Introduction to Finite element method (FEM):

Galerkin based method for PDE's solution using wavelet basis

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Partial differential equations (PDE)

The finite element method also known as Galerkin's method, is a general method for solving PDEs (or ODEs) of the form

$$A(u) = f$$

Where A is a differential operator, f is a given force term and u is the solution.

Second order linear partial differential equations (PDE)

- $$A(u) = -\Delta u = f$$ Poisson's equation (Elliptic)

- $$A(u) = \frac{\partial u}{\partial t} - \Delta u = f$$ Heat equation (Parabolic)

- $$A(u) = \frac{\partial^2 u}{\partial t^2} - \Delta u = f$$ Wave equation (Hyperbolic)

Solving PDEs

Problems

- Analytic solutions can be obtained only for simple geometries in special cases
- Solution should satisfy $u \in C^2(\Omega) \cap C(\overline{\Omega})$

Solution

Find an approximate solution U of the form

$$U(x) = \sum_{j=1}^{N} \xi_j \varphi_j$$

U is a linear combination of basis functions $\{\varphi_j\}_{j=1}^{N}$ with local support.

Notations from functional analysis

- Scalar (inner) product for functions $v, w \in R^1$:

$$(v, w) = \int_\Omega v(x)w(x)dx$$

- $L^2(\Omega)$-norm of a function v:

$$\|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2 dx \right)^{1/2} = \sqrt{(v, v)}$$

- v and w are orthogonal iff $(v, w) = 0$

Galerkin's method $AU = f$ $U(x) = \sum_{j=1}^{N} \xi_j \varphi_j$

Let V be a function space where $V = \mathrm{span}\{\varphi_j\}_{j=1}^{N}$

$$(AU, v) = (f, v) \quad \forall v \in V$$

Variational formulation

$$a(u, v) = (f, v) \qquad \forall v \in V$$

$$a(\cdot,\cdot) = (A(\cdot),\cdot) \qquad a : V \times V \to R$$

Bilinear functional

$$\left(A\sum_{j=1}^{N} \xi_j \varphi_j, v\right) = (f, v)$$

Linear operator →

$$\sum_{j=1}^{N} \xi_j (A\varphi_j, v) = (f, v)$$

Galerkin's method

$$\sum_{j=1}^{N} \xi_j \left(A\varphi_j , \varphi_i \right) = \left(f , \varphi_i \right) \quad i = 1, \ldots, N$$

$$\sum_{j=1}^{N} \xi_j a(\varphi_j , \varphi_i) = \left(f , \varphi_i \right) \quad i = 1, \ldots, N$$

$$\begin{pmatrix} & & \\ & L & \\ & & \end{pmatrix} \begin{pmatrix} \\ \xi \\ \\ \end{pmatrix} = \begin{pmatrix} \\ b \\ \\ \end{pmatrix}$$

$$L_{i,j} = a(\varphi_j , \varphi_i)$$

$$b_i = (f , \varphi_i)$$

Continuous PDE → Linear algebraic system

Example: Helmholtz's equation

1D $$-\varepsilon^2 u'' + u = 1 \quad u \in [0,1], \quad u(0) = u(1) = 0$$

2D $$-\varepsilon^2 \Delta u + u = 1 \quad u \in [0,1]\times[0,1], \quad \partial u = 0$$

- Variational formulation: $a(u,v) = (f,v)$

$$\varepsilon^2 \int_0^1 \nabla u \cdot \nabla v \, dx + \int_0^1 uv \, dx = \int_0^1 v \, dx$$

- Linear system:

$$L_{i,j} = \int_0^1 \nabla \varphi_j \cdot \nabla \varphi_i \, dx + \int_0^1 \varphi_j \varphi_i \, dx, \quad b_i = \int_0^1 \varphi_i$$

How to choose basis $\{\varphi_i\}$?

- Should be simple to differentiate and integrate
- Close to orthogonal basis gives better numerical conditioning

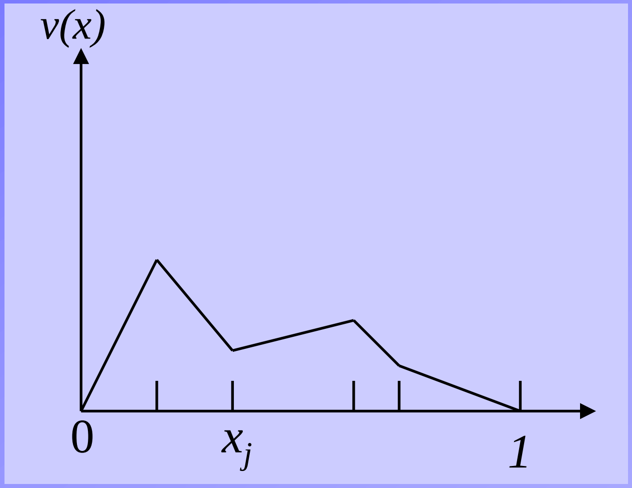
Examples:

- Spectral methods: trigonometric functions (global support, orthogonal)
- Finite element methods (FEM): piecewise polynomials defined on a mesh τ^h (local support, near orthogonal).

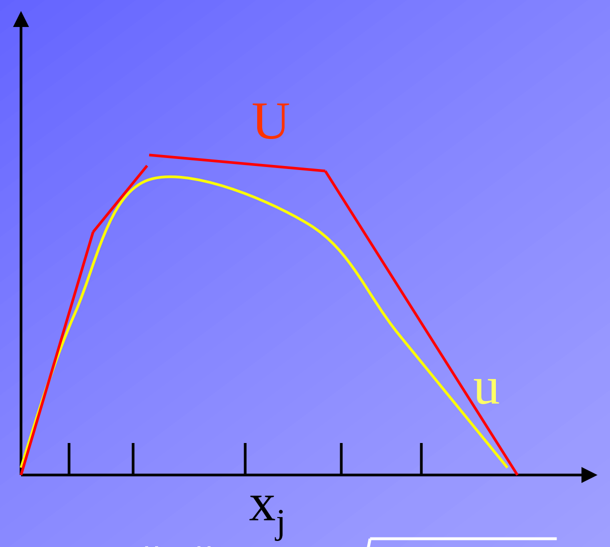
FEM Advantage: sparse matrix → lower computational complexity

Finite element method with piecewise linear functions

- Define intervals $h_j=(x_j,x_{j-1})$ and set $h = \max_j h_j, \quad j = 1,\ldots,N$
- Let V_h be the set of functions v such that:
  - v is linear on each subinterval
  - v is continuous on $[0, 1]$ and $v(0)=v(1)=0$
- The quantity h is a measure of how fine the partition is



Error estimation



- We expect the error $e=U-u$ to decrease if we increase the dimension N of V. This can be achieved by decreasing the mesh size h.

- Let $\|\cdot\|_E$ denote the energy-norm given by $\|v\|_E = \sqrt{a(v,v)}$

Then the piecewise linear finite element solution $U=U(x)$ for the specified equation $-\varepsilon^2 \Delta u + u = 1$ satisfies the error estimate

$$\|e\|_E = \|U-u\|_E \le ch\left(\|u''\|_{L^2(\Omega)} + \|u'\|_{L^2(\Omega)}\right)$$

Wavelets bases

- Collection $\psi_{j,k}$ of scaled and dilated functions

$$\psi_{jk} = 2^{j/2}\psi(2^j x - k)$$

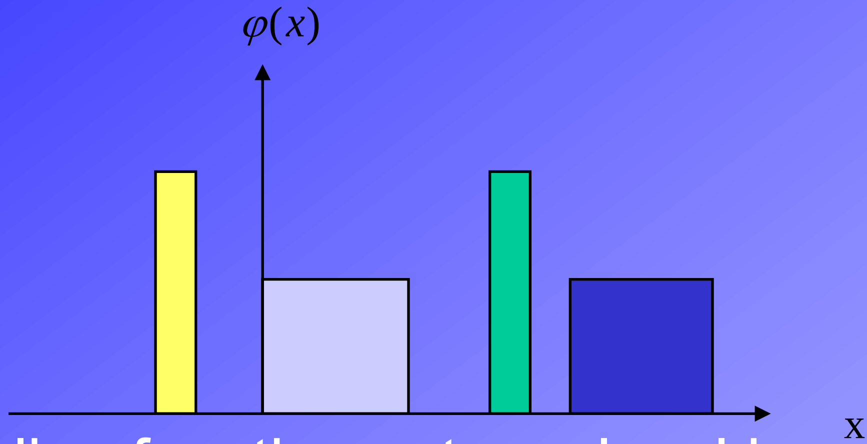
- How is wavelet basis constructed?

  Define the basis functions by translating and stretching (compressing) of functions called the scaling functions.

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$$

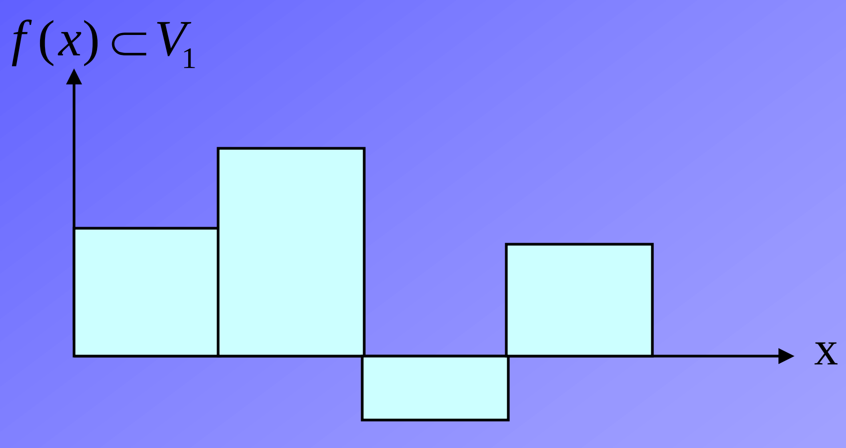
Wavelet bases

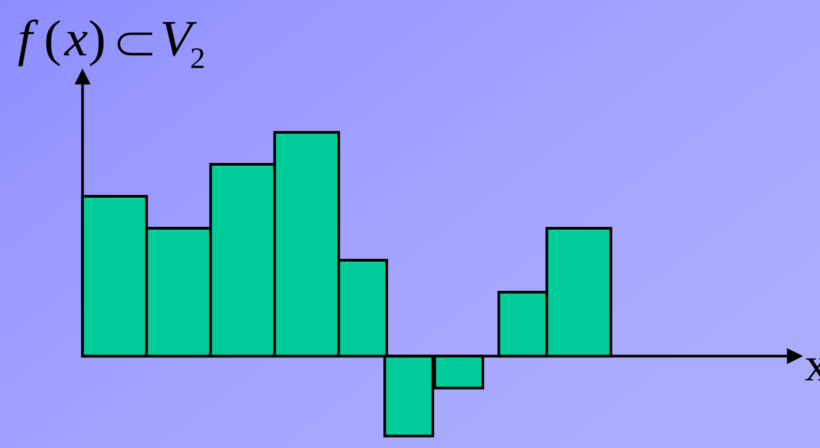
Example: Haar(1910) scaling function $\varphi(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$



The set of scaling functions at any level j can be used to express functions that form a set V_j:

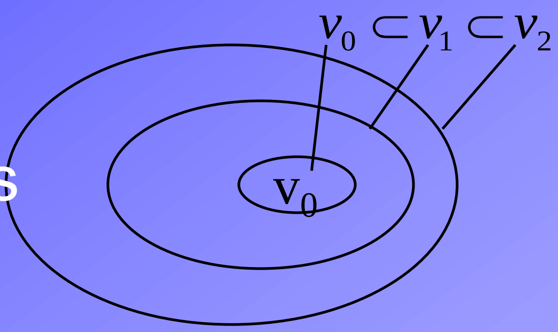
$$f(x) = \sum_k \alpha_k \varphi_{j,k}(x)$$





Wavelet bases

Mallat and Meyer (1986):



Multiresolution of $L^2(R)$:subspaces spanned by the scaling function at low scales are nested within those spanned at high scales.

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$

subject to the following three conditions:

- Completeness $\overline{\bigcup_{n=-\infty}^{\infty} V_n} = L^2(R), \qquad \lim_{n\to-\infty} V_n = \{0\}.$
- Scale similarity $f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}.$
- translation seed The scaling function is orthogonal to its integer translates $\left(\varphi_{j,n}, \varphi_{j,k}\right) = 0$

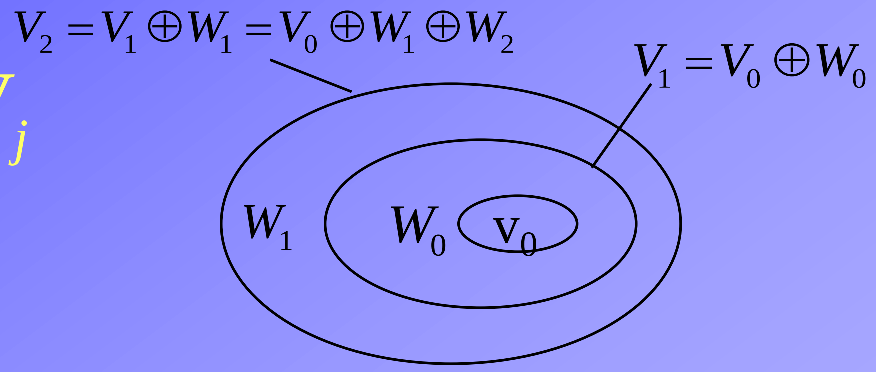
Wavelet bases

- Let $f(x) \in V_1, \quad f_0(x) \in V_0 \qquad f(x) = f_0(x) + e_0(x)$
- W_0 Is the space that contains the residual such that

$$e_0(x) = \sum_k \alpha_k \psi_{0,k}(x-k) \in W_0$$

$\psi(x)$: Wavelet function

- In general $V_{j+1} = V_j \oplus W_j$



- W_j - orthogonal complement, can be expanded in terms of a set of wavelets $\{\psi_{jk}\}$ which must be orthogonal to the scaling functions $\{\varphi_{jk}\}$
- Wavelets are orthonormal basis for $L^2(R)$

$$L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

Wavelet bases

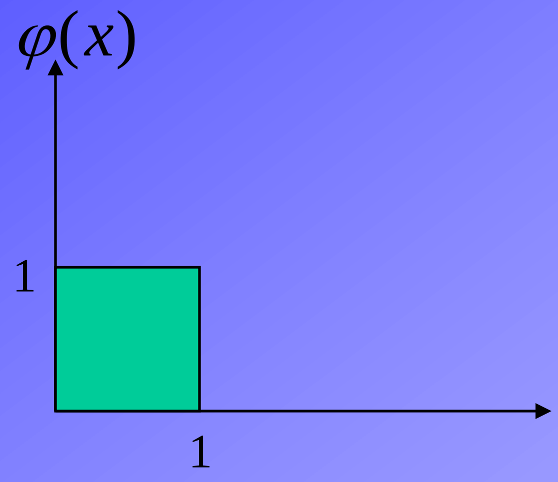
- Wavelets satisfy the requirements:

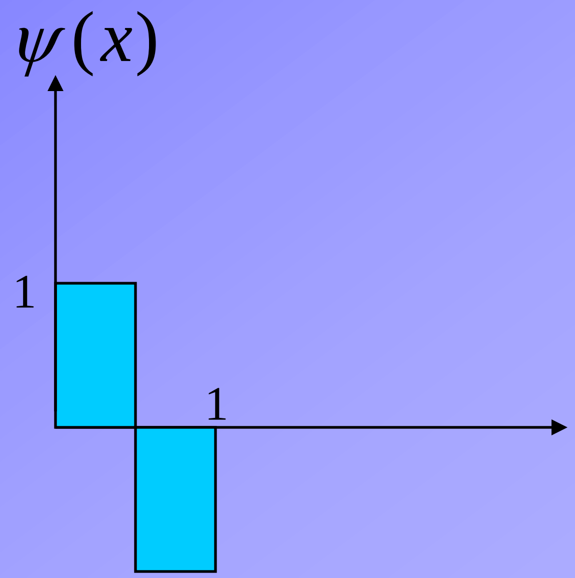
$$\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$$

$$\psi_{j,k} = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

- This leads to: $$\psi(x) = \sum_n g_n \varphi(2x - n)$$

- Example: Haar wavelet $$\psi(x) = \varphi(2x) - \varphi(2x-1)$$

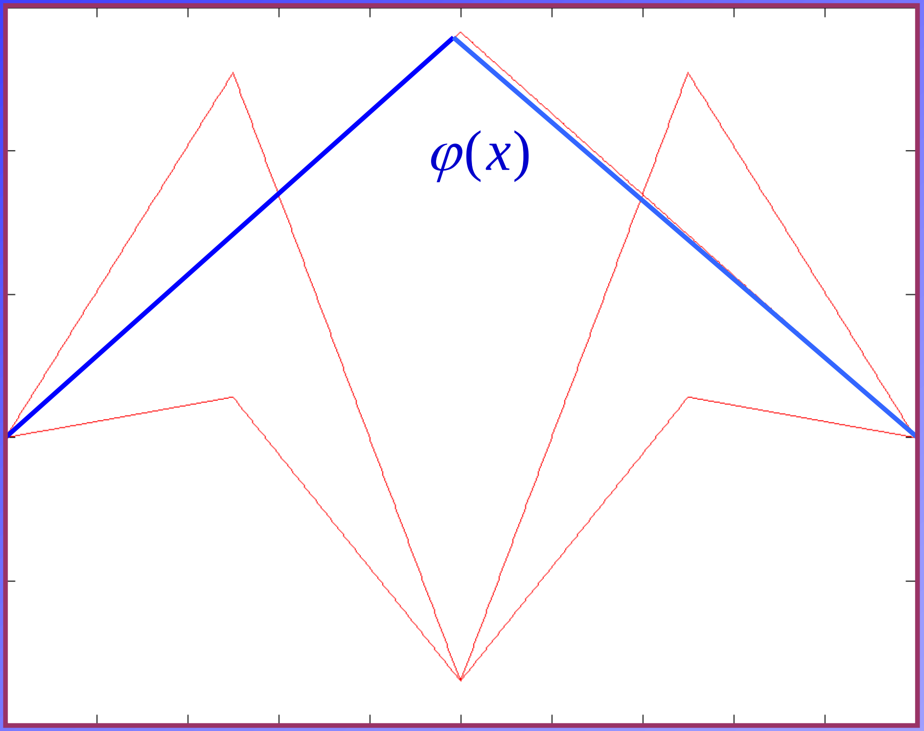




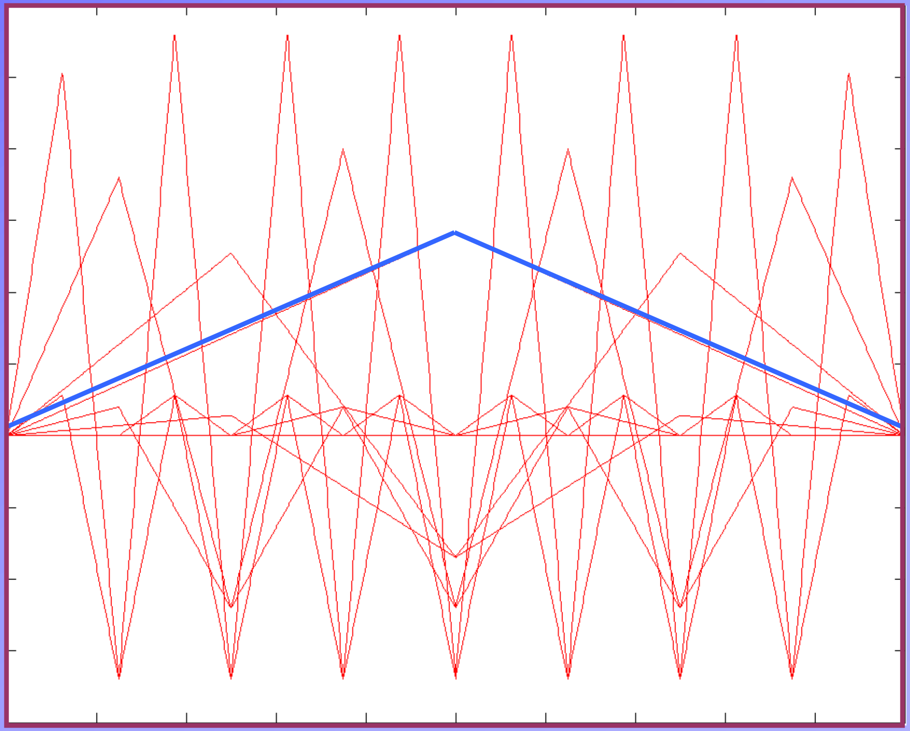
Chui Wang Wavelets

- Piecewise linear, Integration and differentiation are easy to calculate.
- Compactly supported wavelets ⇒ Sparse operator matrix

$V_0 \oplus W_0$

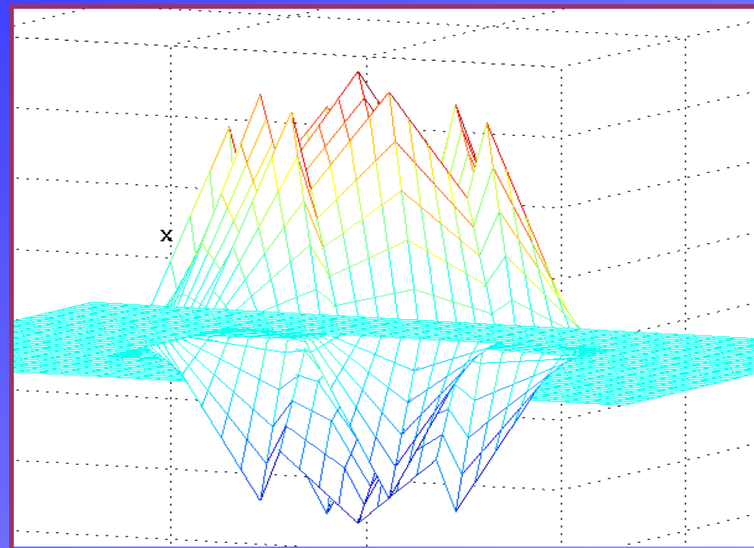


$V_0 \oplus W_0 \oplus W_1 \oplus W_2$

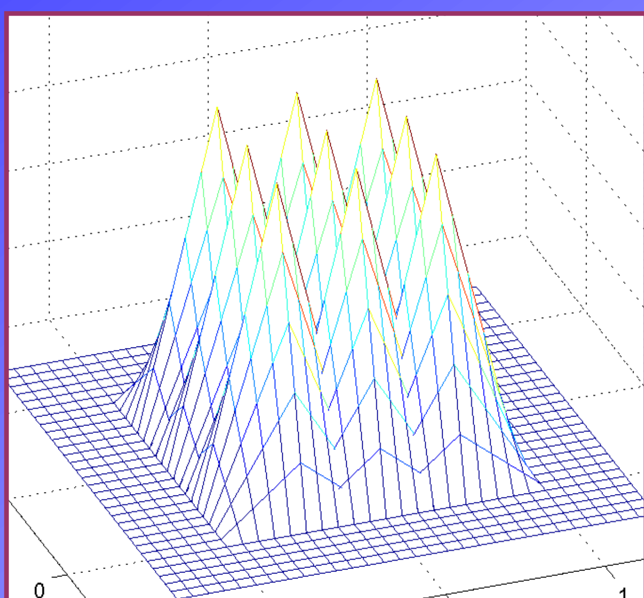


Chui Wang Wavelets 2D

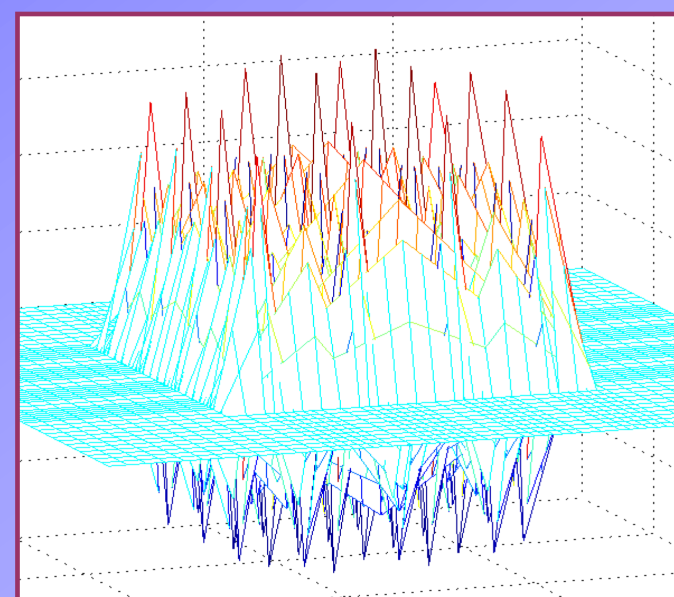
$$V_0 \oplus W_0$$



$$V_2$$



$$V_0 \oplus W_0 \oplus W_1$$



Equation solution

$$-\varepsilon^2 u'' + u = 1 \quad u \in [0,1], \quad u(0) = u(1) = 0$$

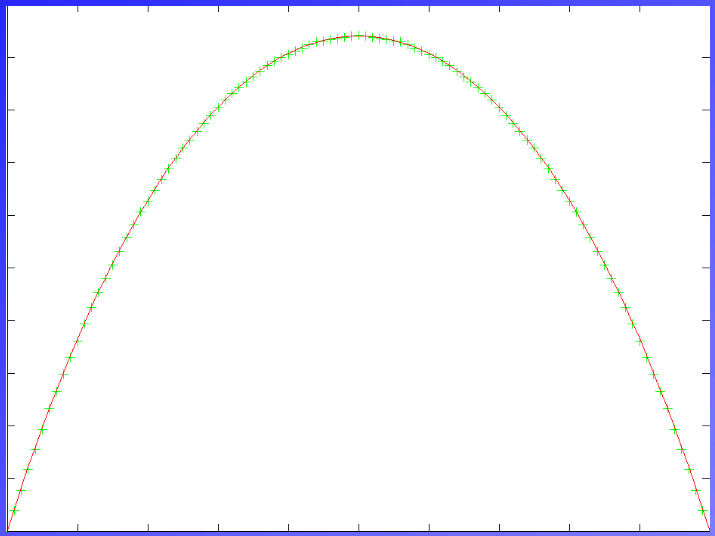
Analytical solution: $$u(x) = ae^{x/\varepsilon} + (-1-a)e^{-x/\varepsilon} + 1$$

Approximated solution $$U(x) = \sum_{j=1}^{N} \xi_j \varphi_j$$

Recall that $$L\xi = b$$

$$L_{i,j} = \int_0^1 \nabla\varphi_j \cdot \nabla\varphi_i \, dx + \int_0^1 \varphi_j \varphi_i \, dx, \quad b_i = \int_0^1 \varphi_i$$

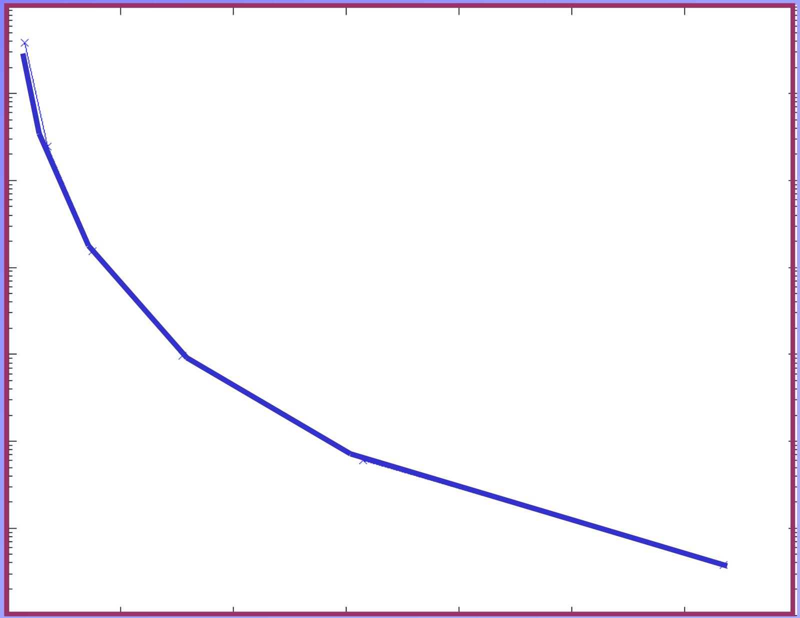
Results 1D



Analytic ——

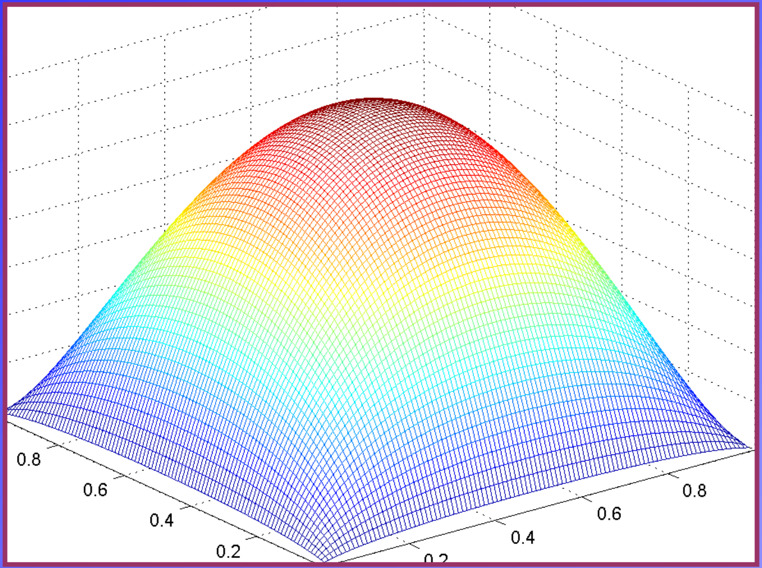
Approximated +

Log(Error) Vs. W level

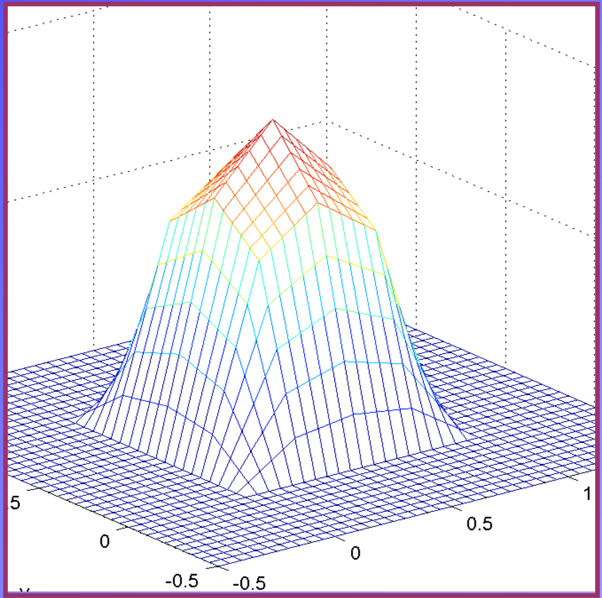


Results: 2D

Analytic



V=1, Error=3.604



V=3, Error=0.0158