Bounded Variation Functions and Γ-Convergence of the Mumford-Shah Functional

Leah Bar

2005

- A. Friedman, "*Foundations of modern analysis*", 1970
- A.N. Kolmogorov and S.V Fomin, "*Introductory Real Analysis*", 1970
- G. Aubert and P. Kornprobst, "*Mathematical Problems in Image Processing*", 2002
- A. Braides, "*Approximation of Free-Discontinuity Problems*", 1998

Monotonic Functions

Consider the indefinite Lebesgue integral

$$F(x) = \int_a^x f(t)dt$$

- if $f(t)$ is nonnegative, $F(x)$ is non decreasing function.
- Summable function $f(t)$ (Lebesgue integral exists) can be decomposed as the sum of two nonnegative functions

$$f(t) = f_+(t) - f_-(t)$$

- The study of Lebesgue integral as a function of upper limit is closely related to the study of monotonic functions.

- **Def:** <u>right-hand limit</u> of function $f(t)$ at x_0: $\lim_{\varepsilon \to 0} f(x_0+\varepsilon) = f(x_0+0)$

  <u>left-hand limit</u> $\lim_{\varepsilon \to 0} f(x_0-\varepsilon) = f(x_0-0)$

- **Def:** A function $f(t)$ is <u>continuous from the right</u> at x_0 if $f(x_0) = f(x_0+0)$

  and <u>continuous from the left</u> if $f(x_0) = f(x_0-0)$

- **Def:** Discontinuity point of the first kind: $f(x_0+0)$ and $f(x_0-0)$

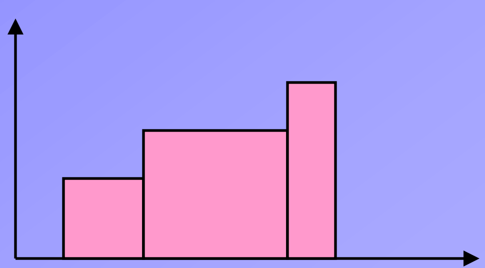
  exist but unequal. $jump = f(x_0+0) - f(x_0-0)$

- **Jump function:** Let $x_1 < x_2 < \ldots < x_n$ countable set in [a,b],

  $h_1, h_2, \ldots, h_n$ corresponding positive numbers

$$f(x) = \sum_{x_n < x} h_n$$

(monotonic function)



- Theorem: Every non decreasing function f on $[a\ b]$ is measurable and bounded, and hence summable.
- Theorem: A non decreasing function can have no more than countably many points of discontinuity.
- Theorem: The jump function is continuous from the left. Moreover, all the discontinuity points of f are of the first kind, with the jump at x_n equal to h_n.
- Theorem: If f is continuous from the left and non decreasing, then f is the sum of a continuous non decreasing function ϕ and a jump function ψ.

Proof: If $x_1, x_2, \ldots$ are the discontinuity points of f, with corresponding jumps $h_1, h_2, \ldots$, let

$$\psi(x) = \sum_{x_n < x} h_n$$

$$\phi(x) = f(x) - \psi(x).$$

$$\phi(x'')-\phi(x')=[f(x'')-f(x')]-[\psi(x'')-\psi(x')],\quad x'<x''.$$

This quantity is non negative ⇒ ϕ is non decreasing.

Moreover, given any point $x\in[a,b]$

$$\phi(x-0)=\lim_{\varepsilon\to 0} f(x-\varepsilon)-\lim_{\varepsilon\to 0}\psi(x-\varepsilon)=f(x-0)-\sum_{x_n<x}h_n$$

$$\phi(x+0)=\lim_{\varepsilon\to 0} f(x+\varepsilon)-\lim_{\varepsilon\to 0}\psi(x+\varepsilon)=f(x+0)-\sum_{x_n\le x}h_n$$

$$\phi(x+0)-\phi(x-0)=f(x+0)-f(x-0)-h=0$$

⇒ ϕ is continuous for every $x\in[a,b]$ ∎

- Theorem: Let f be any function summable on *[a,b]*. Then $\frac{d}{dx}\int_a^x f(t)dt$ exists and is finite for almost all x.

Proof: $f(t)=f_+(t)-f_-(t)$ where f_+ and f_- are non negative summable functions, so that

$$F(x)=\int_a^x f(t)dt=\int_a^x f_+(t)dt-\int_a^x f_-(t)dt=F_1(x)-F_2(x)$$

is the difference between two non decreasing functions F_1 and F_2. But F_1 and F_2 have finite derivatives a.e. so does F. ∎

- Theorem: Let f be any function summable on *[a,b]*. Then

$$\frac{d}{dx}\int_a^x f(t)dt=f(x) \quad a.e.$$

Functions of Bounded Variation

- Def: A function *f* defined on an interval *[a,b]* is of bounded variation if there is a constant *C>0* such that

$$\sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right| \le C$$

for every partition $a = x_0 < x_1 < \cdots < x_n = b \quad \forall n$

- Def: Let *f* be a function of BV, then the total variation of *f* on *[a,b]* is

$$V_a^b(f) = \sup \sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right|$$

Properties:

- $V_a^b(\alpha f) = |\alpha| V_a^b(f)$
- $V_a^b(f+g) = \sum_k \left| f(x_k) + g(x_k) - f(x_{k-1}) - g(x_{k-1}) \right| \le \sum_k \left| f(x_k) - f(x_{k-1}) \right| + \sum_k \left| g(x_k) - g(x_{k-1}) \right| = V_a^b(f) + V_a^b(g)$

⇒ BV is linear space

- Theorem: If $a < b < c$ then $V_a^c(f) = V_a^b(f) + V_b^c(f)$
- Corollary: The function $v(x) = V_a^x(f)$ is non decreasing (since the total variation is non negative).

- Theorem: If f is of BV on $[a,b]$, then f can be represented as the difference between two non decreasing functions on $[a,b]$.

  Proof: Let $v(x) = V_a^x(f)$ and $g = v - f$. If $x' < x''$,

  $g(x'') - g(x') = [v(x'') - v(x')] - [f(x'') - f(x')]$. But

  $|f(x'') - f(x')| \le v(x'') - v(x')$ by the definition of v. ⇨ $g(x)$ is non decreasing.

  □

- Corollary: Every function of BV has a finite derivative a.e.

- Corollary: If f is summable on $[a,b]$, then the indefinite integral

$$\Phi(x) = \int_a^x f(t)dt$$

is a function of BV on $[a,b]$.

- Theorem: Let F be a non decreasing function on $[a,b]$. Then the derivative F' is summable on $[a,b]$ and

$$\int_a^b F'(t)dt \le F(b)-F(a)$$

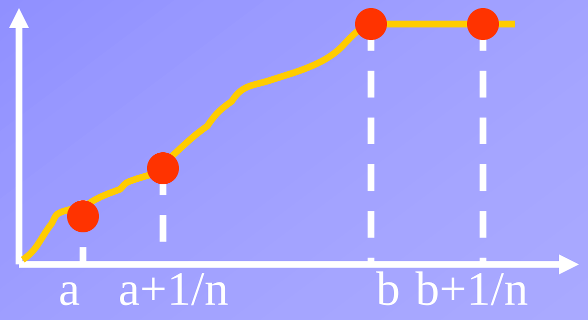
Proof: Let $\Phi_n(t)=\dfrac{F(t+1/n)-F(t)}{1/n}$, clearly

$$F'(t)=\lim_{n\to\infty}\frac{F(t+1/n)-F(t)}{1/n}=\lim_{n\to\infty}\Phi_n(t) \quad a.e.\ \forall t\in[a,b]$$

Since F is summable on $[a,b]$ so is Φ_n.

$$\int_a^b \Phi_n(t)dt = n\int_a^b [F(t+1/n)-F(t)]dt = n\left[\int_{a+1/n}^{b+1/n} F(t)dt-\int_a^b F(t)dt\right] = n\left[\int_b^{b+1/n} F(t)dt-\int_a^{a+1/n} F(t)dt\right]$$

Assume $F(t)=F(b)$ for $b<t<b+1$.



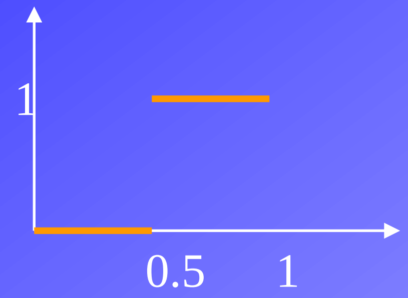
$$n\int_a^{a+1/n} F(t)dt > F(a)$$

$\Rightarrow$ $\displaystyle\int_a^b \Phi_n(t)dt \le F(b)-F(a)$ $\xrightarrow{\text{Fatou}}$ $\displaystyle\int_a^b F'(t)dt \le F(b)-F(a)$ ∎

When $\int_a^b F'(t)dt < F(b) - F(a)$?

- Discontinuous function

$$F(t) = \begin{cases} 0 & 0 \le t \le 0.5 \\ 1 & 0.5 < t \le 1 \end{cases}$$



$$0 = \int_0^1 F'(t)dt < F(1) - F(0) = 1$$

- Cantor function - Continuous function

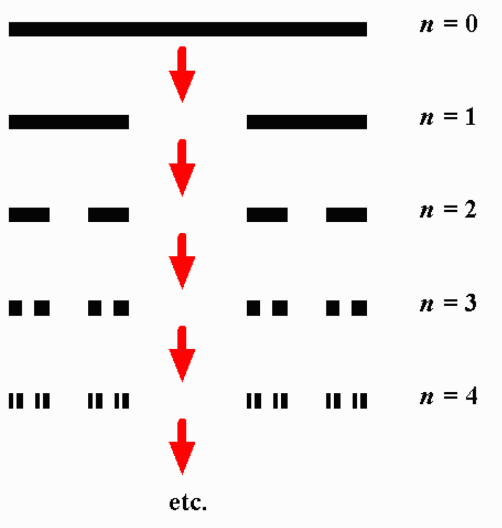
Cantor set: removing the open middle third interval starting with *[0,1].*

Cardinality: continuum (each point can be mapped as i.e. Left,Right,Right,Left....)

Measure: 0

$$\text{removed} = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots \frac{2^{n-1}}{3^n} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = 1$$

[0,1]/removed=0

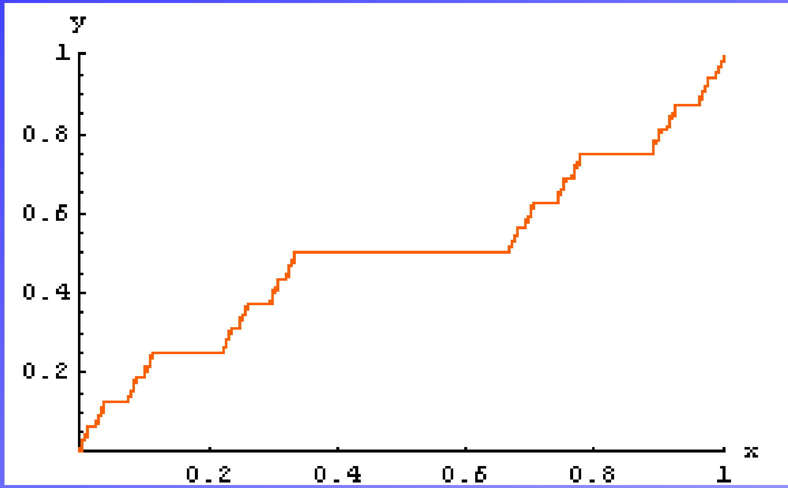


Cantor function

$$F(t)=\frac{2k-1}{2^n} \qquad t\in \underbrace{[a_k^{(n)},b_k^{(n)}]}_{\text{removed intervals}}$$

$$F(t)=\frac{1}{2} \qquad \frac{1}{3}\le t\le \frac{2}{3}$$

$$F(t)=\begin{cases}\frac{1}{4} & \frac{1}{9}\le t\le \frac{2}{9}\\ \frac{3}{4} & \frac{7}{9}\le t\le \frac{8}{9}\end{cases}$$



- The function is defined in the holes + edge points of the cantor set (0,1,1/3,2/3,1/9...).
- Non-decreasing
- Continuous:

Consider the point $1/3$. Obviously is it continuous from the right. Let t_n be an increasing sequence,

$$t_n = \frac{2}{9}, \frac{8}{27}, \frac{26}{81}, \ldots, \frac{3^{n-1}-1}{3^n} \quad n = 2,3,\ldots$$ each such point corresponds to segment $k=2^{n-2}$.

$$\Rightarrow \quad F_{k,n} = \frac{2k-1}{2^n}, \ \lim_{n\to\infty} F_{k,n} = \lim_{n\to\infty} \frac{2\cdot 2^{n-2}-1}{2^n} = \frac{1}{2}$$

Now consider the point $2/3$. Obviously is it continuous from the left.
Let t_n' ne a decreasing sequence,

$$t_n' = \frac{7}{9}, \frac{19}{27}, \frac{55}{81}, \ldots, \frac{2\cdot 3^{n-1}+1}{3^n} \quad n = 2,3,\ldots$$ each such point corresponds to segment $k=2^{n-2}+1$.

$$\Rightarrow \quad F_{k,n} = \frac{2k-1}{2^n}, \ \lim_{n\to\infty} F_{k,n} = \lim_{n\to\infty} \frac{2\cdot\left(2^{n-2}+1\right)-1}{2^n} = \frac{1}{2} \quad \Rightarrow$$ $F(t)$ is continuous

$$F'(t) = 0 \quad a.e. \quad \Rightarrow \quad 0 = \int_0^1 F'(t)dt < F(1) - F(0) = 1$$

Absolutely Continuous Functions

- Def: A function f defined on an interval $[a,b]$ is <u>absolutely continuous</u> if, given any $\varepsilon>0$, there is a $\delta>0$ such that

$$\sum_{k=1}^{n} \left| f(b_k) - f(a_k) \right| < \varepsilon$$

for every finite system of pairwise disjoint subintervals such that

$$\sum_{k=1}^{n} \left(b_k - a_k \right) < \delta$$

- Remarks: absolutely continuous $\Rightarrow$ uniformly continuous

  Cantor function is not absolutely continuous

- Theorem: If f is absolutely continuous on [a,b] f is of BV
- Theorem: absolutely continuous functions set is a linear subspace of BV.

- Theorem: The indefinite integral of a summable function f is absolutely continuous.

$$F(x)=\int_a^x f(t)dt$$

Proof: Given any finite collection of pairwise disjoint intervals (a_k,b_k),

$$\sum_{k=1}^{n}\left|F(b_k)-F(a_k)\right|=\sum_{k=1}^{n}\left|\int_{a_k}^{b_k} f(t)dt\right|\le\sum_{k=1}^{n}\int_{a_k}^{b_k}\left|f(t)\right|dt=\int_{\bigcup_k (a_k,b_k)}\left|f(t)\right|dt$$

but the last expression approaches 0 as the total length of the intervals approaches 0. ■

- Theorem: If F is absolutely continuous on $[a,b]$, then F' is summable on $[a,b]$ and

$$\int_a^x F'(t)dt=F(x)-F(a)$$

The Lebesgue Decomposition

Let f be a function of BV on $[a,b]$. f can be represented as the difference between two non decreasing functions and hence

$$f(x) = \phi(x) + \psi(x)$$

ϕ: continuous of BV; ψ: jump function

Let $\phi_1(x) = \int_a^x \phi'(t)\,dt, \quad \phi_2(x) = \phi(x) - \phi_1(x)$

⇒ ϕ_1 is absolutely continuous, ϕ_2 is continuous of BV s.t.

$$\phi_2'(x) = \phi'(x) - \frac{d}{dx}\int_a^x \phi'(t)\,dt = 0 \quad a.e.$$

A continuous function of BV is <u>singular</u> if its derivative vanishes a.e. (e.g. Cantor function).

⇒

$$f(x) = \phi_1(x) + \phi_2(x) + \psi(x)$$

ϕ_1: absolutely continuous; ϕ_2: singular; ψ: jump function

Basic Definitions in Measure Theory

- Def: Let X be a non empty set and let A be a collection of subsets of X. A is σ-algebra if:

$$\emptyset \in \mathrm{A}, \quad \forall E_h \in \mathrm{A}$$

$$\bigcup_h E_h \in A$$

$$\bigcap_h E_h \in A$$

- Def: Let $\mu : \mathrm{A} \to [0,+\infty]$. μ is positive measure if

$$\mu(\emptyset) = 0$$

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h) \quad \sigma - \text{additivity}$$

- Def: Let X be a non empty set and A be σ-algebra, $m \in \mathrm{N}, m \geq 1$

$\mu : A \to R^m$ is a measure if $\mu(\emptyset) = 0$ and σ-additive. if m=1, μ is real (signed) measure. If m>1, μ is a vector-valued measure. Radon measure if $X=R^n$.

Total Variation: $|\mu|(E) = \sup\left\{\sum_{h=0}^{\infty} |\mu(E_h)| : E_h \in A \text{ pairwise disjoint}, E = \bigcup_{h=0}^{\infty} E_h\right\}$

- Def: Let μ be a positive measure and let ν be a signed measure. ν is absolutely continuous with respect to μ , ν<<μ if $|\mu(E)| = 0 \Rightarrow \upsilon(E) = 0$

  μ and ν are mutually singular $\mu \perp \upsilon$ if there exist sets A,B such that

  $|\mu|(A) = 0, |\upsilon|(B) = 0$ where $X = A \cup B, \quad A \cap B = \varnothing$

- Theorem (Radon-Nikodym): Let (X,A,μ) be measure space with μ a bounded positive measure and ν a signed measure on A, absolutely continuous with respect to μ. Then there exists a real-valued function *f* on X such that

$$\upsilon(E) = \int_E f \, d\mu$$

The function *f* is called the Radon-Nikodym derivative of ν with respect to μ

$$f = \frac{d\upsilon}{d\mu}$$

- Theorem (Lebesgue decomposition): Let μ and ν be bounded signed measures in a measure space (X,A). Then there exist bounded signed measures ν_0 and ν_1 such that

$$\upsilon = \upsilon_0 + \upsilon_{1,} \quad \upsilon_0 \perp \mu \text{ and } \upsilon_1 << \mu$$

The pair (ν_0,ν_1) is uniquely determined.

Proof: $\upsilon_0 \perp \mu$ if there exist disjoint sets A,B such that $|\mu|(A)=0, \quad |\upsilon_0|(B)=0$

where $X = A \cup B, \quad A \cap B = \varnothing$

⇨ $\upsilon_0 \perp |\mu| \Rightarrow \upsilon_0 \perp \mu$

$\upsilon_1 << \mu$ if for any measurable set E $|\mu|(E)=0 \Rightarrow \upsilon_1(E)=0$

⇨ $\upsilon_1 << |\mu| \Rightarrow \upsilon_1 << \mu$

Hence it may be assumed that μ is a positive measure.

We may also assume that ν is positive measure, otherwise it could be decomposed to $\nu = \nu^+ + \nu^-$.

⇨ $\upsilon << \mu + \upsilon$

By the Radon-Nikodym theorem there exists a function f such that

$$\upsilon(E) = \int_E f\, d\mu + \int_E f\, d\upsilon$$

$$0 \le \upsilon(E) = \int_E f\, d\mu + \int_E f\, d\upsilon \le \mu(E) + \upsilon(E) \Rightarrow 0 \le f \le 1 \quad a.e. \quad \text{w.r.t. } \mu+\nu$$

⇒ $0 \le f \le 1 \quad a.e$ w.r.t. ν.

Let $A = \{x; f(x) = 1\}, \quad B = X - A, \quad \upsilon(A) = \int_A d\mu + \int_A d\upsilon = \mu(A) + \upsilon(A).$

since $\upsilon(A) < \infty \Rightarrow \mu(A) = 0.$

Let $\upsilon_0(E) = \upsilon(E \cap A), \quad \upsilon_1(E) = \upsilon(E \cap B)$ ⇒ $\upsilon_0 \perp \mu, \quad \upsilon_0 + \upsilon_1 = \upsilon$

Suppose μ(E)=0, then $\int_{E\cap B} d\upsilon = \int_{E\cap B} f\, d\upsilon, \quad \int_{E\cap B} (1-f)\, d\upsilon = 0$

But *1-f>0* a.e. w.r.t. ν ⇒ $\upsilon(E \cap B) = \upsilon_1(E) = 0$ ⇒ $\nu_1 << \mu$

Uniqueness: suppose another decomposition $\upsilon = \tilde{\upsilon}_0 + \tilde{\upsilon}_1$

Then, the signed measure $\lambda = \upsilon_0 - \tilde{\upsilon}_0 = \upsilon_1 - \tilde{\upsilon}_1$ is both singular and absolutely continuous -- contradiction.

□

The Space of BV Functions

- **Def:** Let Ω be a bounded open subset in R^n and $u \in L^1(\Omega)$,

$$\int_\Omega |Du| = \sup\left\{\int_\Omega u\nabla\cdot\varphi\, dx : \varphi = (\varphi_1, \varphi 2, \ldots, \varphi_N) \in c_0^1, |\varphi|_{L^\infty} \le 1\right\}$$

- **Example 1:** if $u \in c^1(\Omega) \Rightarrow \int_\Omega u\nabla\cdot\varphi\, dx = -\int_\Omega \nabla u\cdot\varphi\, dx$

$$-\nabla u\cdot\varphi \le |\nabla u| \Rightarrow \int_\Omega -\nabla u\cdot\varphi\, dx \le \int_\Omega |\nabla u|\, dx \quad \Longrightarrow \quad \sup_{\|\varphi\|_{L^\infty}\le 1}\left\{-\int_\Omega \nabla u\cdot\varphi\, dx\right\} \le \int_\Omega |\nabla u|\, dx.$$

Let $\varphi_n(x) = -\dfrac{\nabla u(x)}{|\nabla u(x)|} H_n(x)$

where $H_n(x)$ is a disk with radius r_n, smoothly decreased to 0.

$$-\int_\Omega \nabla u\cdot\varphi_n dx = \int_\Omega |\nabla u| H_n(x) dx \; *$$

$$\sup_{\|\varphi\|_{L^\infty}}\left\{-\int_\Omega \nabla u\cdot\varphi_n dx\right\} \ge \left\{-\int_\Omega \nabla u\cdot\varphi_n\, dx\right\},$$

$$\lim_{n\to\infty}\sup_{\|\varphi\|_{L^\infty}}\left\{-\int_\Omega \nabla u\cdot\varphi_n dx\right\} \overset{*}{\ge} \int_\Omega |\nabla u|\, dx \quad \Longrightarrow \quad \int_\Omega |Du| = \int_\Omega |\nabla u(x)| dx$$

- Example 2: if $u(x)=\begin{cases}-1 & -1\le x<0\\ +1 & 0<x\le 1\end{cases}$ then $\int_{-1}^{1} u\varphi'\,dx=-2\varphi(0)=\int_{-1}^{1} Du\,\varphi\,dx$

⇒ $Du=-2\delta(x)$ (distributional derivative), $\int_{-1}^{1}|Du|=2$

- Def: $BV(\Omega)$, is the space of functions of bounded variation as

$$BV(\Omega)=\left\{u\in L^1(\Omega);\int_{\Omega}|Du|<\infty\right\}$$

- If u is a BV function, then Du can be identified as a Radon vector-valued measure.

Let $u \in BV$ and $L : c_0^1(\Omega)^N \to \Re$ the functional defined by $L(\varphi) = \int_\Omega u \nabla \cdot \varphi \, dx$

Then L is linear, and since u is BV, $\sup \left\{ L(\varphi); \varphi \in c_0^1(\Omega)^N, |\varphi|_{L^\infty} \leq 1 \right\} = c < \infty$

⇒ $|L(\varphi)| \leq c |\varphi|_{L^\infty(\Omega)}$ ⇒ Linear bounded functional is also continuous

- Riesz Representation Theorem:

Let Ω be a compact set of R^n. For any positive continuous linear functional L on $c_0^1(\Omega)^N$ there exists a Radon measure μ (positive measure, finite on a compact set of R^n) and a μ-measurable function σ such that $|\sigma(x)| = 1 \; \mu - a.e.$

$$L(\varphi) = -\int_\Omega \sigma \cdot \varphi(x) d\mu, \quad \forall \varphi \;\; (x) \in c_0^1(\Omega)^N$$

⇒ $\int_\Omega u \nabla \cdot \varphi \, dx = -\int_\Omega \varphi(x) \cdot Du$ ⇒ Du is a Radon vector-valued measure

$$Du = \sigma \, d\mu$$

Let $d\mu = dx$, $\nu = Du$,

$$d\upsilon = d\upsilon_{ac} + d\upsilon_s = \frac{d\upsilon_{ac}}{d\mu} d\mu + d\upsilon_s \overset{\text{Radon Nykodim}}{=} \frac{d\upsilon}{d\mu} d\mu + d\upsilon_s$$

$$\Rightarrow \quad d\upsilon = \frac{d(Du)}{dx} dx + d\upsilon_s$$ Let $$\nabla u = \frac{d(Du)}{d\mu}$$

Approximate limits

Let $B(x,r)$ be the ball of center x and radius R, and let $u \in BV(\Omega)$,

$$u^+(x) = \text{ap} - \limsup_{y \to x} u(y) = \inf \left\{ t : \lim_{r \to 0} \frac{dx(\{u > t\} \cap B(x,r))}{r^N} = 0 \right\}$$

$$u^-(x) = \text{ap} - \liminf_{y \to x} u(y) = \sup \left\{ t : \lim_{r \to 0} \frac{dx(\{u < t\} \cap B(x,r))}{r^N} = 0 \right\}$$

Def: u is approximately continuous at x if $u^+(x) = u^-(x)$.

Def: Jump set of u is defined as $S(u) = \left\{ x \in \Omega : \nexists \ \text{ap} - \lim_{y \to x} u(y) \right\}$

$$d\upsilon_s = d\upsilon_s\big|_{S(u)} + d\upsilon_s\big|_{\Omega\backslash S(u)}$$

$$d\upsilon = \nabla u dx + d\upsilon_s$$

$$\left(u^+ - u^-\right) n_u \mathrm{H}^{N-1}\big|_{Su}$$

Cu (Cantor part)

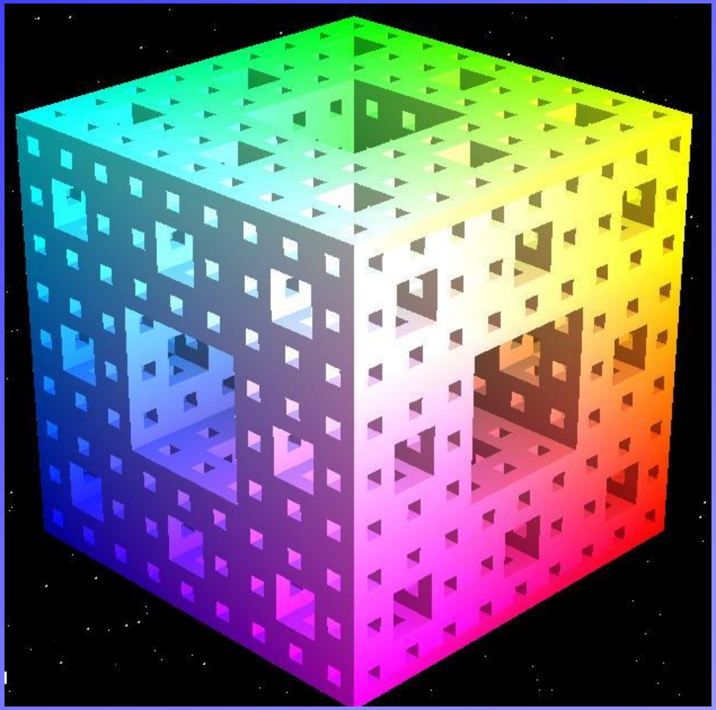
Approximate upper/lower limit

unit normal vector

Hausdorff measure

$$\int_\Omega |Du| = \int_\Omega |\nabla u| dx + \int_{S(u)} \left|u^+ - u^-\right| d\mathrm{H}^{N-1} + \int_{\Omega\backslash S(u)} |Cu|$$

Hausdorff Dim >N-1
(Fractal)



Example: Sierpinski Sponge has Hausdorff dimension of 2.7

Γ-convergence

A sequence $F_j : X \to [-\infty,+\infty]$ Γ-converges to $F : X \to [-\infty,+\infty]$ if:

1. liminf inequality $\quad \forall u_j \to u: \; F(u) \leq \liminf_j F_j(u_j)$
2. existence of recovery sequence $\quad \exists u_j \to u: \; F(u) \geq \limsup_j F_j(u_j)$

- Fundamental theorem of Γ-convergence: Suppose that $F=\Gamma\text{-}\lim_j F_j$, and let a compact set $K \subset X$ exist such that $\inf_x Fj=\inf_K Fj$ for all j.

  then $\exists \min_X F = \lim_j \inf_X F_j$.

  Moreover if u_j is a converging sequence such that $\lim_j Fj(uj)=\lim_j \inf_X Fj$ then its limit is a minimum point for F.

Proof: Let $u_j \subset K$ satisfy $\liminf_j F_j(u_j) = \liminf_j \inf_X F_j$.

There exists a subsequence $\left(u_{j_k}\right)$ converging to some u, such that

$$\lim_k F_{j_k}\left(u_{j_k}\right) = \liminf_j \inf_X F_j. \Rightarrow$$

$$\inf_X F \le F(u) \overset{1)}{\le} \liminf_k F_{j_k}\left(u_{j_k}\right) = \liminf_j \inf_X F_j, \quad *$$

$$\limsup_j \inf_X F_j \le \limsup_j F_j(u_j) \overset{2)}{\le} F(u)$$

This is satisfied for every u and in particular $\limsup_j \inf_X F_j \le \inf_X F(u)$.

$$\inf_X F \ge \limsup_j \inf_X F_j$$

$$\inf_X F \le \liminf_j \inf_X F_j \qquad (*)$$

$$\Rightarrow \quad \inf_X F = \lim_j \inf_X F_j$$

$$\Rightarrow \quad \exists \min_X = \lim_j \inf_X F_j$$

□

Approximation of the Mumford-Shah functional by Elliptic Functional

Let $\Omega \subset R^N$ be bounded open set. $g : \Omega \to R$ is a given function, $u : \Omega \to R$ and $S(u)$ the jump set of u.

$$F(u) = \int_\Omega |u-g|^2 + \beta \int_\Omega |\nabla u|^2 dx + \alpha H^{N-1}(S(u)), \quad u \in BV(\Omega)$$

But if u is a Cantor-like function, $\nabla u = 0 \quad a.e,$ $S(u) = \varnothing$

By the density of u in L^2, $\int_\Omega |u-g|^2 dx < \varepsilon$ we get $\inf_{u \in BV(\Omega)} F(u) = 0$

⇨ trivial solution.

In the Mumford-Shah functional we use the SBV space where the Cantor part $Cu=0$.

Mumford-Shah functional

- Theorem: Let $V:[0,1]\to[0,\infty)$ be a continuous function vanishing only at point 1, and $\psi:[0,1]\to[0,\infty)$ be an increasing lower semi continuous function with $\psi(0)=0, \psi(1)=1, \psi(t)>0$ if $t\neq 0$. Let the functional $G_\varepsilon : L^1(\Omega)\times L^1(\Omega)\to[0,\infty]$

$$G_\varepsilon(u,v)=\begin{cases}\int_\Omega\left(\psi(v)|u'|^2+\frac{1}{\varepsilon}V(v)+\varepsilon|v'|^2\right)dt & u,v\in H^1(\Omega)\\ \infty & \text{otherwise}\end{cases}$$

$$Cv=\int_0^1\sqrt{V(s)}\,ds$$

then G_ε Γ-converge to $G : L^1(\Omega)\times L^1(\Omega)\to[0,\infty]$

$$G(u,v)=\begin{cases}\int_\Omega\left(|u'|^2+4Cv\#\left(S(u)\right)\right)dt & u\in SBV\ (\Omega)\\ \infty & \text{otherwise}\end{cases}$$

- Lemma Given two sequences $t_j^1, t_j^2 \in I$ such that $u_j(t_1^j) \to z_1, \quad u_j(t_2^j) \to z_2$

$$\liminf_j \int_I \left(\frac{1}{\varepsilon_j} V(v_j) + \varepsilon_j |v_j'|^2 \right) dt \geq 2 \left| \int_{z_1}^{z_2} \sqrt{V(s)}\, ds \right|$$

and in particular for *{z1,z2}={0,1}*

$$\liminf_j \int_I \left(\frac{1}{\varepsilon_j} V(v_j) + \frac{\varepsilon_j}{p} |v_j'|^2 \right) dt \geq 2 \left| \int_0^1 \sqrt{V(s)}\, ds \right| = 2Cv$$

Proof: Recall the inequality $a^2 + b^2 \geq 2ab$

$$\int_I \left(\frac{V(v_j)}{\varepsilon_j} + \varepsilon_j |v_j'|^2 \right) dt \geq \left| \int_{t_j^1}^{t_j^2} \left(\frac{V(v_j)}{\varepsilon_j} + \varepsilon_j |v_j'|^2 \right) dt \right|$$

$$\geq 2 \left| \int_{t_j^1}^{t_j^2} \frac{\sqrt{V(v_j)}}{\sqrt{\varepsilon_j}} \sqrt{\varepsilon_j} |v_j'| dt \right| \geq 2 \left| \int_{v_j(t_j^1)}^{v_j(t_j^2)} \sqrt{V(s)}\, ds \right|.$$

Taking $\liminf_j$ of both sides we obtain the claim. This is the cost we pay for every jump that occurs in v.

Assume: $u_j \xrightarrow{L^1} u, \quad v_j \xrightarrow{L^1} v, \quad u_j \to u, \quad v_j \to v \ a.e., \quad \lim_j G_{\varepsilon_j}(u_j, v_j) < \infty$

$\Downarrow$

$v=1 \ a.e.$

Lemma (Edges): $\#S(u) < \infty$ for open $I \subset \Omega$,

$$4Cv\#\big(S(u) \cap I\big) \le \liminf G_{\varepsilon_j}(u_j, v_j, I)$$

Proof: choose $\{t_1, \ldots t_N\} \subset S(u)$ and disjoint intervals $I_i = (a_i, b_i) \subset \Omega, \quad t_i \in I_i.$

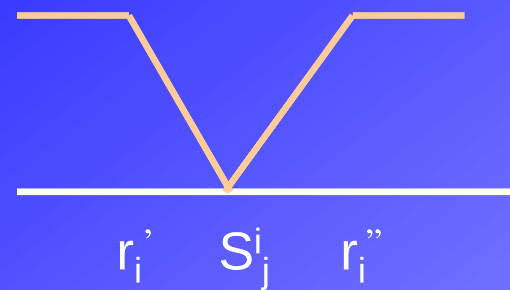
Let $t_i \in I_i^{'} \subset\subset I_i, \quad m_i = \liminf_j \inf_{t \in I_i^{'}} \psi\big(v_j(t)\big).$

If $m_i > 0$, $\displaystyle \int_{I_i^{'}} |u_j^{'}|^2 dt \le \frac{1}{m_i} \int_{I_i^{'}} \psi(v_j)|u_j^{'}|^2 dt \le \frac{c}{m_i}$

By the weak sequential compactness

$\Rightarrow$ $u_j \xrightarrow{\text{weakly in } H^1(I_i^{'})} u, u \in H^1 \subset C \Rightarrow S(u) \cap I_i^{'} = \varnothing$ Contradiction $\Rightarrow$ $m_i = 0$

$\Rightarrow$ $\exists (s_j^i) \subset I_i^{'}$ such that $v_j(s_j^i) \to 0$. But $v_j \to 1$ a.e. So there exist $r_i^{'}, r_i^{''} \in I_i^{'}, \ r_i^{'} < s_j^i < r_i^{''}$ such that $v_j(r_i^{'}) \to 1, v_j(r_i^{''}) \to 1.$



Since there are two jumps 1 to 0 and 0 to 1, we use the previous lemma twice.

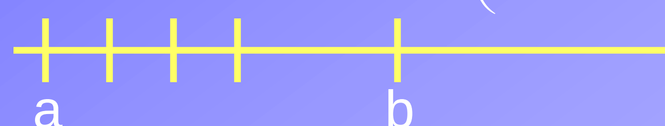
$\Rightarrow \liminf_j G_{\varepsilon_j}(u_j, v_j, I_i) \geq 4Cv$

But $\{t_1, \ldots t_N\} \subset S(u)$ $\Rightarrow$ $\liminf_j G_{\varepsilon_j}(u_j, v_j, I) \geq 4CvN$

and by the arbitrariness of $\{t_i\}$ $\boxed{\liminf_j G_{\varepsilon_j}(u_j, v_j, I) \geq 4Cv\#\left(S(u) \cap I\right)}$ ■

Lemma (segments): If $I \cap S(u) = \varnothing$ then $u \in H^1$ and $\int_I |u'|^2 dt \leq \liminf_j G_{\varepsilon_j}\left(u_j, v_j, I\right)$.

Proof: let $I = (a,b)$, $I \cap S(u) = \varnothing$, $I_N^k = \left(a + (b-a)\frac{k-1}{N}, a + (b-a)\frac{k}{N}\right)$



Let $0<z<1$ and consider the set $J_N^z = \left\{k \in \{1, \ldots N\} : \lim_j \inf_{I_N^k} v_j(t) \leq z\right\}$

Again, by the first lemma,

$$\#\left(J_N^z\right)4\int_z^1 \sqrt{V(s)}ds \le \liminf_j G_{\varepsilon_j}\left(u_j, v_j, I\right) \Rightarrow$$

$$\#\left(J_N^z\right) \le c\left(\int_z^1 \sqrt{V(s)}ds\right)^{-1} \text{ independen t on } N$$

suppose $J_N^z = \{k_i : 1 = 1, \ldots, L \ne L(N)\}, \quad a + (b-a)k_i / N \to t_i \in [a,b]$

Let $S = \{t_1, \ldots t_L\}, \eta > 0, \eta > 1/N$ ⇒ $I_N^k \subset S + [-\eta, \eta]$

⇒

$$\liminf_j \psi(z) \int_{I \setminus (s+[-\eta,\eta]} \left|u_j^{'}\right|^2 dt \le \liminf_j \sum_{k \ne J_N^z} \int_{I_N^k} \psi(z)\left|u_j^{'}\right|^2 dt$$

$$\overset{v_j > z}{\le} \liminf_j \sum_{k \ne J_N^z} \int_{I_N^k} \psi(v_j)\left|u_j^{'}\right|^2 dt \le \liminf_j \int_I \psi(v_j)\left|u_j^{'}\right|^2 dt$$

$$\le \liminf_j G_{\varepsilon_j}(u_j, v_j, I)$$

⇒

By the arbitrariness of η we obtain that u is $H^1(I \setminus S)$, but since $I \cap S(u) = \emptyset$

we have that: $u \in H^1(I), \quad \int_I \left|u_j^{'}\right|^2 dt \le \liminf_j G_{\varepsilon_j}(u_j, v_j, I) \quad (as\ z \to 1)$

□

Finally we integrate the edges and segments:

Set $I_\eta^0 = \Omega \setminus S(u) + [-\eta, \eta], \quad I_\eta^1 = S(u) + (-\eta, \eta) \cap \Omega$

$$\int_{I_\eta} |u'|^2 dt + 4Cv\#\big(S(u)\big) \leq \liminf_j G_{\varepsilon_j}(u_j, v_j, I^0, \eta) + \liminf_j G_{\varepsilon_j}(u_j, v_j, I^1, \eta)$$

$$= \liminf_j G_{\varepsilon_j}(u_j, v_j, \eta)$$

Letting $\eta \to 0$

$$\int_\Omega |u'|^2 dt + 4Cv\#\big(S(u)\big) \leq \liminf_j G_{\varepsilon_j}(u_j, v_j)$$

Construction of recovery sequence

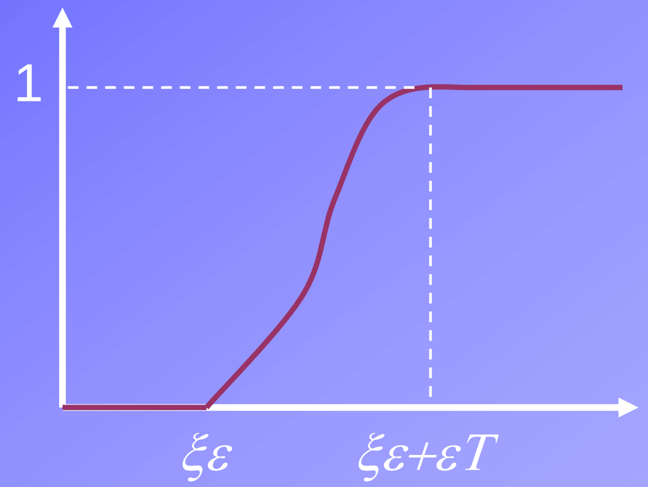
Assume: $\Omega=(-1,1),\ u\in SBV(\Omega),\ u'\in L^2(\Omega),\ S(u)=\{0\},$

$\xi_\varepsilon=o(\varepsilon^2),\ u_\varepsilon\in H^1(\Omega),\ u_\varepsilon(t)=u(t)\text{ if } |t|>\xi_\varepsilon$

Let $v\in H^1(0,T)$ be such that $\int_0^T\left(V(v)+|v'|^2\right)\le 2Cv+\eta,\quad v(0)=0, v(T)=1.$

Set

$$v_\varepsilon(t)=\begin{cases}0 & |t|<\xi_\varepsilon\\ v\left(\dfrac{|t|-\xi_\varepsilon}{\varepsilon}\right) & \xi_\varepsilon<|t|<\xi_\varepsilon+\varepsilon T\\ 1 & |t|>\xi_\varepsilon+\varepsilon T\end{cases}$$



$$G_\varepsilon(u_\varepsilon,v_\varepsilon)=\int_{-1}^{1}\left(\psi(v_\varepsilon)|u_\varepsilon'|^2+\frac{1}{\varepsilon}V(v_\varepsilon)+\varepsilon|v_\varepsilon'|^2\right)dt\le\int_{-1}^{1}\left(|u'|^2\right)dt+2\int_{0}^{1}\left(\frac{1}{\varepsilon}V(v_\varepsilon)+\varepsilon|v_\varepsilon'|^2\right)dt$$

$$=\int_{-1}^{1}\left(|u'|^2\right)dt+2\int_{\xi_\varepsilon}^{\xi_\varepsilon+\varepsilon T}\left(\frac{1}{\varepsilon}V(v(t-\xi_\varepsilon)/\varepsilon)+\varepsilon\,|\,v((t-\xi_\varepsilon)/\varepsilon)'|^2\right)dt+2\int_{0}^{\xi_\varepsilon}\frac{V(0)}{\varepsilon}dt$$

$$=\int_{-1}^{1}\left(|u'|^2\right)dt+\int_{0}^{T}\left(V(v)+|v'|^2\right)dt+2V(0)\frac{\xi_\varepsilon}{\varepsilon}\le\int_{-1}^{1}\left(|u'|^2\right)dt+4Cv+2\eta+2V(0)\frac{\xi_\varepsilon}{\varepsilon}$$

arbitrary η, $\#(S(u))=1$, $\xi_\varepsilon/\varepsilon=o(\varepsilon)$

$$\limsup_{\varepsilon\to 0}{}_j\, G_\varepsilon(u,v)\le\int_{-1}^{1}|u'|^2\,dt+4Cv\#S(u)$$

Weak sequential compactness

Let X be a reflexive Banach space, $K > 0$, and $x_n \in X$ a sequence such that $\|x_n\|_X \le K$. Then there exist $x \in X$ and a subsequence x_{n_j} such that $x_{n_j} \overset{\text{weak}}{\to} x \ (n \to \infty)$.

Back

- Theorem: Let $p>1$, and $p'=p/(p-1)$, let $W : R \to [0,\infty]$ be a continuous function such that $W(z)=0$ iff $z \in \{0,1\}$ let $P_\varepsilon : L^1(\Omega) \to [0,\infty]$ defined by

$$P_\varepsilon(u) = \begin{cases} \frac{1}{\varepsilon p'} \int_\Omega W(u(t))dt + \frac{\varepsilon^{p-1}}{p} \int_\Omega |u'|^p dt & u \in W^{1,p} \\ \infty & \text{otherwise} \end{cases}$$

and let $C_p = \int_0^1 (W(s))^{1/p'} ds$

then $P\varepsilon$ Γ-convergence to CpP with respect to $L^1(\Omega)$-distance.

$$P(u) = \begin{cases} \#(S(u)) & \text{if } u \in \{0,1\}\, a.e. \\ \infty & \text{otherwise.} \end{cases}$$

Proof: for the sake of notations, for all open sets I in R. we set

$$P_\varepsilon(u,I)=\begin{cases}\dfrac{1}{\varepsilon p'}\displaystyle\int_I W(u(t))dt+\dfrac{\varepsilon^{p-1}}{p}\int_I |u'|^p\,dt & u\in W^{1,p}(I)\\ \infty & \text{otherwise}\end{cases}$$

Assume: $u\in L^1(\Omega),\quad u_j\xrightarrow{L^1}u,\quad \sup_j P_{\varepsilon_j}<\infty$

Since $\displaystyle\int_\Omega W(u_j)dt<c\varepsilon_j$ $\Rightarrow$ $u\in\{0,1\}\ a.e.$

Suppose $t_1,\ldots t_N\in S(u)$ then we can find $a_i^\pm\in\Omega$ such that $(a_i^-,a_i^+)\subset\Omega$

$$a_i^-<t_i<a_i^+\le a_{i+1}^- \qquad \lim_j u_j(a_i^\pm)=u(a_i^\pm)\in\{0,1\}$$

$\Rightarrow$ $$P_{\varepsilon_j}(u_j)\ge\sum_{i=1}^N P_{\varepsilon_j}\left(u_j,(a_i^-,a_i^+)\right)$$

Change of Variables $a_i = \frac{a_i^+ + a_i^-}{2\varepsilon_j}, \quad v_j^i(t) = u_j(\varepsilon_j(t + a_i)), \quad T_j^i = \frac{a_i^+ - a_i^-}{2\varepsilon_j}$

$$\Rightarrow \int_{(a_i^-, a_i^+)} W(u_j(s))ds = \int_{(a_i^-, a_i^+)} W\left(v_j^i\left(s/\varepsilon_j - a_i\right)\right)ds = \varepsilon_j \int_{(-T_j^i, T_j^i)} W\left(v_j^i(t)\right)dt$$

$$\int_{(a_i^-, a_i^+)} \left|u_j'\right|^p ds = \int_{(a_i^-, a_i^+)} \left|\frac{1}{\varepsilon_j}\left(v_j^i\left(s/\varepsilon_j - a_i\right)\right)'\right|^p ds = \varepsilon_j^{1-p} \int_{(-T_j^i, T_j^i)} \left(v_j^i(t)\right)' dt$$

$$\Rightarrow P_{\varepsilon_j}\left(u_j, (a_i^-, a_i^+)\right) = \frac{1}{\varepsilon_j p'} \int_{(a_i^-, a_i^+)} W\left(u_j(t)\right)dt + \frac{\varepsilon_j^{p-1}}{p} \int_{(a_i^-, a_i^+)} \left|u_j'\right|^p dt = P_1\left(v_j^i, \left(-T_j^i, T_j^i\right)\right)$$

$$\Rightarrow P_{\varepsilon_j}\left(u_j, (a_i^-, a_i^+)\right) \geq \inf_{T \geq 0} \inf \left\{P_1\left(v, (-T, T)\right) : v \in W^{1,p}(-T, T), v(\pm T) = u_j(a_i^\pm)\right\}$$

By Young's inequality $\frac{a^{p'}}{p'} + \frac{b^p}{p} \geq ab, \quad \frac{1}{p'} + \frac{1}{p} = 1$

$$\frac{1}{p'} W(v) + \frac{1}{p}\left|v'\right|^p \geq W^{1/p'}(v)\,|v'| = \left|\nabla\left(\Phi(v)\right)\right| \quad \text{where } \Phi(z) = \int_0^z W^{1/p'}(s)ds$$

This implies that if $v \in W^{1,p}(a,b)$ then

$$\int_{(a,b)} \left(\frac{1}{p'} W(v) + \frac{1}{p} |v'|^p \right) dt \geq \int_{(a,b)} |\nabla(\Phi(v))| dt \geq |\Phi(v(b)) - \Phi(v(a))|$$

Set $\tilde{c} = \liminf_{(z,w)\to(0,1)} \inf_{T\geq 0} \inf \{ P_1(v,(-T,T)) : v \in W^{1,p}(-T,T), v(-T)=z, v(T)=w \}$

$$\Rightarrow \quad \tilde{c} \geq \liminf_{(z,w)\to(0,1)} |\Phi(w) - \Phi(z)| = \Phi(1) = Cp > 0$$

$$\Rightarrow \quad \liminf_j P_{\varepsilon_j}(u_j) \geq \tilde{c} N \geq CpN$$

By the arbitrariness of $\{t_1, \ldots t_N\} \subset S(u)$

$$\boxed{\liminf_j P_{\varepsilon_j}(u_j) \geq \tilde{c}\,\#(S(u)) \geq Cp\#(S(u))}$$

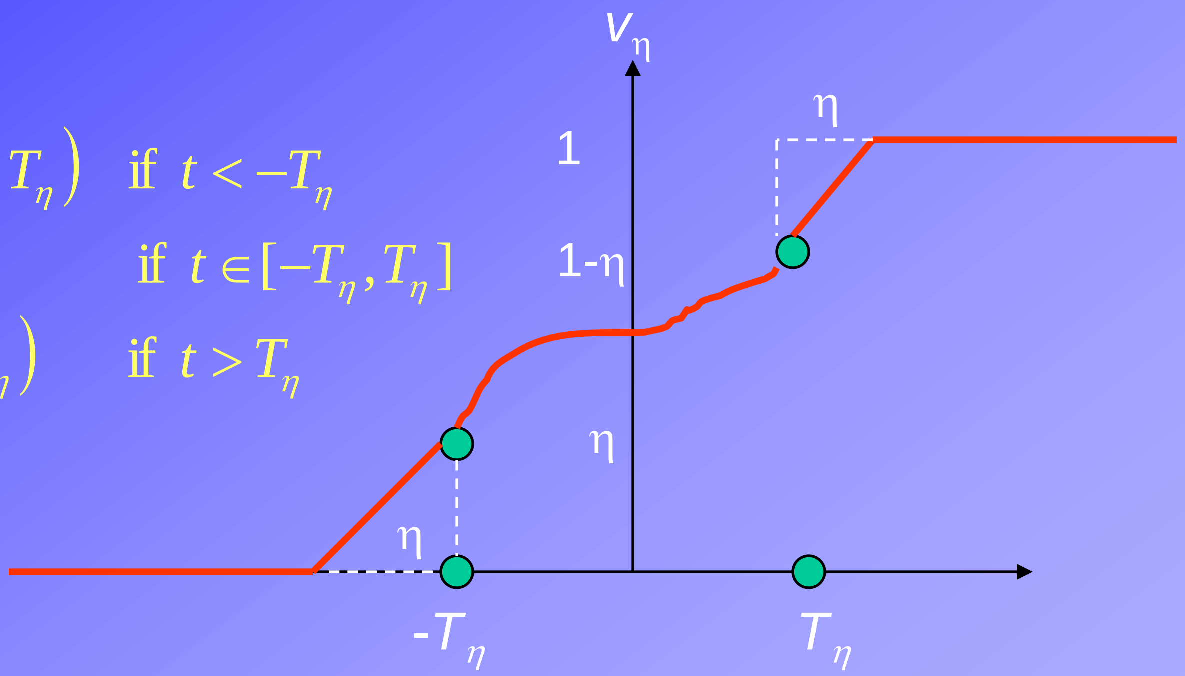
Constructing recovery sequence

For fixed $\eta > 0$, choose $T_\eta > 0$ and $v_\eta \in W^{1,p}(-T_\eta, T_\eta)$ such that

$$P_1\left(v_\eta, (-T_\eta, T_\eta)\right) \le \tilde{c} + \eta, \qquad 0 \le v_\eta\left(-T_\eta\right) \le \eta, \quad 1 - \eta \le v_\eta\left(T_\eta\right) \le 1$$

Extend v_η to R:

$$v_\eta(t) = \begin{cases} 0 \vee \left(v_\eta(-T_\eta) + t + T_\eta\right) & \text{if } t < -T_\eta \\ v_\eta(t) & \text{if } t \in [-T_\eta, T_\eta] \\ 1 \wedge \left(v_\eta(T_\eta) + t - T_\eta\right) & \text{if } t > T_\eta \end{cases}$$



$$P_1(v_\eta, R) = P_1(v_\eta, (-T_\eta, T_\eta)) + P_1(v_\eta, R \setminus [-T_\eta, T_\eta]) \le \tilde{c} + \eta + \eta \frac{2}{p'} \sup\{W(s) : s \in [0,1]\}$$

Change of Variables

$$u_\varepsilon(t)=\begin{cases} v_\eta(t/\varepsilon) & t\in[-\varepsilon(T_\eta+\eta),\varepsilon(T_\eta+\eta)] \\ u(t) & \text{otherwise}\end{cases}$$

$$\Rightarrow\quad \limsup_{\varepsilon\to 0} P_\varepsilon(u_\varepsilon)=P_1(v_\eta,R)\le\tilde{c}+c\eta \overset{\text{arbitrary}\,\eta}{\le} \tilde{c}$$

If u is BV and $u=\{0,1\}$ a.e. then repeating the same argument near each point of $S(u)$

$$\limsup_{\varepsilon\to 0} P_\varepsilon(u_\varepsilon)\le\tilde{c}\,\#\big(S(u)\big)$$

Choose $v\in W^{1,1}_{loc}(R)$ satisfying $\begin{cases} v'=W(v)^p \ a.e. \\ v(-\infty)=0,\ v(+\infty)=1\end{cases}$

$$\Rightarrow$$

$$\frac{1}{p'}W(v)+\frac{1}{p}|v'|^p \overset{v\,\text{increasing}}{=} \left(\frac{1}{p'}+\frac{1}{p}\right)W(v)=W(v)^{1/p'}W(v)^{1/p}=W(v)^{1/p'}|v'|=|\nabla\Phi(v)| \overset{\Phi\,\text{increasing}}{=} \nabla\big(\Phi(v)\big)$$

$$Cp=\lim_{T\to\infty}\big(\Phi(v(T)-\Phi(v(-T)\big)=\lim_{T\to\infty}\int_{(-T,T)}\left(\frac{1}{p'}W(v)+\frac{1}{p}|v'|^p\right)dt\ge\tilde{c}$$

But we proved earlier that $Cp\le\tilde{c}\Rightarrow Cp=\tilde{c}$

$$\Rightarrow\quad \boxed{\limsup_{\varepsilon\to 0} P_\varepsilon(u_\varepsilon)\le Cp\#\big(S(u)\big)}$$

∎