J Glob Optim (2007) 39:113–126
DOI 10.1007/s10898-006-9127-8

ORIGINAL PAPER

On the convexity of a class of quadratic mappings and its application to the problem of finding the smallest ball enclosing a given intersection of balls

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Received: 19 January 2006 / Accepted: 29 November 2006 / Published online: 21 February 2007

Abstract We consider the outer approximation problem of finding a minimum radius ball enclosing a given intersection of at most $n-1$ balls in $\mathbb{R}^n$. We show that if the aforementioned intersection has a nonempty interior, then the problem reduces to minimizing a convex quadratic function over the unit simplex. This result is established by using convexity and representation theorems for a class of quadratic mappings. As a byproduct of our analysis, we show that a class of nonconvex quadratic problems admits a tight semidefinite relaxation.

Keywords Outer approximation problems · Convexity of quadratic mappings · Nonconvex quadratic optimization · Semidefinite relaxation · Strong duality

1 Introduction

The problem of finding a best ellipsoidal approximation of a given closed convex set S is a fundamental problem in control and optimization and has a wide variety of applications, see [17,19] and references therein. Generally speaking, there are two types of these problems: (i) *outer approximation* problems in which we look for the minimum volume ellipsoid containing the set S and (ii) *inner approximation* problems where we seek the maximum volume ellipsoid contained in the set S.

Several outer approximation problems are known to be tractable. Among them are the minimum volume ellipsoid containing a polyhedron given as a convex hull of a finite set of points, and the minimum volume ellipsoid containing a union of ellipsoids. As for inner approximation problems, we recall the maximum volume ellipsoid contained in an intersection of ellipsoids and the maximum volume ellipsoid contained in a polyhedron given as a set of linear inequalities—both are known to be tractable.

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We note that although the inner approximation problem of finding the maximum volume ellipsoid of an intersection of ellipsoids is known to be tractable [17], the related *outer* approximation problem, i.e., that of finding a minimum volume ellipsoid containing an intersection of ellipsoids, is considered difficult. Even the simpler problem of verifying that $\mathcal{E}_0 \supseteq \bigcap_{i=1}^p \mathcal{E}_i$ holds, given ellipsoids $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_p$ is NP-hard [17, Sect 3.7.2]. To better understand the intrinsic difficulty of this simpler problem, note that $\mathcal{E}_0 \supseteq \bigcap_{i=1}^p \mathcal{E}_i$ if and only if the following implication

$$\boldsymbol{x} \in \mathcal{E}_0 \text{ for every } \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \boldsymbol{x} \in \mathcal{E}_1, \boldsymbol{x} \in \mathcal{E}_2, \ldots, \boldsymbol{x} \in \mathcal{E}_p \tag{1}$$

holds true. By describing the ellipsoids as level sets of strictly convex quadratic functions:

$$\mathcal{E}_i = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} + 2\boldsymbol{b}_i^T \boldsymbol{x} + c_i \leq 0\} \quad (\boldsymbol{A}_i \succ \boldsymbol{0}), i = 0, 1, \ldots, p,$$

we arrive at the conclusion that the problem of determining whether an ellipsoid contains the intersection of p ellipsoids is equivalent to verifying whether a given quadratic inequality follows from a set of p quadratic inequalities. In the case $p = 1$, it is well known by the celebrated S-lemma [3,7], that implication (1) is equivalent to solving a linear matrix inequality (LMI) and as such can be verified efficiently by e.g., interior point methods [13]. However, even for $p = 2$ the S-lemma is generally not valid. An exception (when $p = 2$) is the case where the corresponding quadratic inequalities are homogenous ($\boldsymbol{b}_0 = \boldsymbol{b}_1 = \boldsymbol{b}_2 = 0$), see [16].

We are thus led to the conclusion that the *general* outer approximation problem of an intersection of ellipsoids seems to be intractable. This prompts the question whether we can find a class of specially structured ellipsoids for which the outer approximation of an intersection of ellipsoids problem will turn out to be tractable. The most natural choice of structured ellipsoids is of course balls. We note that the ball-version of the outer approximation problem of a *union* of ellipsoids, i.e., the problem of finding a minimum radius ball containing the union of a given set of balls, was extensively studied, see e.g., [21,24] and references therein.

In this paper we consider the ball-version of the outer approximation problem of an intersection of balls, that is, we consider the following problem:

Main problem (minimum radius ball enclosing an intersection of balls): Given a set of balls $B_1, \ldots, B_p$, find a minimum radius ball B_0 enclosing the intersection $\bigcap_{i=1}^p B_i$.

We will show that, as long as the intersection of the balls has a nonempty interior and $p \leq n - 1$ (where n is the dimension of the space), the problem can be solved efficiently. In particular, we will prove (see Sect. 3) that the center and radius of an optimal ball can be explicitly expressed via a p-vector $\boldsymbol{\lambda} \in \mathbb{R}^p$ which is a solution to the following problem of minimizing a convex quadratic function over the unit simplex:

$$\min \left\{ \left\| \sum_{i=1}^p \lambda_i \boldsymbol{a}_i \right\|^2 - \sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2) : \sum_{i=1}^p \lambda_i = 1, \lambda_i \geq 0 \right\},$$

where $\boldsymbol{a}_i$ and r_i are the center and radius of B_i respectively. This result relies on an S-lemma-type result that establishes an LMI characterization of the claim that a ball contains a given intersection of balls.

In order to establish these results we rely on three key results developed in Sect. 2: (i) the image of a mapping comprised of a strictly quadratic function and $n-1$ linear functions is closed and convex and (ii) the image a quadratic mapping has a semidefinite representation if is closed and convex. Based on these results, we are able to show that (iii) a class of nonconvex quadratic problems involving several quadratic functions sharing a similar matrix term admits a tight semidefinite relaxation (SDR). Result (iii) is the key ingredient in analyzing the main problem.

Notation. Vectors are denoted by boldface lowercase letters, e.g., $\boldsymbol{y}$, and matrices by boldface uppercase letters e.g., $\boldsymbol{A}$. We follow the MATLAB convention and use ";" for adjoining scalars, vectors or matrices in a column. For two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \succ \boldsymbol{B}$ ($\boldsymbol{A} \succeq \boldsymbol{B}$) means that $\boldsymbol{A} - \boldsymbol{B}$ is positive definite (semidefinite). $\mathcal{S}^n = \{\boldsymbol{A} \in \mathbb{R}^{n\times n} : \boldsymbol{A} = \boldsymbol{A}^T\}$ is the set of symmetric $n \times n$ matrices, $\mathcal{S}^n_+ = \{\boldsymbol{A} \in \mathbb{R}^{n\times n} : \boldsymbol{A} \succeq \boldsymbol{0}\}$ is the set all $n \times n$ symmetric positive semidefinite matrices and $\mathcal{S}^n_{++} = \{\boldsymbol{A} \in \mathbb{R}^{n\times n} : \boldsymbol{A} \succ \boldsymbol{0}\}$ is the set all $n \times n$ symmetric positive definite matrices. The n-dimensional unit simplex is given by $\Delta_n = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq 0, \sum_{i=1}^n x_i = 1\}$. $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the nonnegative and positive orthant respectively. $\mathbf{E}^r_{ij}$ is the $r \times r$ matrix with one at the ij-th component and zero elsewhere.

2 A class of nonconvex quadratic optimization problems admitting a tight semidefinite relaxation

The primary objective of this section is to analyze a class of nonconvex quadratic optimization problems involving several quadratic functions that share a similar matrix term. Specifically, we consider the class of problems

$$(QP) \quad \begin{array}{ll} \max & f_0(\boldsymbol{x}) \\ \text{s.t.} & l_i \leq f_i(\boldsymbol{x}) \leq u_i, \quad i = 1, 2, \ldots, p, \\ & \boldsymbol{x} \in \mathbb{R}^n, \end{array} \tag{2}$$

where $-\infty \leq l_i \leq u_i \leq \infty$ and $f_i(\boldsymbol{x}) = \alpha_i \boldsymbol{x}^T \mathbf{Q} \boldsymbol{x} + 2\boldsymbol{b}_i^T \boldsymbol{x} + c_i$ with $Q \in S^n_{++}, \boldsymbol{b}_i \in \mathbb{R}^n, c_i \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}, i = 0, 1, \ldots, p$. We assume that $\alpha_0 = 1$. The semidefinite relaxation of the above problem [20] is given by

$$(QPSDR) \quad \begin{array}{ll} \max & \text{Tr}(\boldsymbol{M}_0 \boldsymbol{U}) \\ \text{s.t.} & l_i \leq \text{Tr}(\boldsymbol{M}_i \boldsymbol{U}) \leq u_i, \quad i = 1, 2, \ldots, p, \\ & \boldsymbol{U} \in \mathcal{S}^{n+1}_+, U_{n+1,n+1} = 1, \end{array} \tag{3}$$

where $\boldsymbol{M}_i = \begin{pmatrix} \alpha_i \mathbf{Q} & \boldsymbol{b}_i \\ \boldsymbol{b}_i^T & c_i \end{pmatrix}, \quad i = 0, 1, \ldots, p.$

The main result of this section is that as long as $p \leq n - 1$, the value of problem (QP) is equal to the value of its semidefinite relaxation (QPSDR). The analysis is based on an investigation of the properties of the image of a class of quadratic mappings. Specifically, we show in Sect. 2.1 that the image of a mapping comprised of a strictly convex quadratic function and at most $n - 1$ linear functions is convex. The upper bound, $n - 1$, on the number of linear functions is shown to be tight. In Sect. 2.2 we establish a representation theorem for closed convex images of quadratic mappings which, combined with the results of Sect. 2.1, leads to the main result val(QP) = val(QPSDR).

2.1 Convexity of the image of a class of quadratic mappings

Our approach for establishing the tight SDR result val(QP) = val(QPSDR) is to exploit convexity properties associated with a corresponding class of quadratic mappings. This approach goes back to the work of Fradkov and Yakubovich [7] that utilizes results on the convexity of mappings comprised of two quadratic forms in the real domain and three quadratic forms in the complex domain, in order to prove S-lemma-type results.

Polyak [16] showed that by using a convexity result on the image of three homogenous quadratic forms one can show a strong duality result (which is equivalent under some mild conditions to the property of admitting a tight SDR) for homogenous nonconvex quadratic problems involving two quadratic constraints[1]; this result was also recovered in the work of Ye and Zhang [22]. Polyak [16] also provided an alternative proof of the well known strong duality result for trust region subproblems [4,6,11,12] which is based on the convexity property of mappings comprised of two *nonhomogenous* quadratic functions. Beck and Eldar [2] showed that by comparing the real and complex valued images of quadratic mappings, one can establish a sufficient condition for strong duality of nonconvex quadratic problems with two quadratic constraints. More interesting results concerning various convexity results of quadratic mappings and their relation to optimization problems can be found in the comprehensive survey of Polik and Terlaky [15] and in the paper of Hiriart-Urruty and Torki [8].

We begin by establishing the convexity of the image of mappings comprised of a strictly convex quadratic function and a set of no more than $n-1$ linear functions.

Theorem 2.1 *Let* $f: \mathbb{R}^n \to \mathbb{R}$ *be given by* $f(\boldsymbol{x}) = \boldsymbol{x}^T\mathbf{Q}\boldsymbol{x} + 2\boldsymbol{f}^T\boldsymbol{x} + c$ *where* $\mathbf{Q} \in \mathcal{S}^n_{++}$, $\boldsymbol{f} \in \mathbb{R}^n, c \in \mathbb{R}$ *and let* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$ *with* $m \leq n-1$. *Then the set* $S \subseteq \mathbb{R}^{m+1}$ *given by*

$$S = \{(f(\boldsymbol{x}); \boldsymbol{a}_1^T\boldsymbol{x}; \boldsymbol{a}_2^T\boldsymbol{x}; \ldots; \boldsymbol{a}_m^T\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n\} \tag{4}$$

is closed and convex.

Proof **(convexity)** To show the convexity of S, we consider two vectors

$$(\gamma_1; \boldsymbol{b}), (\gamma_2; \boldsymbol{c}) \in S, \tag{5}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^m$. We will prove that $(\lambda\gamma_1 + (1-\lambda)\gamma_2; \lambda\boldsymbol{b} + (1-\lambda)\boldsymbol{c}) \in S$ for every $\lambda \in [0,1]$. Indeed, let $\lambda \in [0,1]$, then by (4) and (5), there exist two vectors $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathbb{R}^n$ such that

$$\boldsymbol{A}\boldsymbol{x}_0 = \boldsymbol{b}, \boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{c}, f(\boldsymbol{x}_0) = \gamma_1, f(\boldsymbol{x}_1) = \gamma_2,$$

where $\boldsymbol{A}$ is the $m \times n$ matrix whose rows are $\boldsymbol{a}_1^T, \ldots, \boldsymbol{a}_m^T$. Define $\boldsymbol{d} = \lambda\boldsymbol{x}_0 + (1-\lambda)\boldsymbol{x}_1$. Then clearly

$$\boldsymbol{A}\boldsymbol{d} = \lambda\boldsymbol{b} + (1-\lambda)\boldsymbol{c}$$

and, by the convexity of f:

$$f(\boldsymbol{d}) = f(\lambda\boldsymbol{x}_0 + (1-\lambda)\boldsymbol{x}_1) \leq \lambda f(\boldsymbol{x}_0) + (1-\lambda)f(\boldsymbol{x}_1) = \lambda\gamma_1 + (1-\lambda)\gamma_2. \tag{6}$$

[1] Under the assumption that there exists a positive definite linear combination of the corresponding matrices.

The general solution to the linear system

$$\boldsymbol{Ax} = \lambda \boldsymbol{b} + (1-\lambda)\boldsymbol{c}, \quad \boldsymbol{x} \in \mathbb{R}^n, \tag{7}$$

is given by $\boldsymbol{x} = \boldsymbol{Bw} + \boldsymbol{d}, \boldsymbol{w} \in \mathbb{R}^k$, where $\boldsymbol{B} \in \mathbb{R}^{n\times k}$ is a matrix whose columns form a basis (of dimension k) for the null space of $\boldsymbol{A}$. Since $\boldsymbol{A}$ is an $m \times n$ matrix it follows that $k \geq n - m \geq 1$.

Now, consider the function g: $\mathbb{R}^k \to \mathbb{R}$ defined by

$$g(\boldsymbol{w}) \equiv f(\boldsymbol{Bw} + \boldsymbol{d}).$$

Since $\boldsymbol{B}$ has full column rank, we conclude that $\boldsymbol{B}^T\mathbf{Q}\boldsymbol{B}$, which is the hessian of the quadratic function g, is positive definite. Therefore, $g(\|\boldsymbol{w}\|) \to \infty$ as $\|\boldsymbol{w}\|$ tends to ∞. As a result, there must exist $\bar{\boldsymbol{w}} \in \mathbb{R}^k$ for which

$$g(\bar{\boldsymbol{w}}) > \lambda\gamma_1 + (1-\lambda)\gamma_2. \tag{8}$$

Moreover,

$$g(0) = f(\boldsymbol{d}) \overset{(6)}{\leq} \lambda\gamma_1 + (1-\lambda)\gamma_2 \tag{9}$$

Combining (8), (9) with the continuity of g, we conclude that there exists $\alpha \in [0,1)$ such that $g(\alpha\bar{\boldsymbol{w}}) = \lambda\gamma_1 + (1-\lambda)\gamma_2$. By denoting $\boldsymbol{x}_2 = \alpha\boldsymbol{B}\bar{\boldsymbol{w}} + \boldsymbol{d}$ we deduce that

$$(\lambda\gamma_1 + (1-\lambda)\gamma_2, \lambda\boldsymbol{b} + (1-\lambda)\boldsymbol{c}) = (f(\boldsymbol{x}_2), \boldsymbol{Ax}_2) \in S$$

and the convexity of S follows.
(closedness) Let $\{\boldsymbol{u}^k\} \subseteq S$ be a converging sequence, i.e., $\boldsymbol{u}^k \to \boldsymbol{u}^*$ for some $\boldsymbol{u}^* \in \mathbb{R}^{m+1}$. By the definition of S, there exists a sequence $\{\boldsymbol{x}^k\} \subseteq \mathbb{R}^n$ for which

$$\boldsymbol{u}^k = (f(\boldsymbol{x}^k); \boldsymbol{a}_1^T\boldsymbol{x}^k; \ldots; \boldsymbol{a}_m^T\boldsymbol{x}^k).$$

Therefore, $f(\boldsymbol{x}^k) \to u_1^*$ and in particular $\{f(\boldsymbol{x}^k)\}$ is a bounded sequence. Thus, there exist $L \in \mathbb{R}$ such that the sequence $\{\boldsymbol{x}^k\}$ is contained in the nondegenerate ellipsoid $\{\boldsymbol{x} : f(\boldsymbol{x}) \leq L\}$ and hence $\{\boldsymbol{x}^k\}$ is bounded. Therefore, there exists a converging subsequence $\boldsymbol{x}^{k_l} \to \boldsymbol{x}^*$. Combining this with the continuity of f and the linear functions $\boldsymbol{a}_i^T\boldsymbol{x}$, we conclude that

$$\boldsymbol{u}^* = (f(\boldsymbol{x}^*); \boldsymbol{a}_1^T\boldsymbol{x}^*; \ldots; \boldsymbol{a}_m^T\boldsymbol{x}^*) \in S.$$

□

We will now show that the the upper bound, $n-1$, on the number of linear functions in the latter result is tight. Specifically, we will prove that the image of a mapping comprised of a strictly convex quadratic function and n linearly independent functions is *always* nonconvex.

Theorem 2.2 *Let f: $\mathbb{R}^n \to \mathbb{R}$ be a strictly convex quadratic function and let $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ be n linearly independent vectors in $\mathbb{R}^n$. Then the image of f and the n linear functions $\boldsymbol{a}_i^T\boldsymbol{x}$:*

$$T = \{(f(\boldsymbol{x}); \boldsymbol{a}_1^T\boldsymbol{x}; \ldots; \boldsymbol{a}_n^T\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n\}$$

is not convex.

Proof Let $\boldsymbol{A} = (\boldsymbol{a}_1^T; \boldsymbol{a}_2^T; \ldots; \boldsymbol{a}_n^T)$ be the $n \times n$ matrix whose rows are the vectors $\boldsymbol{a}_i^T$. Note that $(\gamma; \boldsymbol{b}) \in T$ if and only if $\gamma = f(\boldsymbol{A}^{-1}\boldsymbol{b})$. Therefore, for every $\boldsymbol{y} \in \mathbb{R}^n$ one has:

$$(f(\boldsymbol{A}^{-1}\boldsymbol{y}); \boldsymbol{y}), (f(-\boldsymbol{A}^{-1}\boldsymbol{y}); -\boldsymbol{y}) \in T.$$

Suppose on the contrary that T is convex. Then the latter relation implies

$$\left(\frac{f(\boldsymbol{A}^{-1}\boldsymbol{y}) + f(-\boldsymbol{A}^{-1}\boldsymbol{y})}{2}, \boldsymbol{0}\right) \in T \text{ for every } \boldsymbol{y} \in T,$$

which is the same as

$$\frac{f(\boldsymbol{A}^{-1}\boldsymbol{y}) + f(-\boldsymbol{A}^{-1}\boldsymbol{y})}{2} = f(\boldsymbol{0}).$$

The left hand side of the above equality tends to ∞ as $\|\boldsymbol{y}\| \to \infty$ leading to the desired contradiction. □

2.2 Tightness of the semidefinite relaxation of problem (QP)

In this section we establish the tight SDR result for the class of problems (QP). Only few classes of problems are known to have this property. The simplest and well known example is the trust region problem, which consists of minimizing an indefinite quadratic function over a ball, and admits an exact semidefinite relaxation (SDR), see [6,12]. Extensions of this problem were considered in [4,11,16,18]. In general, these results cannot be extended to quadratic problems involving two quadratic constraints [22,23]. An exception is the case in which all the functions involved are homogenous quadratic functions [16,22]. In the *complex domain* quadratic problems with two quadratic constraints usually admit a tight SDR, see Beck and Eldar [2] and Huang and Zhang [9]; thus, stronger results hold in the complex domain. Another class of quadratic problems with a tight SDR property is the class of *quadratic matrix programming* problems introduced by Beck [1]; in this latter class of problems, the SDR is tight even in the presence of more than two constraints.

We begin by establishing a result (Lemma 2.2) that provides further evidence for the close connection between convexity of quadratic mappings and results concerning tightness of SDR. In particular, we will show that if the image of a general quadratic mapping is closed and convex, then it can be represented as the image of $\mathcal{S}_+^n \cap \mathcal{A}$ under a linear mapping, where $\mathcal{A}$ is an affine subspace. Such a representation is called a *semidefinite representation*. This result combined with the the convexity result of Theorem 2.1 will be the key ingredient in proving that val(QP) = val(QPSDR). The proof of Lemma 2.2 relies on the following well known Lemma.

Lemma 2.1 ([3, p. 163]) *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be an Hermitian matrix and let $\boldsymbol{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then,*

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^T\boldsymbol{x} + c \geq 0 \text{ for every } \boldsymbol{x} \in \mathbb{R}^n$$

if and only if

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^T & c \end{pmatrix} \succeq \boldsymbol{0}.$$

Lemma 2.2 *Let $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ be quadratic functions given by $f_i(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} + 2\boldsymbol{b}_i^T \boldsymbol{x} + c_i$, where $\boldsymbol{A}_i \in \mathcal{S}^n, \boldsymbol{b}_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$. Suppose that the set*

$$W = \{(f_1(\boldsymbol{x}); f_2(\boldsymbol{x}); \ldots; f_m(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{R}^n\}$$

is closed and convex. Then W is equal to the set F defined by

$$F = \{(Tr(\boldsymbol{M}_1\boldsymbol{U}); Tr(\boldsymbol{M}_2\boldsymbol{U}); \ldots; Tr(\boldsymbol{M}_m\boldsymbol{U})) : \boldsymbol{U} \in \mathcal{S}_+^{n+1}, U_{n+1,n+1} = 1\},$$

where $\boldsymbol{M}_i = \begin{pmatrix} \boldsymbol{A}_i & \boldsymbol{b}_i \\ \boldsymbol{b}_i^T & c_i \end{pmatrix}$.

Proof ($\mathbf{W} \subseteq \mathbf{F}$). Suppose that $\boldsymbol{w} = (w_1; \ldots; w_m) \in W$. Then there exists $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$ such that $\boldsymbol{w} = (w_1; \ldots; w_m) = (f_1(\tilde{\boldsymbol{x}}); \ldots; f_m(\tilde{\boldsymbol{x}}))$. Define

$$\widetilde{\boldsymbol{U}} = (\tilde{\boldsymbol{x}}; 1)(\tilde{\boldsymbol{x}}^T, 1) = \begin{pmatrix} \tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^T & \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{x}}^T & 1 \end{pmatrix}.$$

Clearly, $\widetilde{\boldsymbol{U}} \in \mathcal{S}_+^{n+1}$ and $\widetilde{U}_{n+1,n+1} = 1$. Moreover, $w_i = f_i(\tilde{\boldsymbol{x}}) = \mathrm{Tr}(\boldsymbol{M}_i\widetilde{\boldsymbol{U}})$ for every i and thus $\boldsymbol{w} \in F$.

($F \subseteq W$) Let $\boldsymbol{w} \in F$, that is, $w_i = \mathrm{Tr}(\boldsymbol{M}_i\widetilde{\boldsymbol{U}})$ for some $\widetilde{\boldsymbol{U}} \in \mathcal{S}_+^{n+1}$ satisfying $\widetilde{U}_{n+1,n+1} = 1$. To prove that $\boldsymbol{w} \in W$, assume on the contrary that $\boldsymbol{w} \notin W$. Since $\{\boldsymbol{w}\}$ is a compact set and W is a closed and convex set, then by the strict separation theorem [5, Proposition 2.4.3] we have that there exists $\boldsymbol{a} \in \mathbb{R}^m, \boldsymbol{a} \neq \boldsymbol{0}$ and $\gamma \in \mathbb{R}$ such that

$$\boldsymbol{a}^T\boldsymbol{v} > \gamma \quad \text{for every } \boldsymbol{v} \in W, \tag{10}$$

$$\boldsymbol{a}^T\boldsymbol{w} < \gamma. \tag{11}$$

By the definition of W, (10) can be written as

$$\sum_{i=1}^m a_i f_i(\boldsymbol{x}) > \gamma \text{ for every } \boldsymbol{x} \in \mathbb{R}^n,$$

which is the same as

$$\sum_{i=1}^m a_i(\boldsymbol{x}^T\boldsymbol{A}_i\boldsymbol{x} + 2\boldsymbol{b}_i^T\boldsymbol{x} + c_i) > \gamma \text{ for every } \boldsymbol{x} \in \mathbb{R}^n. \tag{12}$$

Invoking Lemma 2.1, we obtain that (12) is equivalent to the LMI

$$\sum_{i=1}^m a_i\boldsymbol{M}_i \succ \gamma\mathbf{E}_{n+1,n+1}^{n+1}, \tag{13}$$

(recall that $\mathbf{E}_{n+1,n+1}^{n+1}$ is the $(n+1) \times (n+1)$ matrix whose $(n+1, n+1)$-th entry is one and zero elsewhere). On the other hand,

$$\gamma > \boldsymbol{a}^T\boldsymbol{w} = \sum_{i=1}^m a_i w_i = \sum_{i=1}^m a_i \mathrm{Tr}(\boldsymbol{M}_i\widetilde{\boldsymbol{U}}),$$

which can be equivalently written as

$$\mathrm{Tr}\left(\left(\sum_{i=1}^m a_i\boldsymbol{M}_i - \gamma\mathbf{E}_{n+1,n+1}^{n+1}\right)\widetilde{\boldsymbol{U}}\right) < 0. \tag{14}$$

However, $\widetilde{\boldsymbol{U}} \succeq 0$ and by (13), $\sum_{i=1}^{m} a_i \boldsymbol{M}_i - \gamma \mathbf{E}^{n+1}_{n+1,n+1}$ is positive definite. Therefore, since the trace of a product of two positive semidefinite matrices is nonnegative, we conclude that $\mathrm{Tr}((\sum_{i=1}^{m} a_i \boldsymbol{M}_i - \gamma \mathbf{E}^{n+1}_{n+1,n+1})\widetilde{\boldsymbol{U}})$ must be nonnegative, which is a contradiction to (14). □

Combining Theorem 2.1 and Lemma 2.2, we conclude that the image of a quadratic mapping comprised of at most n strictly convex quadratic functions with a similar matrix term is convex and has a semidefinite representation.

Theorem 2.3 *Let $f_i\colon \mathbb{R}^n \to \mathbb{R}, i = 0,\ldots,p$ be $p+1$ quadratic functions given by $f_i(\boldsymbol{x}) = \alpha_i \boldsymbol{x}^T \mathbf{Q}\boldsymbol{x} + 2\boldsymbol{b}_i^T\boldsymbol{x} + c_i$ where $\mathbf{Q} \in \mathcal{S}^n_{++}, \boldsymbol{b}_i \in \mathbb{R}^n, c_i \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}$ with $\alpha_0 = 1$. Suppose that $p \le n-1$. Then*

(i) The set

$$V = \{(f_0(\boldsymbol{x});\ldots;f_p(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{R}^n\}$$

is closed and convex.

(ii) The set V is equal to

$$W = \{(Tr(\boldsymbol{M}_0\boldsymbol{U});\ldots;Tr(\boldsymbol{M}_p\boldsymbol{U})) : \boldsymbol{U} \in \mathcal{S}^{n+1}_{+}, U_{n+1,n+1} = 1\},$$

where $\boldsymbol{M}_i = \begin{pmatrix} \alpha_i\mathbf{Q} & \boldsymbol{b}_i \\ \boldsymbol{b}_i^T & c_i \end{pmatrix}$.

Proof (i). Note that[2] $V = \mathcal{L}(S) + \{(c_0;c_1;\ldots;c_p)\}$ where

$$S = \{(\boldsymbol{x}^T\mathbf{Q}\boldsymbol{x} + 2\boldsymbol{b}_0^T\boldsymbol{x}; (\boldsymbol{b}_1 - \alpha_1\boldsymbol{b}_0)^T\boldsymbol{x}; (\boldsymbol{b}_2 - \alpha_2\boldsymbol{b}_0)^T\boldsymbol{x};\ldots,(\boldsymbol{b}_p - \alpha_p\boldsymbol{b}_0)^T\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n\}$$

and the mapping $\mathcal{L} : \mathbb{R}^{p+1} \to \mathbb{R}^{p+1}$ is the linear transformation given by

$$\mathcal{L}(x_0;x_1;\ldots;x_{k-1}) = (x_0;\alpha_1x_0 + 2x_1;\alpha_2x_0 + 2x_2;\ldots;\alpha_px_0 + 2x_p).$$

By Theorem 2.1 the set S is closed and convex. Since convexity and closedness properties are preserved under linear transformations and translations, the result follows. (ii). Readily follows from the first part of the corollary and Theorem 2.2. □

Finally, based on the latter result we are able to show that val(QP) = val(QPSDR) (see problems (2), (3)).

Corollary 2.1 *The value of the nonconvex quadratic optimization problem (QP) with $p \le n-1$ is equal to the value of its semidefinite relaxation (QPSDR)*

Proof Problem (QP) can be written as following problem in the decision variables $t_i, i = 0,\ldots,p$:

$$\begin{array}{ll} \max & t_0 \\ \text{s.t.} & l_i \le t_i \le u_i, \quad i = 1,2,\ldots,p, \\ & (t_0;t_1;\ldots,t_p) \in C, \end{array} \tag{15}$$

where $C = \{(f_0(\boldsymbol{x});\ldots;f_p(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{R}^n\}$. By Theorem 2.3 (ii), C is also equal to

$$C = \{(\mathrm{Tr}(\boldsymbol{M}_0\boldsymbol{U}),\ldots,\mathrm{Tr}(\boldsymbol{M}_p\boldsymbol{U})) : \boldsymbol{U} \in \mathcal{S}^{n+1}_{++}, U_{n+1,n+1} = 1\},$$

which immediately transforms problem (15) into problem (QPSDR). □

[2] The sum of two sets $A, B \subseteq \mathbb{R}^k$ is given by $A + B = \{a + b : a \in A, b \in B\}$.

We note that higher order relaxations have been recently proposed for nonconvex quadratic problems and for the more general class of polynomial optimization problems. In fact, the semidefinite relaxation considered in corollary 2.1 is the first and simplest relaxation in a hierarchy of relaxations, see e.g.,[3] the works of Lasserre [10] and Parrilo [14].

We end this section by noting that the celebrated MAXCUT problem can be formulated as a problem of the form (QP) with $p = n$. Indeed, recall that the MAXCUT problem can be cast as the bivalent problem [3, p. 173]

$$\begin{aligned} &\max\ \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \\ &\text{s.t.}\quad x_i^2 = 1, i = 1,\ldots,n, \end{aligned} \tag{16}$$

where $\boldsymbol{A}$ is a positive semidefinite matrix. We can also assume without loss of generality that $\boldsymbol{A}$ is *positive definite* since a term of the form $\beta\|\boldsymbol{x}\|^2$ with $\beta > 0$ can be added to the objective function without changing the optimal set of the problem.

Now, consider the relaxed problem of maximizing $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ over a box:

$$\begin{aligned} &\max\ \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \\ &\text{s.t.}\quad -1 \le x_i \le 1, i = 1,\ldots,n. \end{aligned} \tag{17}$$

Problems (16) and (17) have the same optimal value; this is due to the fact that problem (17) consists of maximizing a convex function over a convex set and consequently its solution belongs to the set of extreme points of the box $\{\boldsymbol{x} \in \mathbb{R}^n : -1 \le x_i \le 1\}$ which is just the feasible set of (16). Now, problem (16) is in fact problem (QP) with

$$\mathbf{Q} = \boldsymbol{A}, \boldsymbol{b}_0 = \boldsymbol{0}, c_0 = 0, \alpha_i = c_i = 0, u_i = 1, l_i = -1, \boldsymbol{b}_i = \boldsymbol{e}_i, i = 1,\ldots,n,$$

where $\boldsymbol{e}_i$ is the vector with 1 at the i-th place and zero elsewhere. Although problem (17) is of the form (QP), the tight SDR result of Corollary 2.1 can not be employed here since the assumption $p \le n-1$ fails (p is equal to n). This is not surprising since the MAXCUT is considered to be a hard combinatorial problem and as such is not likely to be equivalent to a convex optimization problem.

3 Finding a minimum radius ball enclosing a given intersection of balls

The main problem (see the introduction) of finding a minimum radius ball $B(\boldsymbol{y}, r)$ enclosing a given intersection of balls can be cast as the minimization problem in the variables $\boldsymbol{y}, r$:

$$\begin{aligned} &\min\ r \\ &\text{s.t.}\quad \textstyle\bigcap_{i=1}^p B(\boldsymbol{a}_i, r_i) \subseteq B(\boldsymbol{y}, r), \\ &\qquad\ \boldsymbol{y} \in \mathbb{R}^n, r \in \mathbb{R}. \end{aligned}$$

Making the change of variables $\gamma = r^2$, the latter problem becomes

$$\begin{aligned} &\min\ \sqrt{\gamma} \\ &\text{s.t.}\quad \textstyle\bigcap_{i=1}^p B(\boldsymbol{a}_i, r_i) \subseteq B(\boldsymbol{y}, \sqrt{\gamma}), \\ &\qquad\ \boldsymbol{y} \in \mathbb{R}^n, \gamma \in \mathbb{R}, \end{aligned}$$

[3] I am indebted to an anonymous reviewer for bringing references [10,14] to my attention.

which has the same optimal set as

$$\begin{array}{ll} \min & \gamma \\ \text{s.t.} & \bigcap_{i=1}^{p} B(\boldsymbol{a}_i, r_i) \subseteq B(\boldsymbol{y}, \sqrt{\gamma}), \\ & \boldsymbol{y} \in \mathbb{R}^n, \gamma \in \mathbb{R}. \end{array} \tag{18}$$

We begin by proving that if (i) $p \leq n-1$ and (ii) the intersection $\bigcap_{i=1}^{p} B(\boldsymbol{a}_i, r_i)$ has a nonempty interior, then the statement $\bigcap_{i=1}^{p} B(\boldsymbol{a}_i, r_i) \subseteq B(\boldsymbol{y}, \sqrt{\gamma})$ can be reformulated as an LMI.

Theorem 3.1 *Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p \in \mathbb{R}^n$ and $r_1, \ldots, r_p \in \mathbb{R}_{++}$. Suppose that the intersection of the balls $B(\boldsymbol{a}_i, r_i), i = 1, \ldots, p$ has a nonempty interior, i.e.,*

$$\operatorname{int}\left(\bigcap_{i=1}^{p} B(\boldsymbol{a}_i, r_i) \right) \neq \emptyset \tag{19}$$

and that $p \leq n-1$. Then the following two statements are equivalent:

(i) $\bigcap_{i=1}^{p} B(\boldsymbol{a}_i, r_i) \subseteq B(\boldsymbol{y}, \sqrt{\gamma})$.

(ii) There exist $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{R}_+$ such that the following LMI is satisfied:

$$\boldsymbol{N}_0 - \sum_{i=1}^{p} \lambda_i \boldsymbol{N}_i \preceq \boldsymbol{0}, \tag{20}$$

where

$$\boldsymbol{N}_0 = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{y} \\ -\boldsymbol{y}^T & \|\boldsymbol{y}\|^2 - \gamma \end{pmatrix}, \quad \boldsymbol{N}_i = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{a}_i \\ -\boldsymbol{a}_i^T & \|\boldsymbol{a}_i\|^2 - r_i^2 \end{pmatrix}, \quad i = 1, \ldots, p. \tag{21}$$

Proof We begin by recalling that, as mentioned in the introduction, statement (i) is equivalent to the validity of the following implication:

$$\|\boldsymbol{x} - \boldsymbol{y}\|^2 \leq \gamma \text{ for every } \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \|\boldsymbol{x} - \boldsymbol{a}_1\|^2 \leq r_1^2, \ldots, \|\boldsymbol{x} - \boldsymbol{a}_p\|^2 \leq r_p^2. \tag{22}$$

Therefore, in the rest of the proof, we will consider the implication (22) instead of statement (i).

(ii) ⇒ (22). Suppose that statement (ii) is satisfied, i.e., there exist $\lambda_i \in \mathbb{R}_+, i = 1, 2, \ldots, p$ such that the LMI (20) is satisfied. Multiplying the LMI from the left by $(\boldsymbol{x}^T, 1)$ and from the right by $(\boldsymbol{x}; 1)$ we obtain

$$(\boldsymbol{x}^T, 1)\boldsymbol{N}_0(\boldsymbol{x}; 1) \leq \sum_{i=1}^{p} \lambda_i (\boldsymbol{x}^T, 1)\boldsymbol{N}_i(\boldsymbol{x}; 1),$$

which can be rewritten as

$$\|\boldsymbol{x} - \boldsymbol{y}\|^2 - \gamma \leq \sum_{i=1}^{p} \lambda_i (\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2).$$

The last inequality and the nonnegativity of λ_i imply that (22) holds true.

(22) ⇒ (ii). Suppose that the implication (22) holds true. Then the value of the optimization problem in the decision variables $\boldsymbol{x}$

$$\begin{array}{ll} \max & \|\boldsymbol{x} - \boldsymbol{y}\|^2 - \gamma \\ \text{s.t.} & \|\boldsymbol{x} - \boldsymbol{a}_i\|^2 \leq r_i^2, \quad i = 1, 2, \ldots, p, \\ & \boldsymbol{x} \in \mathbb{R}^n \end{array} \tag{23}$$

is nonpositive. Now, since $p \le n-1$, then by Corollary 2.1, the value of problem (23) is equal to the value of the SDP problem

$$\begin{array}{ll} \max & \mathrm{Tr}(\boldsymbol{N}_0\boldsymbol{U}) \\ \text{s.t.} & \mathrm{Tr}(\boldsymbol{N}_i\boldsymbol{U}) \le 0, \quad i = 1,2,\ldots,p, \\ & \boldsymbol{U} \in \mathcal{S}_+^{n+1}, U_{n+1,n+1} = 1, \end{array} \tag{24}$$

where $\boldsymbol{N}_i$ is given in (21). Condition (19) implies that there exists $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$ such that

$$\|\tilde{\boldsymbol{x}} - \boldsymbol{a}_i\|^2 - r_i^2 < 0, i = 1,\ldots,p.$$

Therefore, the matrix $\widetilde{\boldsymbol{U}} = \begin{pmatrix} \tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^T & \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{x}}^T & 1 \end{pmatrix}$ satisfies

$$\mathrm{Tr}(\boldsymbol{N}_i\widetilde{\boldsymbol{U}}) < 0, \widetilde{\boldsymbol{U}} \succeq \boldsymbol{0}, \widetilde{U}_{n+1,n+1} = 1.$$

By making small perturbations to the matrix $\widetilde{\boldsymbol{U}}$, we can assume without loss of generality that $\widetilde{\boldsymbol{U}} \succ \boldsymbol{0}$. Hence, problem (24) is strictly feasible and as a result by the conic duality theorem [20,3], the dual problem

$$\begin{array}{ll} \min & \mu \\ \text{s.t.} & \boldsymbol{N}_0 - \sum_{i=1}^p \lambda_i \boldsymbol{N}_i - \mu \mathbf{E}_{n+1,n+1}^{n+1} \preceq \boldsymbol{0}, \\ & \lambda_i \ge 0, i = 1,\ldots,p, \\ & \mu \in \mathbb{R} \end{array} \tag{25}$$

is solvable and has the same optimal value as (24). Thus, there exists a nonpositive $\tilde{\mu}$ and nonnegative $\tilde{\lambda}_i, i = 1,\ldots,p$ such that

$$\boldsymbol{N}_0 - \sum_{i=1}^p \tilde{\lambda}_i \boldsymbol{N}_i - \tilde{\mu}\mathbf{E}_{n+1,n+1}^{n+1} \preceq \boldsymbol{0},$$

which, by the nonpositivity of $\tilde{\mu}$, implies that $\boldsymbol{N}_0 - \sum_{i=1}^p \tilde{\lambda}_i \boldsymbol{N}_i \preceq \boldsymbol{0}$. □

Based on Theorem 3.1, and the reformulation (18) of the main problem, we will now prove our main result.

Theorem 3.2 (quadratic programming formulation of the main problem) *Let $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p \in \mathbb{R}^n$ and $r_1,\ldots,r_p \in \mathbb{R}_{++}$. Suppose that the intersection of the balls $B(\boldsymbol{a}_i, r_i), i = 1,\ldots,p$ has a nonempty interior and that $p \le n-1$. Then the center and radius of a minimum radius ball enclosing the intersection $\bigcap_{i=1}^p B(\boldsymbol{a}_i, r_i)$ are given by*

$$\boldsymbol{y} = \sum_{i=1}^p \lambda_i \boldsymbol{a}_i, \tag{26}$$

$$r = \sqrt{\left\|\sum_{i=1}^p \lambda_i \boldsymbol{a}_i\right\|^2 - \sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2)} \tag{27}$$

respectively, where $\boldsymbol{\lambda} \in \Delta_p$ is an optimal solution of the convex quadratic minimization problem

$$\begin{array}{ll} \min & \left\|\sum_{i=1}^p \lambda_i \boldsymbol{a}_i\right\|^2 - \sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2), \\ \textit{s.t.} & \boldsymbol{\lambda} \in \Delta_p. \end{array}$$

Proof By Theorem 3.1, problem (18) can be rewritten as the following SDP problem

$$
\begin{aligned}
\gamma^* = \min\ & \gamma \\
\text{s.t.}\ & \begin{pmatrix} \left(\sum_{i=1}^p \lambda_i - 1\right) \boldsymbol{I} & \boldsymbol{y} - \sum_{i=1}^p \lambda_i \boldsymbol{a}_i \\ \left(\boldsymbol{y} - \sum_{i=1}^p \lambda_i \boldsymbol{a}_i\right)^T & \gamma - \|\boldsymbol{y}\|^2 + \sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2) \end{pmatrix} \succeq \boldsymbol{0}, \\
& \boldsymbol{y} \in \mathbb{R}^n, \gamma \in \mathbb{R}, \boldsymbol{\lambda} \in \mathbb{R}^p_+.
\end{aligned}
\tag{28}
$$

Here we denoted the optimal squared radius by γ^* which is positive by its definition. Note that any feasible solution $(\boldsymbol{y}, \boldsymbol{\lambda}, \gamma)$ of problem (28) satisfies in particular $\sum_{i=1}^p \lambda_i \geq 1$. We will now show that in fact every *optimal* solution satisfies $\sum_{i=1}^p \lambda_i = 1$. Suppose on the contrary that there exists an optimal solution $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\lambda}}, \tilde{\gamma})$ of problem (28) satisfying $\sum_{i=1}^p \tilde{\lambda}_i > 1$. In that case, we can by invoke Schur's complement and conclude that problem (28) reduces to

$$
\begin{aligned}
\gamma^* = \min\ & \gamma \\
\text{s.t.}\ & \textstyle\sum_{i=1}^p \lambda_i > 1, \\
& \gamma \geq \|\boldsymbol{y}\|^2 - \textstyle\sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2) + \frac{\|\boldsymbol{y} - \sum_{i=1}^p \lambda_i \boldsymbol{a}_i\|^2}{\sum_{i=1}^p \lambda_i - 1}, \\
& \boldsymbol{y} \in \mathbb{R}^n, \boldsymbol{\lambda} \in \mathbb{R}^p_+, \gamma \in \mathbb{R},
\end{aligned}
$$

which can be equivalently written as

$$
\begin{aligned}
\gamma^* = \min\ & \|\boldsymbol{y}\|^2 - \textstyle\sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2) + \frac{\|\boldsymbol{y} - \sum_{i=1}^p \lambda_i \boldsymbol{a}_i\|^2}{\sum_{i=1}^p \lambda_i - 1} \\
\text{s.t.}\ & \textstyle\sum_{i=1}^p \lambda_i > 1, \\
& \boldsymbol{y} \in \mathbb{R}^n, \gamma \in \mathbb{R}.
\end{aligned}
\tag{29}
$$

Fixing $\boldsymbol{\lambda}$ and minimizing the latter problem with respect to $\boldsymbol{y}$ we obtain that $\boldsymbol{y} = \frac{1}{\sum_{i=1}^p \lambda_i} \sum_{i=1}^p \lambda_i \boldsymbol{a}_i$ at any optimal solution. Plugging this expression back into the objective function of problem (29), we obtain that $\tilde{\boldsymbol{\lambda}}$ is an optimal solution of

$$
\begin{aligned}
\gamma^* = \min\ & \varphi(\boldsymbol{\lambda}) \equiv \frac{1}{\sum_{i=1}^p \lambda_i} \| \textstyle\sum_{i=1}^p \lambda_i \boldsymbol{a}_i \|^2 - \sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2) \\
\text{s.t.}\ & \textstyle\sum_{i=1}^p \lambda_i > 1, \\
& \boldsymbol{\lambda} \in \mathbb{R}^p_+.
\end{aligned}
\tag{30}
$$

Note that the objective function $\varphi(\boldsymbol{\lambda})$ is homogenous of order 1, i.e., satisfying $\varphi(\alpha\boldsymbol{\lambda}) = \alpha\varphi(\boldsymbol{\lambda})$ for every $\alpha \in \mathbb{R}_+, \boldsymbol{\lambda} \in \mathbb{R}^p$. Denote $\hat{\boldsymbol{\lambda}} = \alpha\tilde{\boldsymbol{\lambda}}$, where α is some real number in the interval $(1/\sum_{i=1}^p \lambda_i, 1)$. Then $\hat{\boldsymbol{\lambda}}$ is a feasible solution of (30) and satisfies $\varphi(\hat{\boldsymbol{\lambda}}) = \varphi(\alpha\tilde{\boldsymbol{\lambda}}) = \alpha\varphi(\tilde{\boldsymbol{\lambda}}) < r^*$ contradicting the optimality of $\tilde{\boldsymbol{\lambda}}$. We have thus proven that $\sum_{i=1}^p \lambda_i = 1$ at any optimal solution so that (28) can be written as

$$
\begin{aligned}
\gamma^* = \min\ & \gamma \\
\text{s.t.}\ & \begin{pmatrix} \boldsymbol{0} & \boldsymbol{y} - \sum_{i=1}^p \lambda_i \boldsymbol{a}_i \\ \left(\boldsymbol{y} - \sum_{i=1}^p \lambda_i \boldsymbol{a}_i\right)^T & \gamma - \|\boldsymbol{y}\|^2 + \sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2) \end{pmatrix} \succeq \boldsymbol{0}, \\
& \boldsymbol{y} \in \mathbb{R}^n, \gamma \in \mathbb{R}, \boldsymbol{\lambda} \in \Delta_p.
\end{aligned}
\tag{31}
$$

The LMI constraint in the above problem reduces to

$$\boldsymbol{y} = \textstyle\sum_{i=1}^p \lambda_i \boldsymbol{a}_i, \tag{32}$$

$$\gamma \geq \|\boldsymbol{y}\|^2 - \sum_{i=1}^p \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2), \tag{33}$$

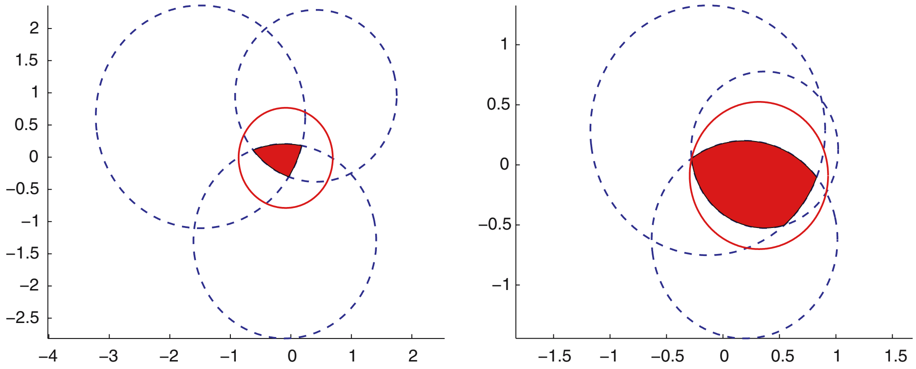

Fig. 1 Three balls in the plane

which implies that problem (31) can be converted into

$$\begin{aligned} \gamma^* = \min \ & \left\| \sum_{i=1}^{p} \lambda_i \boldsymbol{a}_i \right\|^2 - \sum_{i=1}^{p} \lambda_i (\|\boldsymbol{a}_i\|^2 - r_i^2), \\ \text{s.t.} \ & \boldsymbol{\lambda} \in \Delta_p. \end{aligned}$$

The optimal center $\boldsymbol{y}$ is given by (32). □

It is interesting to note that the ball $B(\boldsymbol{y}, r)$ with $\boldsymbol{y}$ and r given by (26) and (27) contains the intersection $\bigcap_{i=1}^{p} B(\boldsymbol{a}_i, r_i)$ even when $p \geq n$. However, only when $p \leq n-1$ this ball is guaranteed to be the smallest one possible. In Fig. 1 two examples of intersections of three balls (dashed lines) are given. In both examples it is clear that the enclosing ball (solid line)—computed by Theorem 3.2—is not the smallest.

Acknowledgments I would like to thank two anonymous referees for their insightful comments which helped to improve the presentation of the paper. I also wish to thank Aharon Ben-Tal for several motivating discussions on the general subject of the paper.

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