{F})$, $B \in \mathbb{F}^{n\times r}$ and $c \in \mathbb{R}$. In the case $\mathbb{F} = \mathbb{R}$, the QM function (2) takes the familiar form of a QM function over the real domain,

$$f(Z) = \mathrm{Tr}(Z^T A Z) + 2\mathrm{Tr}(B^T Z) + c.$$

If $B = 0_{n\times r}$ and $c = 0$, then f is called a *homogenous quadratic matrix function* or a *quadratic matrix form*. The *homogenized quadratic matrix function* of f is denoted by $f^H : \mathbb{F}^{(n+r)\times r} \to \mathbb{R}$ and given by

$$f^H(Z; T) \equiv \mathrm{Tr}(Z^* A Z) + 2\Re(\mathrm{Tr}(T^* B^* Z)) + \frac{c}{r}\mathrm{Tr}(T^* T), \quad Z \in \mathbb{F}^{n\times r},\ T \in \mathbb{F}^{r\times r}, \tag{3}$$

which is a homogenous QM function of order r corresponding to the matrix

$$M(f) \equiv \begin{pmatrix} A & B \\ B^* & \frac{c}{r} I_r \end{pmatrix}. \tag{4}$$

The operator M will be used throughout the paper.

A QM function of order one is merely a standard-form quadratic function: $f(z) = z^*Az + 2\Re(b^*z) + c$. Moreover, in this case f^H stands for the usual homogenized version of a quadratic function, i.e., $f^H : \mathbb{F}^{n+1} \to \mathbb{R}$, $f^H(z; w) = z^*Az + 2\Re(b^*z\bar{w}) + \|w\|^2$.

We note that every QM function of order r is also a "standard" quadratic function with nr variables. The latter observation follows directly from the relation

$$\mathrm{Tr}(Z^*AZ) + 2\Re(\mathrm{Tr}(B^*Z)) + c = \mathrm{vec}(Z)^*(I_r \otimes A)\mathrm{vec}(Z) + 2\Re(\mathrm{vec}(B)^*\mathrm{vec}(Z)) + c,$$

where $\mathrm{vec}(D)$ denotes the vector obtained by stacking the columns of D and $\otimes$ is the Kronecker product. The right-hand side presentation of the QM function is called the *vectorized QM function* [1]. We refer the reader to the discussion in [1] on various relations between the matricial and vectorized presentation of functions and optimization problems; however, in this paper, we will not focus on the "vectorized" counterparts of our results.

Quadratic matrix programming (QMP) problems are problems in which the goal is to minimize a QM objective function subject to equality and inequality QM constraints,

$$\begin{aligned} \text{(QMP)} \quad \min \quad & f_0(Z), \\ \text{s.t.} \quad & f_i(Z) \le \alpha_i, \quad i \in \mathcal{I}, \\ & f_j(Z) = \alpha_j, \quad j \in \mathcal{E}, \\ & Z \in \mathbb{F}^{n\times r}, \end{aligned} \tag{5}$$

where $f_i : \mathbb{F}^{n\times r} \to \mathbb{R}$, $i \in \mathcal{I} \cup \mathcal{E} \cup \{0\}$, are QM functions of order r given by

$$f_i(Z) = \mathrm{Tr}(Z^*A_iZ) + 2\Re(\mathrm{Tr}(B_i^*Z)) + c_i, \quad Z \in \mathbb{F}^{n\times r},$$

with $A_i \in \mathcal{H}^n(\mathbb{F})$, $B_i \in \mathbb{F}^{n\times r}$, $c_i \in \mathbb{R}$, $i \in \{0\} \cup \mathcal{I} \cup \mathcal{E}$. The index sets $\{0\}, \mathcal{I}, \mathcal{E}$ are pairwise disjoint sets of nonnegative integers.

2.2 Semidefinite Relaxation and Dual of the QMP Problem

The following lemma presents a homogenized version of the QMP problem which utilizes the homogenization procedure ($f \to f^H$) described in Sect. 2.1. This lemma is a straightforward extension of Lemma 3.1 from [1] and its proof is therefore omitted.

Lemma 2.1 *Consider the following homogenized version of the QMP problem* (5):

$$\begin{aligned} \min \quad & f_0^H(Z;T), \\ \text{s.t.} \quad & f_i^H(Z;T) \le \alpha_i, \quad i \in I, \\ & f_j^H(Z;T) = \alpha_j, \quad j \in \mathcal{E}, \\ & T^*T = I_r, \\ & Z \in \mathbb{F}^{n\times r}, \quad T \in \mathbb{F}^{r\times r}. \end{aligned} \tag{6}$$

(i) *Suppose that the QMP problem* (5) *is solvable and let* $\widehat{Z}$ *be an optimal solution of* (*QMP*). *Then problem* (6) *is solvable,* $(\widehat{Z}; I_r)$ *is an optimal solution of* (6) *and* $\mathrm{val}(QMP) = \mathrm{val}(6)$.
(ii) *Suppose that problem* (6) *is solvable and let* $(\widehat{Z}; \widehat{T})$ *be an optimal solution of* (6). *Then problem* (*QMP*) *is solvable,* $\widehat{Z}\widehat{T}^*$ *is an optimal solution of* (*QMP*) *and* $\mathrm{val}(QMP) = \mathrm{val}(6)$.

In order to be able to present problem (6) as a QMP problem, we will make use of the following technical lemma (recall that $[U]_r$ denotes the southeast $r \times r$ submatrix of U):

Lemma 2.2 *Let* n, r *be positive integers and let* $U \in \mathcal{H}^{n+r}(\mathbb{F})$. *Then,* $[U]_r = I_r$ *if and only if*

$$\begin{aligned} &\mathrm{Tr}(L_i U) = 2, \quad i = 1, \dots, r, \\ &\mathrm{Tr}(N_{ij}^k U) = 0, \quad 1 \le i < j \le r,\ k = 1, \dots, \theta(\mathbb{F}), \end{aligned}$$

where

$$\begin{aligned} N_{ij}^1 &= \begin{pmatrix} 0_{n\times n} & 0_{n\times r} \\ 0_{r\times n} & E_{ij}^r + E_{ji}^r \end{pmatrix}, \qquad N_{ij}^2 = \begin{pmatrix} 0_{n\times n} & 0_{n\times r} \\ 0_{r\times n} & iE_{ij}^r - iE_{ji}^r \end{pmatrix}, \\ L_i &= \begin{pmatrix} 0_{n\times n} & 0_{n\times r} \\ 0_{r\times n} & 2E_{ii}^r \end{pmatrix}. \end{aligned} \tag{7}$$

By denoting $W = (Z; T) \in \mathbb{F}^{(n+r)\times r}$ and using Lemma 2.2, together with the fact that $T^*T = I_r$ if and only if $TT^* = I_r$, we conclude that problem (6) can be written as

$$\begin{aligned} \min \quad & \mathrm{Tr}(M(f_0)WW^*), \\ \text{s.t.} \quad & \mathrm{Tr}(M(f_i)WW^*) \le \alpha_i, \quad i \in I, \\ & \mathrm{Tr}(M(f_j)WW^*) = \alpha_j, \quad j \in \mathcal{E}, \\ & \mathrm{Tr}(L_i WW^*) = 2, \quad i = 1, \dots, r, \\ & \mathrm{Tr}(N_{ij}^k WW^*) = 0, \quad 1 \le i < j \le r,\ k = 1, \dots, \theta(\mathbb{F}), \\ & W \in \mathbb{F}^{(n+r)\times r}, \end{aligned}$$

where the operator M is defined in (4). Making the change of variables $U = WW^*$, problem (6) becomes

$$\begin{aligned}
\min \quad & \mathrm{Tr}(M(f_0)U), \\
\text{s.t.} \quad & \mathrm{Tr}(M(f_i)U) \le \alpha_i, \quad i \in I, \\
& \mathrm{Tr}(M(f_j)U) = \alpha_j, \quad j \in \mathcal{E}, \\
& \mathrm{Tr}(L_i U) = 2, \quad i = 1, \ldots, r, \\
& \mathrm{Tr}(N_{ij}^k U) = 0, \quad 1 \le i < j \le r,\ k = 1, \ldots, \theta(\mathbb{F}), \\
& U \in \mathcal{H}_+^{n+r}(\mathbb{F}), \\
& \mathrm{rank}(U) \le r.
\end{aligned}$$

Omitting the "hard" constraint $\mathrm{rank}(U) \le r$, and invoking Lemma 2.2, we arrive at the following semidefinite relaxation of the QMP problem (5):

$$\begin{aligned}
(\mathrm{SDR}) \quad \min_U \quad & \mathrm{Tr}(M(f_0)U), \\
\text{s.t.} \quad & \mathrm{Tr}(M(f_i)U) \le \alpha_i, \quad i \in I, \\
& \mathrm{Tr}(M(f_j)U) = \alpha_j, \quad j \in \mathcal{E}, \\
& [U]_r = I_r, \\
& U \in \mathcal{H}_+^{n+r}(\mathbb{F}).
\end{aligned} \tag{8}$$

The dual problem to the semidefinite relaxation problem (SDR) is given by

$$\begin{aligned}
(\mathrm{D}) \quad \max_{\lambda_i, \Phi} \quad & -\sum_{i \in \mathcal{I} \cup \mathcal{E}} \lambda_i \alpha_i - \mathrm{Tr}(\Phi), \\
\text{s.t.} \quad & M(f_0) + \sum_{i \in \mathcal{I} \cup \mathcal{E}} \lambda_i M(f_i) + \begin{pmatrix} 0_{n \times n} & 0_{n \times r} \\ 0_{r \times n} & \Phi \end{pmatrix} \succeq 0, \\
& \Phi \in \mathcal{H}^r(\mathbb{F}), \\
& \lambda_i \ge 0, \quad i \in \mathcal{I}, \\
& \lambda_j \in \mathbb{R}, \quad j \in \mathcal{E}.
\end{aligned} \tag{9}$$

It is interesting to note that (D) is, in fact, the standard dual problem of (QMP). Therefore, (D) is the dual problem of both (QMP) and (SDR).

2.3 Tightness of the Semidefinite Relaxation of the QMP Problem

Consider the following general-form SDP problem:

$$\begin{aligned}
\min \quad & \mathrm{Tr}(C_0 U), \\
\text{s.t.} \quad & \mathrm{Tr}(C_i U) \le \alpha_i, \quad i \in \mathcal{I}_1, \\
& \mathrm{Tr}(C_j U) = \alpha_j, \quad j \in \mathcal{E}_1, \\
& U \in \mathcal{H}_+^n(\mathbb{F}),
\end{aligned} \tag{10}$$

where $\mathcal{I}_1$ and $\mathcal{E}_1$ are disjoint index sets, $C_i \in \mathcal{H}^n(\mathbb{F})$ for $i \in \{0\} \cup \mathcal{I}_1 \cup \mathcal{E}_1$ and $\alpha_i \in \mathbb{R}$ for $i \in \mathcal{I}_1 \cup \mathcal{E}_1$. In the real case ($\mathbb{F} = \mathbb{R}$), Pataki [18, 19] showed that, if $|\mathcal{I}_1| + |\mathcal{E}_1| \leq \varphi(r+1; \mathbb{R})$ and if the SDP problem (10) is solvable, then there exists a rank-r solution of the problem. In Appendix A, we present a simple extension of Pataki's procedure (termed "algorithm RED") that considers both the real and complex domains. The validity of this procedure implies the following theorem:

Theorem 2.1 *Suppose that problem* (10) *is solvable and that* $|\mathcal{I}_1| + |\mathcal{E}_1| \leq \varphi(r + 1; \mathbb{F}) - 1$, *where r is a positive integer. Then problem* (10) *has a solution U for which* $\operatorname{rank}(Z) \leq r$.

A different rank reduction algorithm for SDP problems can be found in Huang and Zhang [12].

One of the main reasons for considering QMP problems, which are a special class of general QCQPs, is that they enjoy more powerful results in the context of tight SDR and strong duality results. The following theorem, which is the main result of this section, shows that QMP problems with at most r constraints in the real domain, or $2r$ constraints in the complex domain, have—under some mild conditions—a tight SDR and the duality gap is zero. This result is based on the constructed SDR problem (8) and Theorem 2.1.

Theorem 2.2 (Tight Semidefinite Relaxation for the QMP Problem) *If problem* (*SDR*) *is solvable and* $|\mathcal{I}| + |\mathcal{E}| \leq \theta(\mathbb{F})r$, *then problem* (*QMP*) *is solvable and* $\operatorname{val}(SDR) = \operatorname{val}(QMP)$.

Proof It is sufficient to show that problem (SDR) has a solution with rank smaller or equal to r. The number of constraints in (SDR) is equal to $|\mathcal{I}| + |\mathcal{E}| + \varphi(r; \mathbb{F})$. Thus, using the inequality $|\mathcal{I}| + |\mathcal{E}| \leq \theta(\mathbb{F})r$, we conclude that the number of constraints in (SDR) is bounded above by

$$\theta(\mathbb{F})r + \varphi(r; \mathbb{F}) = \theta(\mathbb{F})\left(r + \frac{r(r-1)}{2}\right) + r = \theta(\mathbb{F})\left(\frac{(r+1)r}{2}\right) + r$$
$$= \varphi(r+1; \mathbb{F}) - 1.$$

Invoking Theorem 2.1, the result follows. □

In order to guarantee the solvability of the SDR problem, some kind of a regularity condition must be imposed. For example, the condition

$$\exists \gamma_i \in \mathbb{R},\ i \in \mathcal{I} \cup \mathcal{E} \quad \text{for which } \gamma_i \geq 0, i \in \mathcal{I} \quad \text{such that } A_0 + \sum_{i \in \mathcal{I} \cup \mathcal{E}} \gamma_i A_i \succ 0 \quad (11)$$

implies that the dual problem (9) is strictly feasible (for details see e.g. [1, Lemma 3.2]); this together with the feasibility of the QMP problem implies, by the conic duality theorem [14], the solvability of the SDR problem and that val(SDR) = val(D). This is summarized in the following corollary.

Corollary 2.1 *Consider the QMP problem* (5) *with* $|\mathcal{I}|+|\mathcal{E}| \leq \theta(\mathbb{F})r$, *its semidefinite relaxation* (*SDR*) (*problem* (8)) *and its dual* (*D*) (*problem* (9)). *Suppose that condition* (11) *holds true and that the QMP problem is feasible. Then*, *problems* (*QMP*) *and* (*SDR*) *are solvable and* $\mathrm{val}(QMP) = \mathrm{val}(SDR) = \mathrm{val}(D)$.

Remark 2.1 In the special case $r = 1$, Corollary 2.1 recovers the well-known strong duality/tightness of SDR results for QCQP problems with a single quadratic constraint over the real domain (see e.g. [4, 7, 8, 10]) and the corresponding result on QCQP problems with two quadratic constraints over the complex domain [1, 12].

3 Convexity of the Image of Quadratic Matrix Mappings

In this section, we establish several results regarding the convexity of the image of several quadratic maps defined by QM functions. In particular, we will show in Sect. 3.1 that the image of $\mathbb{F}^{n\times r}$ under a map comprised of $\theta(\mathbb{F})r$ QM functions of order r is always convex, and under some further assumptions, a mapping comprised of $\theta(\mathbb{F})r + 1$ QM functions of order r is convex. Moreover, we present a semidefinite presentation for each of the sets for which convexity is proved. In Sect. 3.2, the connection to general-form optimization problems involving QM functions is discussed.

3.1 Semidefinite Representation and Convexity of the Image of $\mathbb{F}^{n\times r}$ under a QM Mapping

We begin by showing that the image of $\mathbb{F}^{n\times r}$ under a QM mapping comprised of $\theta(\mathbb{F})r$ QM functions is always convex. This is a consequence of the strong duality result of Theorem 2.2. We present also a semidefinite representation of this convex set, i.e., we provide an explicit description of the set as the image of $\mathcal{H}^n_+(\mathbb{F}) \cap \mathcal{A}$ under a linear mapping, where $\mathcal{A}$ is an affine subspace.

Theorem 3.1 (Convexity of the Image of $\mathbb{F}^{n\times r}$ under $\theta(\mathbb{F})r$ QM Functions) *Let* $g_1, g_2, \ldots, g_m$ *be* m *QM functions of order* r *given by*

$$g_i(Z) = \mathrm{Tr}(Z^* A_i Z) + 2\Re(\mathrm{Tr}(B_i^* Z)) + c_i, \quad Z \in \mathbb{F}^{n\times r}, i = 1, 2, \ldots, m,$$

where $A_i \in \mathcal{H}^n(\mathbb{F})$, $B_i \in \mathbb{F}^{n\times r}$ *and* $c_i \in \mathbb{R}$. *Suppose that* $m \leq \theta(\mathbb{F})r$. *Then*, *the set*

$$F \equiv \{(g_1(Z); \ldots; g_m(Z)) : Z \in \mathbb{F}^{n\times r}\}$$

is convex and equal to the set

$$W \equiv \{(\mathrm{Tr}(M(g_1)U); \ldots; \mathrm{Tr}(M(g_m)U)) : U \in \mathcal{H}^{n+r}_+(\mathbb{F}), [U]_r = I_r\}.$$

Proof Since obviously the set W is convex, it is enough to show that $F = W$. The inclusion $F \subseteq W$ is clear. We will show that the reverse inclusion ($W \subseteq F$) holds true. Let $(\beta_1; \ldots; \beta_m) \in W$ and consider the QMP problem

$$\begin{aligned} \min \quad & 0, \\ \text{s.t.} \quad & f_j(Z) = \beta_j, \quad j = 1, \ldots, m, \\ & Z \in \mathbb{F}^{n \times r}, \end{aligned} \tag{12}$$

and its SDR

$$\begin{aligned} \min \quad & 0, \\ \text{s.t.} \quad & \operatorname{Tr}(M(f_j)U) = \beta_j, \quad j = 1, \ldots, m, \\ & [U]_r = I_r, \\ & U \in \mathcal{H}_+^{n+r}(\mathbb{F}). \end{aligned} \tag{13}$$

Note that since the objective functions of problems (12) and (13) are identically zero, then the notion of solvability of these problems coincide with the notion of *feasibility*. Since $(\beta_1; \ldots; \beta_m) \in W$, we conclude that problem (13) is feasible. Thus, by Theorem 2.2, problem (12) is also feasible (= solvable). Hence, there exists $Z \in \mathbb{F}^{n \times r}$ such that $\beta_j = f_j(Z)$, $j = 1, \ldots, m$, which implies that $(\beta_1; \ldots; \beta_m) \in F$. □

Our next objective is to prove the convexity of the image of $\mathbb{F}^{n \times r}$ under $\theta(\mathbb{F})r + 1$ QM functions. Such a convexity result was proven already by Polyak for the case $r = 1, \mathbb{F} = \mathbb{R}$ [4]:

Theorem 3.2 [4, Theorem 2.2] *Let $f_i(x) = x^T A_i x + 2b_i^T x + c_i$, $i = 1, 2$ with $A_i \in \mathcal{H}^n(\mathbb{R})$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$. Suppose that $n \geq 2$ and that there exist $\mu_1, \mu_2 \in \mathbb{R}$ such that $\mu_1 A_1 + \mu_2 A_2 \succ 0$. Then, the set*

$$\{(f_1(x), f_2(x)) : x \in \mathbb{R}^n\}$$

is closed and convex.

Note that the convexity result of Polyak's theorem is established under some conditions (Polyak also provides examples demonstrating the necessity of these conditions): the existence of a positive definite linear combination of the corresponding matrices and the restriction $n \geq 2$. We will show that under the exact same conditions, the image of $\mathbb{F}^{n \times r}$ under $\theta(\mathbb{F})r + 1$ QM functions is closed and convex; we will also provide a semidefinite representation of these sets.

We begin by stating Theorem 3.3 below that establishes a result on the convexity of the image of several *homogenous* QM functions. In the real domain, Barvinok proved this result under very similar conditions [20, Theorem 1.2]. However, it seems that the result for both the real and complex domains is not stated explicitly in the literature. We therefore provide a complete proof of this theorem in Appendix B.

Theorem 3.3 (Convexity of the Image of Homogenous QM Mappings) *Let $A_1, \ldots, A_k \in \mathcal{H}^n(\mathbb{F})$, where $k \leq \varphi(r + 1; \mathbb{F})$. Suppose that there exist $\mu_i \in \mathbb{R}, i = 1, \ldots, k$*

such that $\sum_{i=1}^k \mu_i A_i \succ 0$ *and that* $n \geq r+2$. *Then, the set*

$$F = \{(\mathrm{Tr}(Z^* A_1 Z); \ldots; \mathrm{Tr}(Z^* A_k Z)) : Z \in \mathbb{F}^{n\times r}\}$$

is closed and convex and equal to

$$W = \{(\mathrm{Tr}(A_1 U); \ldots; \mathrm{Tr}(A_k U)) : U \in \mathcal{H}_+^n(\mathbb{F})\}.$$

Proof The proof basically follows the line of analysis of [4], see the details in Appendix B. □

We are now ready to prove the main result of this section.

Theorem 3.4 (Convexity of the Image of $\theta(\mathbb{F})r+1$ QM Mappings) *Let* $g_1, g_2, \ldots, g_m$ *be* m *QM functions of order* r *given by*

$$g_i(Z) = \mathrm{Tr}(Z^* A_i Z) + 2\Re(\mathrm{Tr}(B_i^* Z)) + c_i, \quad Z \in \mathbb{F}^{n\times r},\ i = 1, 2, \ldots, m,$$

where $A_i \in \mathcal{H}^n(\mathbb{F})$, $B_i \in \mathbb{F}^{n\times r}$ *and* $c_i \in \mathbb{R}$. *Suppose that* $m \leq \theta(\mathbb{F})r + 1, n \geq 2$ *and that there exist* $\mu_i \in \mathbb{R}, i = 1, \ldots, m$, *such that*

$$\sum_{i=1}^m \mu_i A_i \succ 0. \tag{14}$$

Then, the set

$$F \equiv \{(g_1(Z); \ldots; g_m(Z)) : Z \in \mathbb{F}^{n\times r}\}$$

is closed and convex and $F = W$, *where* W *is given by*

$$W = \{(\mathrm{Tr}(M(g_1)U); \ldots; \mathrm{Tr}(M(g_m)U)) : U \in \mathcal{H}_+^{n+r}(\mathbb{F}), [U]_r = I_r\}.$$

Proof Let $h_1, h_2, \ldots, h_{m+\varphi(r;\mathbb{F})} : \mathbb{F}^{(n+r)\times r} \to \mathbb{R}$ be the $m + \varphi(r;\mathbb{F})$ homogenous QM functions comprised of

(i) the m QM functions $h_i(W) = g_i^H(W) = \mathrm{Tr}(W^* M(g_i) W), i = 1, \ldots, m$,
(ii) the r QM functions $\phi_i(W) = \mathrm{Tr}(W^* L_i W), i = 1, \ldots, r$, where L_i is given in (7),
(iii) the $\theta(\mathbb{F})\frac{r(r-1)}{2}$ QM functions $\psi_{ij}^k(W) = \mathrm{Tr}(W^* N_{ij}^k W), 1 \leq i < j \leq r, k = 1, \ldots, \theta(\mathbb{F})$, where N_{ij}^k is given in (7).

Consider the set

$$R = \{(h_1(Z; T); \ldots; h_{m+\varphi(r;\mathbb{F})}(Z; T)) : Z \in \mathbb{F}^{n\times r}, T \in \mathbb{F}^{r\times r}\} \subseteq \mathbb{R}^{m+\varphi(r;\mathbb{F})},$$

which is the image of $\mathbb{F}^{n+r}$ under $m + \varphi(r;\mathbb{F})$ homogenous QM functions, and let

$$\mathcal{A} = \left\{w \in \mathbb{R}^{m+\varphi(r;\mathbb{F})} : w_{m+1} = \cdots = w_{m+r} = 2,\ w_{m+r+1} = \cdots = w_{m+\varphi(r;\mathbb{F})} = 0\right\}.$$

Let $\mathbb{P}:\mathbb{R}^{m+\varphi(r;\mathbb{F})}\to\mathbb{R}^m$ be the linear transformation that maps each vector in $\mathbb{R}^{m+\varphi(r;\mathbb{F})}$ onto its first m components. Lemma 2.2 implies that the set $\mathbb{P}(\mathcal{A}\cap R)$ can be written as

$$\mathbb{P}(\mathcal{A}\cap R)=\{(g_1^H(Z;T);\dots;g_m^H(Z;T)):T^*T=I_r,T\in\mathbb{F}^{r\times r},Z\in\mathbb{F}^{n\times r}\}.$$

We begin by showing that

$$\mathbb{P}(\mathcal{A}\cap R)=F. \tag{15}$$

($F\subseteq\mathbb{P}(\mathcal{A}\cap R)$)—Let $w\in F$. Then there exists $Z\in\mathbb{F}^{n\times r}$ such that $w_i=g_i(Z)$. Therefore, $w_i=g_i^H(Z;T)$ with $T=I_r$ which implies that $w\in\mathbb{P}(\mathcal{A}\cap R)$.
($\mathbb{P}(\mathcal{A}\cap R)\subseteq F$)—Let $w\in\mathbb{P}(\mathcal{A}\cap R)$. Then there exist $Z\in\mathbb{F}^{n\times r},T\in\mathbb{F}^{r\times r}$ such that $T^*T=I_r$ and $w_i=g_i^H(Z;T)$. Using the relation $g_i^H(Z;T)=g_i^H(ZT^*;I_r)=g_i(ZT^*)$, we conclude that $w\in F$.

We will now show that the conditions of Theorem 3.3 are satisfied for the homogenous QM functions $h_1,\dots,h_{m+\varphi(r;\mathbb{F})}$. Let $\Phi=(\phi_{ij})_{i,j=1}^r\in\mathcal{H}^r(\mathbb{F})$ be any matrix satisfying

$$\Phi\succ\left(\sum_{i=1}^m\mu_iB_i\right)^*\left(\sum_{i=1}^m\mu_iA_i\right)^{-1}\left(\sum_{i=1}^m\mu_iB_i\right)-\frac{1}{r}\sum_{i=1}^m c_i, \tag{16}$$

and consider the linear combination

$$\sum_{i=1}^m\mu_iM(g_i)+\frac{1}{2}\sum_{i=1}^r\phi_{ii}L_i+\sum_{i,j\in\{1,\dots,r\},i<j}\Re(\phi_{ij})N_{ij}^1+\sum_{i,j\in\{1,\dots,r\},i<j}\Im(\phi_{ij})N_{ij}^2$$
$$=\begin{pmatrix}\sum_{i=1}^m\mu_iA_i & \sum_{i=1}^m\mu_iB_i\\ (\sum_{i=1}^m\mu_iB_i)^* & \frac{1}{r}\sum_{i=1}^m\mu_ic_i+\Phi\end{pmatrix}. \tag{17}$$

Using (14) and (16) combined with the Schur complement, we conclude that matrix (17) is positive definite. Therefore, we have proven that there exists a positive definite linear combination of the matrices associated with $h_1,\dots,h_{m+\varphi(r;\mathbb{F})}$. Moreover,

$$m+\varphi(r;\mathbb{F})\le\theta(\mathbb{F})r+1+\theta(\mathbb{F})\frac{r(r-1)}{2}+r=\theta(\mathbb{F})\frac{r(r+1)}{2}+r+1=\varphi(r+1;\mathbb{F}).$$

Therefore, the conditions of Theorem 3.3 are satisfied and we can deduce that R is closed, convex and equal to

$$S=\{(\mathrm{Tr}(M(h_1)U);\dots;\mathrm{Tr}(M(h_{m+\varphi(r;\mathbb{F})})U)):U\in\mathcal{H}_+^{n+r}(\mathbb{F})\}.$$

The set $\mathcal{A}\cap R$—being an intersection of two closed and convex sets—is closed and convex, which implies that $\mathbb{P}(\mathcal{A}\cap R)=\mathbb{P}(\mathcal{A}\cap S)$ is also closed and convex. Moreover,

$$\mathbb{P}(\mathcal{A}\cap S)=\{(\mathrm{Tr}(M(g_1)U),\dots,\mathrm{Tr}(M(g_m)U)):[U]_r=I_r,U\in\mathcal{H}_+^n(\mathbb{F})\}$$

and as such is equal to W. To conclude, we have shown that $F = \mathbb{P}(\mathcal{A} \cap R) = W$ is closed and convex. $\square$

Substituting $r = 1$ in Theorem 3.4, we arrive at the following corollary whose first part is a recovery of Polyak's theorem (Theorem 3.2).

Corollary 3.1

(i) *Let* $f_i(x) = x^T A_i x + 2b_i^T x + c_i, i = 1, 2,$ *with* $A_i \in \mathcal{H}^n(\mathbb{R}), b_i \in \mathbb{R}^n$ *and* $c_i \in \mathbb{R}$. *Suppose that* $n \geq 2$ *and that there exist* $\mu_1, \mu_2 \in \mathbb{R}$ *such that* $\mu_1 A_1 + \mu_2 A_2 \succ 0$. *Then, the set*

$$\{(f_1(x), f_2(x)) : x \in \mathbb{R}^n\}$$

is closed and convex and equal to

$$\{(\mathrm{Tr}(M(f_1)U), \mathrm{Tr}(M(f_2)U)) : U \in \mathcal{H}_+^{n+1}(\mathbb{R}), U_{n+1,n+1} = 1\}.$$

(ii) *Let* $f_i(z) = z^* A_i z + 2\Re(b_i^* z) + c_i, i = 1, 2, 3,$ *with* $A_i \in \mathcal{H}^n(\mathbb{C}), b_i \in \mathbb{C}^n$ *and* $c_i \in \mathbb{R}$. *Suppose that* $n \geq 2$ *and that there exist* $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ *such that* $\mu_1 A_1 + \mu_2 A_2 + \mu_3 A_3 \succ 0$. *Then, the set*

$$\{(f_1(z), f_2(z), f_3(z)) : z \in \mathbb{C}^n\}$$

is closed and convex and equal to

$$\{(\mathrm{Tr}(M(f_1)U), \mathrm{Tr}(M(f_2)U), \mathrm{Tr}(M_3 U)) : U \in \mathcal{H}_+^{n+1}(\mathbb{C}), U_{n+1,n+1} = 1\}.$$

3.2 Convex Counterparts of Problems Involving QM Functions

Theorems 3.1 and 3.4 can be used to generate large classes of nonconvex problems, possibly different from QMP problems, for which there exists an equivalent convex problem. For example, consider the following nonconvex problem:

$$\min_Z \{\psi_0(f_1(Z), f_2(Z), \ldots, f_m(Z)) : \psi_i(f_1(Z), f_2(Z), \ldots, f_m(Z)) \leq 0,$$
$$i = 1, 2, \ldots, p\},$$

where f_i are QM functions of order r and $\psi_i : \mathbb{R}^r \to \mathbb{R}$ are p convex functions over $\mathbb{R}^r$. The last problem can be cast as the problem

$$\min_{t_i} \{\psi_0(t_1, t_2, \ldots, t_m) : \psi_i(t_1, t_2, \ldots, t_m) \leq 0, i = 1, 2, \ldots, p, (t_1; t_2; \ldots; t_r) \in C\}, \tag{18}$$

where C is the set $\{(f_1(Z); \ldots; f_m(Z)) : Z \in \mathbb{F}^{n \times r}\}$. By Theorems 3.1 and 3.4, C is convex, provided that either $m \leq \theta(\mathbb{F})r$ or $m \leq \theta(\mathbb{F})r + 1$ and the conditions of Theorem 3.4 are satisfied. In this case, (18) is a convex optimization problem.

We can also recover the tightness-of-SDR result of Theorem 2.2 by using Theorem 3.4. Indeed, consider the QMP problem (5). Then, the QMP problem can be written as

$$\min_{t_i} \{t_0 : t_i \leq \alpha_i, t_j = \alpha_j, i \in \mathcal{I}, j \in \mathcal{E}, (t_0; t_1; t_2; \ldots; t_m) \in C\}, \tag{19}$$

where $C = \{(f_0(Z); \ldots; f_m(Z)) : Z \in \mathbb{F}^{n\times r}\}$. Assume that (i) $m \le \theta(\mathbb{F})r$, (ii) condition (11) is satisfied, (iii) the SDR problem (8) is solvable and (iv) $n \ge 2$. Then, by Theorem 3.4, the set C is also equal to

$$\{(\mathrm{Tr}(M(f_0)U); \ldots; \mathrm{Tr}(M(f_m)U)) : U \in \mathcal{H}_+^{n+r}(\mathbb{F}), [U]_r = I_r\}.$$

Substituting this presentation of C back into (19), we obtain the SDR problem (8). Note that, in this line of proof the additional restriction $n \ge 2$ was added showing that there is a cost for using the result on the image space in order to obtain the tightness-of-SDR result. Moreover, this proof is highly nonconstructive in the sense that it does not provide a procedure for obtaining an optimal solution of the QMP problem from its SDR counterpart. This is in contrast to the proof of Theorem 2.2, which uses the constructive RED procedure.

4 S-lemma on QM Functions

The celebrated S-lemma has many applications in several areas and is a key tool in control and optimization; see the comprehensive survey [13]. The nonhomogenous version of the S-lemma that takes into account both the real and complex domains was derived by Fradkov and Yakubovich [3]. We state their result explicitly.

Lemma 4.1 (Real and Complex S-lemma [3]) *Let* $f_i(z) = z^* A_i z + 2\Re(b_i^* z) + c_i, i = 0, \ldots, \theta(\mathbb{F})$, *with* $A_i \in \mathcal{H}^n(\mathbb{F}), b_i \in \mathbb{F}^n$ *and* $c_i \in \mathbb{R}, i = 0, \ldots, \theta(\mathbb{F})$. *Suppose that there exists* $\tilde{z} \in \mathbb{F}^n$ *such that* $f_i(\tilde{z}) < 0, i = 1, \ldots, \theta(\mathbb{F})$. *Then, the following two statements are equivalent*:

(i) $f_0(z) \le 0$ *for every* $z \in \mathbb{F}^n$ *satisfying* $f_i(z) \le 0, i = 1, \ldots, \theta(\mathbb{F})$.
(ii) *There exists* $\lambda_i \ge 0, i = 1, \ldots, \theta(\mathbb{F})$ *such that*

$$\begin{pmatrix} A_0 & b_0 \\ b_0^* & c_0 \end{pmatrix} \preceq \sum_{i=1}^{\theta(\mathbb{F})} \lambda_i \begin{pmatrix} A_i & b_i \\ b_i^* & c_i \end{pmatrix}.$$

In this section, we derive an S-lemma-type result on QM functions that gives an LMI characterization for a statement on the implication of a QM constraint of order r from $\theta(\mathbb{F})r$ QM constraints of order r. This result can be regarded as an extension of the classical S-lemma result (Theorem 4.1). The second part of this section is devoted to the presentation of two applications: the first is concerned with the solution of linear systems immune to implementation errors, and the second describes a tractable robust counterpart of a class of quadratic problems with unstructured uncertainty.

4.1 Extended S-Lemma on QM Functions

In this section, we prove an extended version of the S-lemma applied to QM functions. We use the following lemma that gives an LMI characterization of the claim that a certain QM function is nonnegative. This result was derived in [1] over the real domain; the fact that it is also valid over the complex domain is straightforward.

Lemma 4.2 [1, Lemma 4.2] *Let f be a QM function given in* (2). *Then, the following three statements are equivalent*:

(i) $f(Z) \geq 0$ *for every* $Z \in \mathbb{F}^{n\times r}$.
(ii) *There exists* $\Phi \in \mathcal{H}^r(\mathbb{F})$ *for which* $\mathrm{Tr}(\Phi) \leq 0$ *such that*

$$\begin{pmatrix} A & B \\ B^* & \frac{c}{r} I_r + \Phi \end{pmatrix} \succeq 0.$$

(iii)

$$\begin{pmatrix} I_r \otimes A & \mathrm{vec}(B) \\ \mathrm{vec}(B)^* & c \end{pmatrix} \succeq 0.$$

We are now ready to state and prove the extended S-lemma.

Lemma 4.3 (S-Lemma on QM Functions) *Let $f_i(Z) = \mathrm{Tr}(Z^* A_i Z) + 2\Re(\mathrm{Tr}(B_i^* Z)) + c_i, i = 0, \dots, m$, be $m+1$ QM functions of order r with $A_i \in \mathcal{H}^n(\mathbb{F}), b_i \in \mathbb{F}^n$ and $c_i \in \mathbb{R}$. Assume that $m \leq \theta(\mathbb{F})r$ and that there exist $\mu_i \geq 0, i = 1, \dots, \theta(\mathbb{F})r$ such that*

$$-A_0 + \sum_{i=1}^{m} \mu_i A_i \succ 0. \tag{20}$$

Furthermore, suppose that there exists $\widehat{Z} \in \mathbb{F}^{n\times r}$ for which

$$f_i(\widehat{Z}) < 0, \quad i = 1, \dots, \theta(\mathbb{F})r. \tag{21}$$

Then, the following four statements are equivalent:

(i) $f_0(Z) \leq 0$ *for every* $Z \in \mathbb{F}^{n\times r}$ *such that* $f_i(Z) \leq 0, i = 1, \dots, m$.
(ii) *There exist* $\lambda_i \geq 0, i = 1, \dots, m$ *such that*

$$f_0(Z) - \sum_{i=1}^{m} \lambda_i f_i(Z) \leq 0, \quad \textit{for every } Z \in \mathbb{F}^{n\times r}.$$

(iii) *There exist* $\lambda_i \geq 0, i = 1, \dots, m$ *such that*

$$\begin{pmatrix} I_r \otimes (A_0 - \sum_{i=1}^m \lambda_i A_i) & \mathrm{vec}(B_0 - \sum_{i=1}^m \lambda_i B_i) \\ \mathrm{vec}(B_0 - \sum_{i=1}^m \lambda_i B_i)^* & c_0 - \sum_{i=1}^m \lambda_i c_i \end{pmatrix} \preceq 0.$$

(iv) *There exist* $\lambda_i \geq 0, i = 1, \dots, m$ *and* $\Phi \in \mathcal{H}^r(\mathbb{F})$ *with* $\mathrm{Tr}(\Phi) \leq 0$ *such that*

$$\begin{pmatrix} A_0 - \sum_{i=1}^m \lambda_i A_i & B_0 - \sum_{i=1}^m \lambda_i B_i \\ (B_0 - \sum_{i=1}^m \lambda_i B_i)^* & \frac{1}{r}(c_0 - \sum_{i=1}^m \lambda_i c_i) I_r - \Phi \end{pmatrix} \preceq 0.$$

Proof Consider the QMP problem.

$$\begin{aligned} \max \quad & f_0(Z), \\ \text{s.t.} \quad & f_i(Z) \le 0, \quad i = 1, \dots m, \\ & Z \in \mathbb{F}^{n\times r}. \end{aligned} \tag{22}$$

The SDR of (22) is the problem

$$\begin{aligned} \max \quad & \operatorname{Tr}(M(f_0)U), \\ \text{s.t.} \quad & \operatorname{Tr}(M(f_i)U) \le 0, \quad i = 1, \dots, m, \\ & [U]_r = I_r, \\ & U \in \mathcal{H}_+^{n+r}(\mathbb{F}), \end{aligned} \tag{23}$$

and its dual is given by the SDP

$$\begin{aligned} \min \quad & \operatorname{Tr}(\Phi), \\ \text{s.t.} \quad & -M(f_0) + \sum_{i=1}^m \lambda_i M(f_i) + \begin{pmatrix} 0_{n\times n} & 0_{n\times r} \\ 0_{r\times n} & \Phi \end{pmatrix} \succeq 0, \\ & \Phi \in \mathcal{H}^r(\mathbb{F}), \\ & \lambda_i \ge 0, \quad i = 1, \dots, m. \end{aligned} \tag{24}$$

We begin by showing that both the SDR (23) and its dual (24) are strictly feasible. The strict feasibility of the dual problem (24) follows immediately from condition (20). To show that problem (23) is strictly feasible, consider $\widehat{U} = (\widehat{Z}; I_r)(\widehat{Z}^*, I_r)$. Then by (21), we have that $\widehat{U} \in \mathcal{H}_+^n(\mathbb{F})$ satisfies

$$[\widehat{U}]_r = I_r, \qquad \operatorname{Tr}(M(f_i)\widehat{U}) < 0, \quad i = 1, \dots, m.$$

Let $\Delta \in \mathcal{H}_{++}^n(\mathbb{F})$ be any $n \times n$ positive-definite matrix. Define

$$\widetilde{\Delta} = \begin{pmatrix} \Delta & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{pmatrix}.$$

Then $\widetilde{\Delta} \in \mathcal{H}_+^{n+r}(\mathbb{F})$. Consider the matrix $W = \hat{U} + \alpha\widetilde{\Delta}$, where α is a positive number. Obviously $[W]_r = I_r$. Moreover, for small enough α, we have $\operatorname{Tr}(M(f_i)W) < 0, i = 1, \dots, m$. From its definition, W is positive semidefinite. To prove that W is positive *definite*, all that remains to show is that for $a \in \mathbb{F}^{n+r}$, $a^*Wa = 0$, if and only if $a = 0$. Suppose indeed that $a^*Wa = 0$ for $a \in \mathbb{F}^{n+r}$. Then,

$$0 = a^*Wa = a^*\widehat{U}a + \alpha a^*\widetilde{\Delta}a = 0,$$

and thus $a^*\widehat{U}a = 0$ and $a^*\widetilde{\Delta}a = 0$. Denote $a = (a_1; a_2)$ where $a_1 \in \mathbb{F}^n$ and $a_2 \in \mathbb{F}^r$. By the definition of $\widetilde{\Delta}$, the equality $a^*\widetilde{\Delta}a = 0$ is equivalent to $a_1^*\Delta a_1 = 0$, which, by the positive definiteness of Δ, implies that $a_1 = 0$. Using this and $a^*\widehat{U}a = 0$, we conclude that $a_2^*[\widehat{U}]_r a_2 = 0$. Since $[\widehat{U}]_r = I_r$ we deduce that $a_2 = 0$. In conclusion, $a = 0$ and we have proven that W satisfies

$$W \in \mathcal{H}_{++}^{n+r}(\mathbb{F}), \quad \mathrm{Tr}(M(f_i)W) < 0, \qquad [W]_r = I_r,$$

which implies that W is a strictly feasible solution of (23).

Since both the SDR (23) and its dual (24) are strictly feasible, we can invoke the conic duality theorem [14] and conclude that both problems are solvable and val(23) = val(24). The conditions of Theorem 2.2 are satisfied since (23) is solvable and $m \le \theta(\mathbb{F})r$. Therefore, the QMP problem (22) is solvable and val(22) = val(23) = val(24).

Statement (i) is equivalent to the claim that the value of the optimization problem (22) is nonpositive. By the preceding discussion, this is equivalent to the statement that the value of the dual problem (24) is nonpositive, which, by solvability of the dual problem, is the same as:

$$\begin{aligned} &\text{There exists } \lambda_i \ge 0 \text{ and } \Phi \in \mathcal{H}^r(\mathbb{F}),\ \text{with } \mathrm{Tr}(\Phi) \le 0,\\ &\text{for which} - M(f_0) + \sum_{i=1}^m \lambda_i M(f_i) + \begin{pmatrix} 0_{n\times n} & 0_{n\times r} \\ 0_{r\times n} & \Phi \end{pmatrix} \succeq 0. \end{aligned}$$

Noting that the latter is in fact statement (iv), we conclude that statement (i) and (iv) are equivalent. Finally, invoking Lemma 4.2, we conclude that statements (ii), (iii) and (iv) are equivalent. □

Remark 4.1 In the case $r = 1$, we recover the S-lemma (Lemma 4.1) with the exception that the condition (20) is assumed to hold.

Remark 4.2 The technique used to prove Lemma 4.3 is based on convex duality. An alternative approach is to invoke the convexity result on the image of QM functions (Theorem 3.4) together with a separation argument. This methodology was used, for example, in the derivation of the classical S-lemma [3] and in Polyak [4]. However, this line of analysis will necessarily yield a weaker result, since the restriction $n \ge 2$ of Theorem 3.4 will have to be imposed in the S-lemma result. The restriction $n \ge 2$, although necessary in the convexity result on the image of QM mappings (Theorem 3.4), is not necessary in the S-lemma result (4.3).

4.2 Applications I: Solutions of Linear Systems Immune to Implementation Errors

Many problems in data fitting and estimation give rise to a linear system of the form

$$AZ \approx B, \tag{25}$$

where $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{m\times r}$ and $Z \in \mathbb{F}^{n\times r}$ is an unknown variable matrix. Frequently, the case $r = 1$ is considered. The situation in which $r > 1$ is considered, for example, in the multiple observations setting in which we are given r linear systems $Az_k \approx b_k, k = 1, \dots, r$ with $b_k \in \mathbb{F}^m$. By denoting $B = (b_1, \dots, b_k)$ and $Z = (z_1, \dots, z_r)$, we arrive at the model (25). The latter model was analyzed, for instance, in the context of total least squares solutions where it is called "multidimensional total least squares" [21]. Also, there are different estimation problems, in which the Z is indeed a matrix rather than a vector, see e.g. [22] for an example with $r = n$.

Due to expected implementation errors, we would like to construct a solution to (25) for which the worst case data error $\|A(Z + \Delta) - B\|^2$ over all possible $\Delta \in \mathcal{U}$ is minimized,

$$\min_Z \max_{\Delta \in \mathcal{U}} \|A(Z + \Delta) - B\|^2. \tag{26}$$

The solution Z is a robust solution to the system (25) in the sense that it is immune to implementation errors. The tractability of the minmax problem (26) depends, of course, on the choice of the uncertainty set $\mathcal{U}$. Here, we consider $\mathcal{U}$ to be an intersection of several unstructured ellipsoids,

$$\mathcal{U} = \{\Delta : \|C_i \Delta\|^2 \le \rho_i, i = 1, \dots, m\}.$$

We assume that there exist μ_i, $i = 1, \dots, m$, such that $\sum_{i=1}^m \mu_i C_i^* C_i \succ 0$ and that $m \le \theta(\mathbb{F}) r$. To give an example of such a structure, note that in the case $r = n$, this structure of $\mathcal{U}$ can model the situation in which each row of the perturbation matrix Δ has a Euclidean norm bound (by substituting $C_i = E_{ii}^n$).

The inner maximization problem

$$\max_{\Delta \in \mathcal{U}} \|A(Z + \Delta) - B\|^2$$

can be written as

$$\min\{t : \|A(Z + \Delta) - B\|^2 \le t \ \forall \Delta \in \mathcal{U}\}.$$

Invoking Lemma 4.3, we conclude that the statement

$$\|A(Z + \Delta) - B\|^2 \le t, \quad \forall \Delta \in \mathcal{U},$$

holds true if and only if there exist $\lambda_i \ge 0$ such that

$$\begin{pmatrix} I_r \otimes (A^* A - \sum_{i=1}^m \lambda_i C_i^* C_i) & \operatorname{vec}(A^*(AZ - B)) \\ \operatorname{vec}(A^*(AZ - B))^* & \|AZ - B\|^2 - t + \sum_{i=1}^m \lambda_i \rho_i^2 \end{pmatrix} \preceq 0,$$

which is equivalent to saying that there exist $\lambda_i \ge 0$ such that

$$\begin{pmatrix} (I_r \otimes A^*)(I_r \otimes A) - \sum_{i=1}^m \lambda_i (I_r \otimes C_i^*)(I_r \otimes C_i) & (I_r \otimes A^*)\operatorname{vec}(AZ - B) \\ \operatorname{vec}(AZ - B)^*(I_r \otimes A) & \|AZ - B\|^2 - t + \sum_{i=1}^m \lambda_i \rho_i^2 \end{pmatrix}$$
$$\preceq 0.$$

Using the Schur complement, the latter LMI is transformed to

$$\begin{pmatrix} I_{mr} & I_r \otimes A & \operatorname{vec}(AZ - B) \\ I_r \otimes A^* & \sum_{i=1}^m \lambda_i (I_r \otimes C_i^*)(I_r \otimes C_i) & 0 \\ \operatorname{vec}(AZ - B)^* & 0 & t - \sum_{i=1}^m \lambda_i \rho_i^2 \end{pmatrix} \succeq 0. \tag{27}$$

Therefore, problem (26) can be recast as the following SDP in the variables t, λ_i, Z:

$$\begin{aligned}
\min \quad & t, \\
\text{s.t.} \quad & \begin{pmatrix} I_{mr} & I_r \otimes A & \mathrm{vec}(AZ - B) \\ I_r \otimes A^* & \sum_{i=1}^m \lambda_i (I_r \otimes C_i^*)(I_r \otimes C_i) & 0 \\ \mathrm{vec}(AZ-B)^* & 0 & t - \sum_{i=1}^m \lambda_i \rho_i^2 \end{pmatrix} \succeq 0, \\
& t \in \mathbb{R}, \\
& \lambda_i \in \mathbb{R}_+, \quad i = 1, \dots, m, \\
& Z \in \mathbb{F}^{n \times r}.
\end{aligned}$$

4.3 Applications II: Robust Quadratic Problems with Unstructured Uncertainty

Consider a second-order cone problem of the form:

$$\begin{aligned}
\text{(Q)} \quad \min_z \quad & \Re(a^* z), \\
\text{s.t.} \quad & \|A_i z + b_i\| \le c_i, \quad i = 1, \dots, k, \\
& z \in \mathbb{F}^n,
\end{aligned}$$

where $A_i \in \mathbb{F}^{r \times n}, b_i \in \mathbb{F}^r, a \in \mathbb{F}^n$ and $c_i \in \mathbb{R}, i = 1, \dots, k$. The constraints in (Q) are more specific than those considered in the general form of second-order cone problems (see e.g., [14]) in which a linear term is introduced in the right-hand side of each constraint. However, in many applications the natural constraints are norm-type constraints so that the model (Q) captures a substantial amount of "real-life" situations.

Assume now that for each $i = 1, \dots, k$, the data (A_i, b_i) is uncertain and is only known to reside in some uncertainty set $\mathcal{U}_i$. The robust counterpart of the problem (Q) is the optimization problem

$$\begin{aligned}
\text{(RQ)} \quad \min_z \quad & \Re(a^* z), \\
\text{s.t.} \quad & \|A_i z + b_i\| \le c_i \quad \forall (A_i, b_i) \in \mathcal{U}, \ i = 1, \dots, k, \\
& z \in \mathbb{F}^n.
\end{aligned}$$

The tractability of the robust counterpart strongly relies on the choice of the uncertainty set $\mathcal{U}$. For example, in the structured case, it is well known that if $\mathcal{U}$ is an ellipsoid, then (RQ) can be recast as an SDP; however, in the case when $\mathcal{U}$ is given by an intersection of ellipsoids, then (RQ) is generally not tractable [23]. We will now show that when $\mathcal{U}$ is given by an intersection of at most $\theta(\mathbb{F})r$ unstructured ellipsoids, the problem can be recast as an SDP. Define

$$\mathcal{U}_i = \{(A, b) = (A_i^0, b_i^0) + \Delta^* : \Delta \in \mathbb{F}^{(n+1)\times r}, \|C_j \Delta\|^2 \le \rho_j, j = 1, \dots, m\},$$

where we assume—as in the previous application—that $m \le \theta(\mathbb{F})r$ and that there exist $\mu_i \in \mathbb{R}, i = 1, \dots, m$ such that $\sum_{j=1}^m \mu_j C_j^* C_j \succ 0$.

Note that a vector $z \in \mathbb{F}^n$ satisfies the ith constraint if and only if

$$\|A_i z + b_i\| \le c_i, \quad \text{for every } (A_i, b_i) \in \mathcal{U}_i,$$

which is the same as

$$\begin{aligned} &\|(A_i^0, b_i^0)\tilde{z} + \Delta^* \tilde{z}\|^2 \le c_i^2, \\ &\text{for every } \Delta \in \mathbb{F}^{n\times r} \text{ satisfying } \|C_j \Delta\|^2 \le \rho_j, \quad j = 1, \dots, m. \end{aligned}$$

Here, $\tilde{z} \equiv (z; 1)$. The latter implication can be written in terms of QM functions as follows:

$$\begin{aligned} &\mathrm{Tr}(\Delta^* E \Delta) + 2\Re(\mathrm{Tr}(F^* \Delta)) + g \le 0, \\ &\text{for every } \Delta \in \mathbb{F}^{n\times r} \text{ satisfying } \mathrm{Tr}(\Delta^* C_j^* C_j \Delta) - \rho_j^2 \le 0, \quad j = 1, \dots, m, \end{aligned} \tag{28}$$

where

$$E = \tilde{z}\tilde{z}^*, \qquad F = \tilde{z} w_i^*, \qquad g = \mathrm{Tr}(w_i w_i^*) - c_i^2$$

with $w_i \equiv (A_i^0, b_i^0)\tilde{z}$. By Lemma 4.3, we have that (28) is equivalent to the following statement:

$$\begin{aligned} &\text{There exist } \lambda_j^i \ge 0, \ j = 1, \dots, m \text{ such that} \\ &\begin{pmatrix} I_r \otimes (E - \sum_{i=1}^m \lambda_j^i C_j^* C_j) & \mathrm{vec}(F) \\ \mathrm{vec}(F)^* & g + \sum_{i=1}^m \lambda_j^i \rho_j^2 \end{pmatrix} \preceq 0. \end{aligned} \tag{29}$$

Using the identities

$$I_r \otimes E = (I_r \otimes \tilde{z})(I_r \otimes \tilde{z})^*, \qquad \mathrm{vec}(F) = (I_r \otimes \tilde{z}) w_i, \qquad g = w_i^* w_i - c_i^2,$$

we deduce that the LMI (29) is the same as

$$\begin{pmatrix} (I_r \otimes \tilde{z})(I_r \otimes \tilde{z})^* - \sum_{j=1}^m \lambda_j^i (I_r \otimes C_j)^* (I_r \otimes C_j) & (I_r \otimes \tilde{z}) w_i \\ w_i^* (I_r \otimes \tilde{z})^* & w_i^* w_i - c_i^2 + \sum_{j=1}^m \lambda_j^i \rho_j^2 \end{pmatrix}, \\ \preceq 0,$$

which, by the Schur complement, transforms to

$$\begin{pmatrix} I_r & (I_r \otimes \tilde{z})^* & w_i \\ I_r \otimes \tilde{z} & \sum_{j=1}^m (I_r \otimes C_j)^* (I_r \otimes C_j) & 0 \\ w_i^* & 0 & c_i - \sum_{j=1}^m \lambda_j^i \rho_j^2 \end{pmatrix} \succeq 0,$$

so that problem (RQ) is equivalent to the following SDP problem in the variables z, λ_j^i:

$$
\begin{aligned}
\text{(RQ)} \quad \min \quad & \Re(a^* z), \\
\text{s.t.} \quad & \begin{pmatrix} I_r & (I_r \otimes \tilde{z})^* & w_i = (A_i^0, b_i^0)\tilde{z} \\ I_r \otimes \tilde{z} & \sum_{j=1}^m (I_r \otimes C_j)^*(I_r \otimes C_j) & 0 \\ \tilde{z}^*(A_i^0, b_i^0)^* & 0 & c_i - \sum_{j=1}^m \lambda_j^i \rho_j^2 \end{pmatrix} \\
& \succeq 0, \quad i = 1, \dots, k, \\
& \tilde{z} = (z; 1) \in \mathbb{F}^{n+1}, \\
& \lambda_j^i \geq 0, \quad i = 1, \dots, k, \ j = 1, \dots, m.
\end{aligned}
$$

5 Uniformly Quadratic Problems

Consider the following class of QCQPs:

$$
\begin{aligned}
\text{(UQ)} \quad \min \quad & f_0(z), \\
\text{s.t.} \quad & f_i(z) \leq 0, \qquad i \in \mathcal{I}, \\
& 0 f_j(z) = 0, \quad j \in E, \\
& z \in \mathbb{F}^p,
\end{aligned}
$$

where the functions $f_i : \mathbb{F}^p \to \mathbb{R}, i \in \mathcal{I} \cup \mathcal{E} \cup \{0\}$ are given by

$$
f_i(z) = a_i z^* Q z + 2\Re(b_i^* z) + c_i,
$$

with $a_i \in \mathbb{R}, b_i \in \mathbb{F}^d, c_i \in \mathbb{R}$ and $Q \in \mathcal{H}_{++}^p(\mathbb{F})$.

Such problems will be called *uniformly quadratic problems*. In this section, we will show that this special class of (possibly) nonconvex problems (UQ) admits a tight SDR and has a zero duality gap as long as the number of constraints $|\mathcal{I}| + |\mathcal{E}|$ is smaller or equal to $\theta(\mathbb{F})p$ (and under some very mild conditions). This result is an improvement and extension of a related result [17, Corollary 2.1] which was derived for the real case. In [17], it was shown that, as long as the number of constraints is no larger than $p - 1$, then the problem admits a tight SDR. Here, we improve the result by allowing p constraints and extending it to the complex domain.

5.1 Strong Duality for the Class of Uniform Quadratic Problems

The dual problem of (UQ) is the problem

$$
\begin{aligned}
\text{(DUQ)} \quad \max \quad & -t, \\
\text{s.t.} \quad & M(f_0) + \sum_{i \in \mathcal{I} \cup \mathcal{E}} \lambda_i M(f_i) + \begin{pmatrix} 0_{d\times d} & 0_{d\times 1} \\ 0_{1\times d} & t \end{pmatrix} \succeq 0, \\
& \lambda_i \in \mathbb{R}_+, \quad i \in \mathcal{I}, \\
& \lambda_i \in \mathbb{R}, \quad i \in \mathcal{E}
\end{aligned}
$$

(recall that $M(f_i) = \left(\begin{smallmatrix} a_i Q & b_i \\ b_i^* & c_i \end{smallmatrix}\right)$).

We assume that problem (UQ) is feasible and that condition (11) is satisfied, which, for problem (UQ), translates to

$$\text{Either there exists } k \in \mathcal{E} \text{ for which } a_k \neq 0 \text{ or there exists } k \in \mathcal{I} \text{ for which } a_k > 0. \tag{30}$$

Theorem 5.1 (Strong Duality of Uniformly Quadratic Problems) *Assume that problem* (*UQ*) *is feasible and that condition* (30) *is satisfied. Furthermore, suppose that* $m \leq \theta(\mathbb{F})p$. *Then, problem* (*UQ*) *is solvable and* $\text{val}(UQ) = \text{val}(DUQ)$.

Proof Making the change of variables

$$w = z^* Q^{1/2} \quad (w \in \mathbb{F}^{1\times p}), \tag{31}$$

problem (UQ) becomes

$$\begin{aligned} \min \quad & g_0(w), \\ \text{s.t.} \quad & g_i(w) \leq 0, \quad i \in \mathcal{I}, \\ & g_j(w) = 0, \quad j \in \mathcal{E}, \\ & w \in \mathbb{F}^{1\times p}, \end{aligned} \tag{32}$$

where $g_k : \mathbb{F}^{1\times p} \to \mathbb{R}$ are given by $g_k(w) = a_k \text{Tr}(w^* w) + 2\Re(\text{Tr}(d_k^* w)) + c_k$, $k \in \mathcal{I} \cup \mathcal{E}$ and $d_k = b_k^* Q^{-1/2} \in \mathbb{F}^{1\times p}$. Since the linear change of variables (31) is a bijection, problems (32) and (UQ) have the same optimal value. Note that problem (32) is a QMP problem of order $r = p$ (and $n = 1$) and therefore, since all the conditions of Corollary 2.1 are satisfied, we conclude that strong duality holds for the transformed problem (32) whose value is hence equal to

$$\begin{aligned} \max \quad & -\text{Tr}(\Phi), \\ \text{s.t.} \quad & \begin{pmatrix} a_0 + \sum \lambda_i a_i & d_0 + \sum \lambda_i d_i \\ d_0^* + \sum \lambda_i d_i^* & \frac{1}{p}(c_0 + \sum \lambda_i c_i) + \Phi \end{pmatrix} \succeq 0, \\ & \Phi \in \mathcal{H}^p(\mathbb{F}), \\ & \lambda_i \in \mathbb{R}_+, \quad i \in \mathcal{I}, \\ & \lambda_j \in \mathbb{R}, \quad j \in \mathcal{E}, \end{aligned}$$

where all the summations are over $i \in \mathcal{I} \cup \mathcal{E}$. The latter maximization problem is, of course, equivalent to the maximization problem

$$\begin{aligned}
\max \quad & -t, \\
\text{s.t.} \quad & \operatorname{Tr}(\Phi) \le t, \\
& \begin{pmatrix} a_0 + \sum \lambda_i a_i & d_0 + \sum \lambda_i d_i \\ d_0^* + \sum \lambda_i d_i^* & \frac{1}{d}(c_0 + \sum \lambda_i c_i) + \Phi \end{pmatrix} \succeq 0, \\
& \Phi \in \mathcal{H}^d(\mathbb{F}), \\
& \lambda_i \in \mathbb{R}_+, \quad i \in \mathcal{I}, \\
& \lambda_j \in \mathbb{R}, \quad j \in \mathcal{E}.
\end{aligned} \tag{33}$$

Making the change of variables $\widetilde{\Phi} = \Phi - \frac{t}{p} I_p$, problem (33) becomes

$$\begin{aligned}
\max \quad & -t, \\
\text{s.t.} \quad & \operatorname{Tr}(\widetilde{\Phi}) \le 0, && (34) \\
& \begin{pmatrix} a_0 + \sum \lambda_i a_i & d_0 + \sum \lambda_i d_i \\ d_0^* + \sum \lambda_i d_i^* & \frac{1}{p}(c_0 + \sum \lambda_i c_i + t) + \tilde{\Phi} \end{pmatrix} \succeq 0, && (35) \\
& \widetilde{\Phi} \in \mathcal{H}^p(\mathbb{F}), \\
& \lambda_i \in \mathbb{R}_+, \quad i \in \mathcal{I}, \\
& \lambda_j \in \mathbb{R}, \quad j \in \mathcal{E}.
\end{aligned}$$

Using Lemma 4.2 (equivalence of (iii) and (iv)), we can rewrite the above problem by replacing constraints (34) and (35) with a different LMI not depending on a matrix $\widetilde{\Phi}$:

$$\begin{aligned}
\max \quad & -t, \\
\text{s.t.} \quad & \begin{pmatrix} (a_0 + \sum_{i=1}^m \lambda_i a_i) I_p & d_0^* + \sum_{i=1}^m \lambda_i d_i^* \\ d_0 + \sum_{i=1}^m \lambda_i d_i & c_0 + \sum_{i=1}^m \lambda_i c_i + t \end{pmatrix} \succeq 0, \\
& \lambda_i \in \mathbb{R}_+, \quad i \in I, \\
& \lambda_j \in \mathbb{R}, \quad j \in \mathcal{E}.
\end{aligned}$$

Multiplying the LMI in the above problem from the left and right by

$$\begin{pmatrix} Q^{1/2} & 0_{p\times 1} \\ 0_{1\times p} & 1 \end{pmatrix},$$

we arrive at the problem

$$\begin{aligned}
\max \quad & -t, \\
\text{s.t.} \quad & \begin{pmatrix} (a_0 + \sum_{i=1}^m \lambda_i a_i) Q & b_0 + \sum_{i=1}^m \lambda_i b_i \\ b_0^* + \sum_{i=1}^m \lambda_i b_i^* & t \end{pmatrix} \succeq 0, \\
& \lambda_i \in \mathbb{R}_+, \quad i \in \mathcal{I}, \\
& \lambda_j \in \mathbb{R}, \quad j \in \mathcal{E},
\end{aligned}$$

which is just the dual problem (DUQ). □

5.2 Chebyshev Center of the Intersection of at Most $\theta(\mathbb{F})p$ Balls in $\mathbb{F}^p$

As an application of Theorem 5.1, we consider the problem of finding the Chebyshev center of the intersection of several balls. We recall that the Chebyshev center of a set Ω is defined as the center of the minimum radius ball enclosing the Ω [24]. Finding the Chebyshev center is generally a hard problem; exceptions for this "hardness" statement are when Ω is (i) a finite set of points (ii) a union of balls or ellipsoids, see [25] and references therein. The Chebyshev center of the intersection of several balls is the vector $\hat{z}$ which is the solution of the minmax problem

$$\min_{\hat{z}} \max_{z\in\Omega} \|z-\hat{z}\|^2, \tag{36}$$

where

$$\Omega = \{z \in \mathbb{F}^p : \|z-a_i\|^2 \le r_i^2, i=1,\dots,m\}. \tag{37}$$

We assume that $m \le \theta(\mathbb{F})p$ and that Ω is nonempty. Problem (36) can be written as

$$\min_{\hat{z}} \left\{ \max_{z\in\Omega} \{\|z\|^2 - 2\Re(z^*\hat{z})\} + \|\hat{z}\|^2 \right\}. \tag{38}$$

Note that the inner maximization problem in (38) is a uniform quadratic problem which satisfies the conditions of Theorem 5.1 and therefore can be replaced by the dual minimization problem

$$\begin{aligned} \min \quad & t, \\ \text{s.t.} \quad & \begin{pmatrix} (-1+\sum_{i=1}^m \lambda_i) I_p & \hat{z} - \sum_{i=1}^m \lambda_i a_i \\ \hat{z}^* - \sum_{i=1}^m \lambda_i a_i^* & t \end{pmatrix} \succeq 0, \\ & \lambda_i \in \mathbb{R}_+, \quad i=1,\dots,m. \end{aligned}$$

We thus conclude that the Chebyshev center problem (38) can be recast as the convex minimization problem

$$\begin{aligned} \min_{t,\lambda_i,\hat{z}} \quad & t + \|\hat{z}\|^2, \\ \text{s.t.} \quad & \begin{pmatrix} (-1+\sum_{i=1}^m \lambda_i) I_p & \hat{z} - \sum_{i=1}^m \lambda_i a_i \\ \hat{z}^* - \sum_{i=1}^m \lambda_i a_i^* & t \end{pmatrix} \succeq 0, \\ & \lambda_i \in \mathbb{R}_+, \quad i=1,\dots,m, \end{aligned}$$

which transforms to the SDP

$$\begin{aligned} \min_{t,s,\lambda_i,\hat{z}} \quad & t + s, \\ \text{s.t.} \quad & \begin{pmatrix} (-1+\sum_{i=1}^m \lambda_i) I_p & \hat{z} - \sum_{i=1}^m \lambda_i a_i \\ \hat{z}^* - \sum_{i=1}^m \lambda_i a_i^* & t \end{pmatrix} \succeq 0, \\ & \begin{pmatrix} I_p & \hat{z} \\ \hat{z}^* & s \end{pmatrix} \succeq 0, \\ & \lambda_i \in \mathbb{R}_+, \quad i=1,\dots,m. \end{aligned} \tag{39}$$

We summarize the above derivation in the following theorem.

Theorem 5.2 *Consider problem* (36) *of finding the Chebyshev center of the set* Ω *given by* (37). *Assume that* Ω *is nonempty and* $m \leq \theta(\mathbb{F})p$. *Then, the value of problem* (36) *is equal to the value of problem* (39). *Moreover, if* $(t, s, \lambda_i, \hat{z})$ *is an optimal solution of* (39), *then* $\hat{z}$ *is an optimal solution of the minmax problem* (36).

Remark 5.1 It can be shown, by following the analysis in [17], that if Ω has a nonempty interior, then the Chebyshev center is given by

$$\hat{z} = \sum_{i=1}^{m} \lambda_i a_i,$$

where $(\lambda_1, \ldots, \lambda_m)$ is an optimal solution of the following convex quadratic minimization problem over the unit simplex:

$$\min\left\{\left\|\sum_{i=1}^{m} \lambda_i a_i\right\|^2 - \sum_{i=1}^{m} \lambda_i (\|a_i\|^2 - r_i^2) : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0\right\}.$$

The corresponding radius of the minimum ball enclosing Ω is given by

$$\sqrt{\left\|\sum_{i=1}^{m} \lambda_i a_i\right\|^2 - \sum_{i=1}^{m} \lambda_i (\|a_i\|^2 - r_i^2)}.$$

Some examples of Chebyshev centers of intersection of balls are given in Fig. 1. The Chebyshev centers were found by using Theorem 5.2.

Appendix A: Low-Rank Solution of Real and Complex SDP Problems

The underlying assumption that guarantees the validity of the process is that problem (10) is solvable and that $|\mathcal{I}_1| + |\mathcal{E}_1| \leq \varphi(r+1; \mathbb{F}) - 1$.

Algorithm RED
Input: Z_0—an optimal solution to problem (10).
Output: An optimal solution $\widehat{Z}$ to problem (10) satisfying $\operatorname{rank}(\widehat{Z}) \leq r$.

1. If $\operatorname{rank}(Z_0) \leq r$, then go to step 3. Else, go to step 2.
2. While $\operatorname{rank}(Z_0) > r$, repeat steps (a)–(e):
   (a) Set $d \leftarrow \operatorname{rank}(Z_0)$.
   (b) Compute a decomposition of Z_0: $Z_0 = UU^*$, where $U \in \mathbb{F}^{n \times d}$.
   (c) Find a nontrivial solution T_0 for the set of homogenous linear equations in the $d \times d$ *Hermitian* variables matrix T $(T = T^*)$,

$$\operatorname{Tr}(U^* C_i U T) = 0, \quad i \in \mathcal{I}_1 \cup \mathcal{E}_1.$$

(d) If $T_0 \succeq 0$, then set $W \leftarrow -T_0$. Else set $W \leftarrow T_0$.
(e) Set $Z_0 \leftarrow U(I + \beta W)U^*$, where $\beta = -1/\lambda_{\min}(W)$.

3. Set $\widehat{Z} \leftarrow Z_0$ and Stop.

The linear system of step (c) has a nonzero solution, since the relations $|\mathcal{I}_1| + |\mathcal{E}_1| \le \varphi(r+1; \mathbb{F}) - 1, d > r$ imply that the homogenous system has more variables than equations.

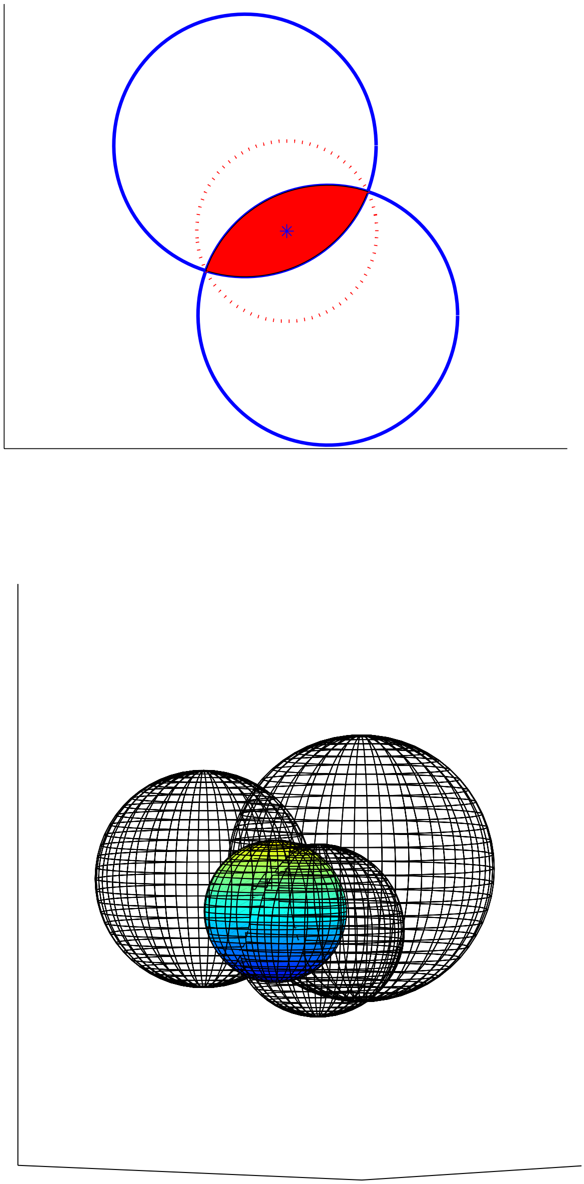

Fig. 1 The Chebyshev center of the intersection of p balls in $\mathbb{R}^p$ ($p = 2, 3$). In the *upper figure*, the *filled area* is the intersection of *two circles* and the Chebyshev center of this area is denoted by “*”; the *dotted circle* is the corresponding minimum enclosing ball. In the *lower figure*, the filled-face ball is the minimum enclosing ball of the three faceless balls

Appendix B: Proof of Theorem 3.3

We begin by stating a result of Au-Yeung and Poon [16, Theorem 2]. This result is as a generalization of Brickman's theorem [26] on the numerical range of two quadratic forms.

Theorem B.1 *Let $A_1, \dots, A_k \in \mathcal{H}^n(\mathbb{F})$ and let r be a positive integer for which $n \geq r+2$ and $k \leq \varphi(r+1; \mathbb{F}) - 1$. Then, the set*

$$\{(\mathrm{Tr}(Z^* A_1 Z), \dots, \mathrm{Tr}(Z^* A_k Z)) : \|Z\|_F^2 = 1, Z \in \mathbb{F}^{n\times r}\}$$

is convex.

In [19], Polyak showed how to use Brickman's theorem in order to derive a convexity result on a mapping comprised of three homogenous quadratic forms. We use a similar line of analysis (with the necessary modifications) in order to prove our result on the convexity of no more than $\varphi(r+1; \mathbb{F})$ homogenous QM functions over the real or complex domain.

Proof of Theorem 3.3 Denote by $f : \mathbb{F}^{n\times r} \to \mathbb{R}^m$ the vector function

$$f(Z) = (f_1(Z); \dots; f_k(Z)),$$

where $f_i(Z) = \mathrm{Tr}(Z^* A_i Z)$. Our goal is to show that $\{f(Z) : Z \in \mathbb{F}^{n\times r}\} \subseteq \mathbb{R}^m$, which is the image of $\mathbb{F}^{n\times r}$ under f, is closed and convex. Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be any invertible linear transformation of the form

$$T(x_1; \dots; x_k) = \left(*; \dots; *; \sum_{i=1}^{k} \mu_i x_i\right).$$

Since closedness and convexity properties are invariant under linear transformations over $\mathbb{R}^k$, we conclude that it is enough to prove the desired properties (closedness and convexity) on the set $G = \{g(z) \mid z \in \mathbb{F}^{n\times r}\}$ with $g = Tf$. Note that

$$g_k(Z) = \sum_{i=1}^{k} \mu_i f_i(Z) > 0, \quad \text{for every } Z \neq 0.$$

By applying an appropriate linear transformation on $\mathbb{F}^{n\times r}$, we conclude that we can assume, without loss of generality, that $g_k(Z) = \|Z\|_F^2$. By Theorem B.1, we have that the set

$$H = \{(g_1(Z); \dots; g_{k-1}(Z)) \ : \ \|Z\|_F^2 = 1\} \subseteq \mathbb{R}^{k-1}$$

is convex. Moreover, G can be represented as

$$G = \{\lambda Q \mid \lambda \geq 0\}, \tag{40}$$

where $Q = \{(h; 1) : h \in H\}$, i.e., G is the conic hull of the convex set Q and as such is convex, see [27, p. 14]. All that is left is to show that G, given by (40), is closed.

Note that H, being an image of the unit sphere by a continuous function, is a compact set. This implies that Q is also compact. To show the closedness of G, consider a sequence of points $\{\lambda_k q^k\}$ from G, such that $\lambda_k \geq 0$ and $q^k \in Q$. We will prove that, if

$$\lambda_k q^k \to a, \tag{41}$$

then $a \in G$. Indeed, the compactness of Q implies that $\{q^k\}$ must have a subsequence $\{q^{k_l}\}$ that converges to a point $\hat{q} \in Q$ and in particular $\hat{q}_m = 1$. We can thus write, for large enough l,

$$\lambda_{k_l} = \frac{\lambda_{k_l} q_m^{k_l}}{q_m^{k_l}} \to a_m \equiv \hat{\lambda},$$

which implies $\lambda^{k_l} q^{k_l} \to \hat{\lambda}\hat{q}$. Combining this with (41) we have $a = \hat{\lambda}\hat{q} \in G$.

Finally, we will show that $F = W$. The inclusion $F \subseteq W$ is clear so only the converse inclusion $W \subseteq F$ will be proven. Let $w = (\mathrm{Tr}(A_1 U); \dots; \mathrm{Tr}(A_k U)) \in W$ with $U \in \mathcal{H}_+^n(\mathbb{F})$. The positive semidefinite matrix U has a decomposition $U = \frac{1}{n}\sum_{i=1}^n z_i z_i^*$ with $z_i \in \mathbb{F}^n$. Then,

$$U = \frac{1}{n}\sum_{i=1}^n Z_i Z_i^*,$$

with $Z_i = (z_i, 0_{n\times(r-1)}) \in \mathbb{F}^{n\times r}$. Therefore, $w = \frac{1}{n}\sum_{i=1}^n w^i$, where

$$w^i = (\mathrm{Tr}(Z_i^* A_1 Z_i); \dots; \mathrm{Tr}(Z_i^* A_k Z_i)) \in F,$$

and we conclude that w, being a convex combination of points from F, also belongs to F. □

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