

CONVERGENCE OF AN ITERATIVE METHOD FOR VARIATIONAL DECONVOLUTION AND IMPULSIVE NOISE REMOVAL*

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Abstract. Image restoration, i.e. the recovery of images that have been degraded by blur and noise, is a challenging inverse problem. A unified variational approach to edge-preserving image deconvolution and impulsive noise removal has been recently suggested by the authors and shown to be effective. It leads to a minimization problem that is iteratively solved by alternate minimization for both the recovered image and the discontinuity set. The variational formulation yields a nonlinear integro-differential equation. This equation was linearized by fixed point iteration. In this paper, we analyze and prove the convergence of the iterative method.

Key words. variational image deconvolution, convergence analysis

AMS subject classifications. 49N45, 65K10

1. Introduction. In the variational approach for image processing and analysis it is customary to define the desired results, e.g. denoised image, segmenting curve, optical flow vector field, as the minimum of a functional. The object of interest is then found by calculating the critical point of the functional. The critical point satisfies a necessary condition which is the Euler-Lagrange differential equation(s). It has been realized in the last decades that non-quadratic terms in the functional are needed and lead to much better results in a variety of image analysis problems. These formulations revolutionized many domains and are state-of-the-art methods. However, the introduction of non-quadratic terms comes with a price: The Euler-Lagrange differential equations become non-linear. It is therefore harder to analyze the equations and to numerically solve them.

One of the most popular ways to deal with these non-linear differential equations is to approximate them by a series of linear equations. The series of solutions of these linear equations is supposed to converge to the solution of the non-linear PDE. The linearization is obtained by time/iteration delay of the non-linear part. Clearly, if the series converges, then it converges to the solution of the non-linear equation. The question is: does it converge?

In the variational approach for image restoration and denoising, the cost functional consists of a data fidelity term and a regularizer. Non-quadratic regularizers, such as Total Variation (TV), are known to facilitate edge preserving restoration [16, 8, 15], that is difficult to accomplish with quadratic regularization. Non-quadratic regularization terms induce nonlinear differential equations. Vogel and Oman [17] approached the TV restoration problem by fixed point iteration, where the nonlinear term was lagged by one iteration. Later, Dobson and Vogel [9] proved the convergence of that scheme in the case of the TV denoising problem.

In the present study we prove the convergence of an effective algorithm for image deconvolution in the presence of impulsive noise, that includes the fixed point

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iteration method. The problem statement, the objective functional and the resulting differential equations are entirely different than in [9]; yet their work has been a source of inspiration.

2. Problem Statement. Consider the problem of image deblurring in the presence of impulsive noise. A unified variational technique, based on Mumford-Shah regularization, was recently developed [3, 4]. It recovers the degraded image, preserves and extracts the edges, and removes the impulsive noise. Specifically, the pair (u, K) is the minimizer of the cost functional

$$\mathcal{F}(u, K) = \int_{\Omega} \sqrt{(h * u - g)^2 + \eta} \, dx + G(u, K) \quad (2.1)$$

where

$$\begin{aligned} G(u, K) &= \beta \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \alpha \mathcal{H}^{N-1}(K), \\ u &\in H^1(\Omega \setminus K), K \subset \Omega \text{ closed in } \Omega \end{aligned} \quad (2.2)$$

Here α and β are positive scalars, Ω is an open bounded set on $\mathbb{R}^n$, $g : \Omega \rightarrow \mathbb{R}^N \in L^\infty$ and $u : \Omega \rightarrow \mathbb{R}^N$ are respectively the observed and original images, where g is normalized to the hypercube $[0, 1]^N$, h is the blur kernel such that $h(x) > 0$ and $\int h(x) \, dx = 1$, K denotes the discontinuity set, $\mathcal{H}^{N-1}(K)$ is the total edge length expressed as the $N - 1$ dimensional Hausdorff measure, and $0 < \eta \ll 1$ is a small scalar.

The first term in (2.1) stands for robust data fidelity. The widely used quadratic term is incompatible with the impulsive noise. It assigns too much weight to distant points. The effect of such outlier data is reduced by using a robust function, modified L^1 norm in our case. The use of L^1 fidelity norm for impulse noise removal was first considered by Nikolova [14].

The regularizer $G(u, K)$ reflects the underlying piecewise-smooth model of images, following Mumford and Shah [12, 13]. The existence of minimizers to the original Mumford-Shah segmentation problem has been proved by De Giorgi *et al* [11] by setting

$$G(u) = \int_{\Omega} \beta |\nabla u|^2 \, dx + \alpha \mathcal{H}^{N-1}(S_u), \quad (2.3)$$

where S_u is the discontinuity set of u and $u \in SBV(\Omega)$, the special class of bounded variation such that the Cantor part of the BV measure is zero [5].

Ambrosio and Tortorelli [1, 2] introduced a variational approximation $G_\epsilon(u, v)$ to $G(u)$ via an elliptic functional where $\epsilon \rightarrow 0^+$. The discontinuity set S_u was represented by the characteristic function $(1 - \chi_{S_u})$, which was approximated by a smooth auxiliary function v :

$$G_\epsilon(u, v) = \beta \int_{\Omega} (v^2 + o_\epsilon) |\nabla u|^2 \, dx + \alpha \int_{\Omega} \left(\epsilon |\nabla v|^2 + \frac{(v - 1)^2}{4\epsilon} \right) \, dx, \quad (2.4)$$

and o_ϵ is a non-negative constant tending to zero faster than ϵ . They proved that a weak solutions exist such that $u, v \in H^1(\Omega)$, and that $G_\epsilon(u, v)$ Γ -converges to $G(u)$. Maximum principle was proved as well. We give a proof below as this is needed for the sequel.

Replacing the Mumford-Shah regularization terms in (2.1) by G_ϵ yields

$$\begin{aligned}\mathcal{F}(u, v) &:= \int \Theta(u, v) dx \\ &:= \int_{\Omega} \sqrt{(h * u - g)^2 + \eta} \, dx + \beta \int_{\Omega} (v^2 + o_\epsilon) |\nabla u|^2 \, dx + \alpha \int_{\Omega} \left(\epsilon |\nabla v|^2 + \frac{(v-1)^2}{4\epsilon} \right) dx.\end{aligned}\tag{2.5}$$

By the stability property of the Γ -convergence under continuous perturbations,

$$\Gamma - \lim_{\epsilon} (G_\epsilon + F) = G + F,$$

where F is a continuous function, the existence of minimizers to 2.5 is assured. In this case the fidelity term F is continuous with respect to u .

Minimization with respect to u and v is carried out using the first Gâteaux variation of the functional with respect to u and v subject to the Neumann boundary conditions $\partial v / \partial n = 0$ and $\partial u / \partial n = 0$ on $\partial\Omega$, where n denotes the normal to the boundary:

$$\delta_v(u, v; \psi) = 0, \quad \delta_u(u, v; \phi) = 0$$

where

$$\begin{aligned}\delta_v(u, v; \psi) &= \int_{\Omega} \left(\Theta_v(u, v) \psi + \sum_{i=1}^N \Theta_{v_{x_i}}(u, v) \psi_{v_{x_i}} \right) dx \\ &= \int_{\Omega} \left(2\beta v |\nabla u|^2 \psi + \alpha \left(\frac{v-1}{2\epsilon} \right) \psi + 2\alpha \epsilon (v_x \psi_x + v_y \psi_y) \right) dx,\end{aligned}$$

and

$$\begin{aligned}\delta_u(u, v; \phi) &= \int_{\Omega} \left(\Theta_u(u, v) \phi + \sum_{i=1}^N \Theta_{u_{x_i}}(u, v) \phi_{v_{x_i}} \right) dx \\ &= \int_{\Omega} \left(\frac{(h * u - g)(h * \phi)}{\sqrt{(h * u - g)^2 + \eta}} + 2\beta(v^2 + o_\epsilon)(u_x \phi_x + u_y \phi_y) \right) dx.\end{aligned}$$

The corresponding Euler-Lagrange (E-L) equations, under the assumption that $u, v \in C^2(\Omega)$, take the form

$$2\beta v |\nabla u|^2 + \alpha \left(\frac{v-1}{2\epsilon} \right) - 2\epsilon \alpha \nabla^2 v = 0 \tag{2.6}$$

$$h(-x) * \frac{(h * u - g)}{\sqrt{(h * u - g)^2 + \eta}} - 2\beta \nabla \cdot ((v^2 + o_\epsilon) \nabla u) = 0. \tag{2.7}$$

Since the objective functional (2.5) is convex, lower bounded and coercive with respect to either u or v , following [6, 7] the alternate minimization (AM) approach is applied: in each step n of the iterative procedure, we minimize with respect to one function and keep the other one fixed.

Obviously, Eq. (2.6) is a linear partial differential equation with respect to v . In contrast, (2.7) is a nonlinear integro-differential equation. *Linearization* of this equation is carried out using the fixed point iteration scheme, as in [6, 17]. Specifically,



FIG. 2.1. *Image deblurring in the presence of salt and pepper noise. Left: Source image blurred by a pill-box kernel of radius 3, and contaminated by noise of density 0.1. Right: Recovered image using the algorithm analyzed in this paper.*

an internal iterative process, indexed by an additional counter l , is carried out for the calculation of u^{n+1} . We set u^l in the denominator, and $u = u^{l+1}$ elsewhere, where l is the current iteration number. Adopting the notations of [9], equation (2.7) can thus be modified in the form

$$\mathcal{A}(v, u^l)u^{l+1} = \mathcal{G}(u^l), \quad l = 0, 1, \dots \quad (2.8)$$

where $\mathcal{A}$ is the linear integro-differential operator

$$\mathcal{A}(v, u^l) = \mathcal{A}_1(u^l) + \mathcal{A}_2(v, u^l)$$

such that

$$\mathcal{A}_1(u^l)u^{l+1} := h(-x) * \frac{h * u^{l+1}}{\sqrt{(h * u^l - g)^2 + \eta}}, \quad (2.9)$$

$$\mathcal{A}_2(v, u^l)u^{l+1} = \mathcal{A}_2(v)u^{l+1} := -2\beta \nabla \cdot ((v^2 + o_\epsilon) \nabla u^{l+1}), \quad (2.10)$$

and

$$\mathcal{G}(u^l) := h(-x) * \frac{g}{\sqrt{(h * u^l - g)^2 + \eta}}. \quad (2.11)$$

3. Iterative Algorithm. The iterative algorithm for minimizing the functional (2.5) is:

1. Solve the elliptic equation for v^{n+1}

$$(2\beta |\nabla u^n|^2 + \frac{\alpha}{2\epsilon} - 2\alpha \epsilon \nabla^2) v^{n+1} = \frac{\alpha}{2\epsilon} \quad (3.1)$$

2. Set $u^{n+1,0} = u^n$ and solve for u^{n+1} (iterating on l)

$$\mathcal{A}(v^{n+1}, u^{n+1,l})u^{n+1,l+1} = \mathcal{G}(u^{n+1,l}) \quad (3.2)$$

3. if $(\|u^{n+1} - u^n\|_{L_2} < \varepsilon_1 \|u^n\|_{L_2})$ stop.

Here $0 < \varepsilon_1 \ll 1$. Figure 2.1 exemplifies the performance of the suggested algorithm. The 256×256 *Lena* image was blurred by a pill-box kernel of radius 3 and contaminated by salt and pepper noise of density 0.1 (shown left). The outcome of the proposed algorithm is shown on the right. In this example the parameters were set to $\beta = 0.1$, $\alpha = 0.5$, $\epsilon = 0.1$ and $\eta = 10^{-4}$. The algorithm was implemented in the MATLAB[®] environment on a Pentium[®] 4 computer. The convergence tolerance of $\varepsilon_1 = 10^{-4}$ was reached with 4 external iterations (over n). The number of internal iterations (over l) was 10, and running time was 2.35 minutes.

4. Convergence of the Alternate Minimization Scheme. First, we show the convergence of the alternate minimization (AM) algorithm. In the work of Charbonnier *et al* [8], the restoration problem (in the presence of Gaussian noise) is approached via the half quadratic regularization with the alternate minimization method. In their formulation, the functional consists of an L_2 data fidelity and a potential function $\varphi(\nabla u)$ which serves as a regularizer. In the half quadratic regularization two auxiliary variables $b = (b_x, b_y)$ are introduced in order to make the manipulation of the regularizer simpler (quadratic in u). The E-L equations for the auxiliary variables are *algebraic*. Although in our case the cost functional and differential equations are different in both the fidelity and regularization terms, and the equation for the auxiliary variable is differential rather than algebraic, their proof for the AM convergence can be adapted to our problem.

Before we proceed we need few technical results. Let us first note that we work on a closed simply connected subset of $\mathbb{R}^n$ denoted by Ω . The boundary of Ω is denoted by $\partial\Omega$ and is assumed to satisfy the interior sphere condition: for every point $x \in \partial\Omega$ there exists an open interior sphere $B(y, R) \subset \Omega$ such that $x \in \partial B(y, R)$.

LEMMA 1. *Let $v(x)$ satisfy equation (2.6) where $u(x)$ is fixed, and let $\partial\Omega$ satisfy an interior sphere condition. Then $0 < v(x) \leq 1 \quad \forall x \in \Omega$.*

Proof. Equation (2.6) can be rewritten as

$$-\nabla^2 v + \left(\frac{\beta}{\epsilon\alpha} |\nabla u|^2 + \frac{1}{4\epsilon^2} \right) v = \frac{1}{4\epsilon^2} \quad (4.1)$$

Define the function

$$p(x) := - \left(\frac{\beta}{\epsilon\alpha} |\nabla u|^2 + \frac{1}{4\epsilon^2} \right).$$

Clearly $p(x) < 0$, $\forall x \in \Omega$. $p(x)$ can be substituted in the linear operator of the left hand side of (4.1). The operator now satisfies

$$Lv := (\nabla^2 + p(x)) v = -\frac{1}{4\epsilon^2} < 0, \quad \forall x \in \Omega.$$

By the *strong maximum principle* of E. Hopf [10] it is readily seen that since $Lv < 0$ in Ω , v cannot achieve non-positive minimum. Indeed, assume in contrary that a minimum at x_0 such that $v(x_0) \leq 0$ exists. Lemma 3.4 in [10] asserts that a necessary condition for a minimum of v to belong to $\partial\Omega$ is that the normal derivative at the minimum point satisfies $\frac{\partial v}{\partial n}(x_0) < 0$, in contradiction to the Neumann boundary condition. Assuming next that x_0 belongs to the interior of Ω , then minimality implies $\nabla v(x_0) = 0$ and $\nabla^2 v(x_0) \geq 0$. By the negativity of $v(x_0)$ and of $p(x_0)v(x_0)$ it follows that $Lv \geq 0$, in contradiction to the negativity of L at all points. Therefore, $v(x)$ cannot achieve a non-positive minimum.

Similarly, the maximum of $v(x)$ cannot belong to $\partial\Omega$ because of the the Neumann boundary conditions (Lemma 3.4 in [10]). The maximum is attained therefore at an interior point of Ω . Maximality at a point x_0 implies $\nabla v(x_0) = 0$ and $\nabla^2 v(x_0) \leq 0$. Together with the positivity of $v(x_0)$, it follows that

$$\frac{1}{4\epsilon^2}v(x_0) = \frac{1}{4\epsilon^2} + \nabla^2 v(x_0) - \frac{\beta}{\epsilon\alpha}|\nabla u|^2 v(x_0) \leq \frac{1}{4\epsilon^2}.$$

□

THEOREM 1. *The sequence $\mathcal{F}_n := \mathcal{F}(u^n, v^{n+1})$ is convergent, $\|v^{n+1} - v^n\|_{H^1(\Omega)} \rightarrow 0$ and $\|u^{n+1} - u^n\|_{H^1(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. According to the algorithm, v^{n+1} is the minimizer of $\mathcal{F}(u^n, v)$ and u^{n+1} is the minimizer of $\mathcal{F}(u, v^{n+1})$. Hence,

$$\mathcal{F}(u^n, v^{n+1}) \leq \mathcal{F}(u^n, v) \quad \forall v, \forall n, \quad (4.2)$$

and

$$\mathcal{F}(u^{n+1}, v^{n+1}) \leq \mathcal{F}(u, v^{n+1}) \quad \forall u, \forall n. \quad (4.3)$$

By the definition of $\mathcal{F}_n$

$$\mathcal{F}_{n-1} - \mathcal{F}_n = \mathcal{F}(u^{n-1}, v^n) - \mathcal{F}(u^n, v^{n+1}).$$

Adding and subtracting $\mathcal{F}(u^n, v^n)$ yields

$$\mathcal{F}_{n-1} - \mathcal{F}_n = [\mathcal{F}(u^n, v^n) - \mathcal{F}(u^n, v^{n+1})] + [\mathcal{F}(u^{n-1}, v^n) - \mathcal{F}(u^n, v^n)]. \quad (4.4)$$

But equation (4.2) with $v = v^n$ and equation (4.3) with $u = u^{n-1}$ respectively yield

$$\mathcal{F}(u^n, v^n) - \mathcal{F}(u^n, v^{n+1}) \geq 0, \quad (4.5)$$

and

$$\mathcal{F}(u^{n-1}, v^n) - \mathcal{F}(u^n, v^n) \geq 0. \quad (4.6)$$

Thus, $\mathcal{F}_n$ is a real monotone decreasing sequence, where

$$\mathcal{F}_{n-1} - \mathcal{F}_n \geq 0 \quad \forall n. \quad (4.7)$$

Since $\mathcal{F}_n \geq 0 \quad \forall n$, the real monotone sequence is bounded, and therefore convergent. Let us define

$$\psi^n(x) := \frac{v^n(x) - v^{n+1}(x)}{\varepsilon_0}.$$

By the calculus of variations derivation,

$$\begin{aligned} \mathcal{F}(u^n, v^n) - \mathcal{F}(u^n, v^{n+1}) &= \varepsilon_0 \delta_v(u^n, v^{n+1}; \psi^n) + \\ &\quad \frac{\varepsilon_0^2}{2} \delta_v^2(u^n, v^{n+1}; \psi^n) + o(\varepsilon_0^3), \end{aligned} \quad (4.8)$$

where δ_v and δ_v^2 are the first and second Gâteaux variations respectively. The first term vanishes since v^{n+1} satisfies the Euler-Lagrange equation (2.6), (3.1). As for the

second variation,

$$\begin{aligned}\delta_v^2(u^n, v^{n+1}; \psi^n) &= \int \left(\sum_{i,j=1}^N \Theta_{v x_i v x_j} \psi_{x_i}^n \psi_{x_j}^n + 2 \sum_{i=1}^N \Theta_{v v x_i} \psi^n \psi_{x_i}^n + \Theta_{vv} (\psi^n)^2 \right) dx \\ &= 2 \int \left(\left[\beta |\nabla u^n|^2 + \frac{\alpha}{4\epsilon} \right] (\psi^n)^2 + \alpha \epsilon |\nabla \psi^n|^2 \right) dx \geq 0,\end{aligned}$$

thus

$$\mathcal{F}(u^n, v^n) - \mathcal{F}(u^n, v^{n+1}) = \varepsilon_0^2 \int \left(\left[\beta |\nabla u^n|^2 + \frac{\alpha}{4\epsilon} \right] (\psi^n)^2 + \alpha \epsilon |\nabla \psi^n|^2 \right) dx + o(\varepsilon_0^3). \quad (4.9)$$

Substituting (4.6) and (4.9) in (4.4) yields

$$\begin{aligned}\mathcal{F}_{n-1} - \mathcal{F}_n &\geq \varepsilon_0^2 \int \left(\left[\beta |\nabla u^n|^2 + \frac{\alpha}{4\epsilon} \right] (\psi^n)^2 + \alpha \epsilon |\nabla \psi^n|^2 \right) dx \\ &\geq C_1 \left(\|\psi^n\|_{L^2(\Omega)}^2 + \|\nabla \psi^n\|_{L^2(\Omega)}^2 \right) = C_1 \|\psi^n\|_{H^1(\Omega)}^2,\end{aligned}$$

and since $\mathcal{F}_n$ converges, we conclude that $\|\psi^n\|_{H^1(\Omega)} \rightarrow 0$, or $\|v^{n+1} - v^n\|_{H^1(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$.

In the same manner, let us define

$$\phi^n(x) := \frac{u^{n-1} - u^n}{\varepsilon_0}.$$

Hence

$$\begin{aligned}\mathcal{F}(u^{n-1}, v^n) - \mathcal{F}(u^n, v^n) &= \varepsilon_0 \delta_u(u^n, v^n; \phi^n) + \\ &\quad \frac{\varepsilon_0^2}{2} \delta_u^2(u^n, v^n; \phi^n) + o(\varepsilon_0^3).\end{aligned} \quad (4.10)$$

The first derivative vanishes since u^n satisfies the Euler-Lagrange equation (2.7), (3.2), and

$$\delta_u^2(u^n, v^n; \phi^n) = \int_{\Omega} \left(\frac{(h * \phi^n)^2 \eta}{((h * u^n - g)^2 + \eta)^{3/2}} + 2\beta ((v^n)^2 + o_\epsilon) |\nabla \phi^n|^2 \right) dx. \quad (4.11)$$

Since both terms are positive and $0 < v \leq 1$,

$$\delta_u^2(u^n, v^n; \phi^n) \geq \frac{1}{2} C_1 \|\nabla \phi^n\|_{L^2(\Omega)}^2 + \frac{1}{2} C_1 \|\nabla \phi^n\|_{L^2(\Omega)}^2.$$

By the Neumann-Poincaré inequality

$$\|\nabla \phi^n\|_{L^2(\Omega)}^2 \geq C \|\phi^n - \phi_\Omega^n\|_{L^2(\Omega)}^2$$

where ϕ_Ω^n is the mean value of ϕ^n in Ω . Substituting (4.8) and (4.10) in (4.4) yields

$$\mathcal{F}_{n-1} - \mathcal{F}_n = \frac{\varepsilon_0^2}{2} \left(\delta_v^2(u^n, v^{n+1}; \psi^n) + \delta_u^2(u^n, v^n; \phi^n) \right) + o(\varepsilon_0^3), \quad (4.12)$$

and since $\mathcal{F}_n$ converges, $\|\psi^n\|_{H^1(\Omega)} \rightarrow 0$ and $\delta_u^2(u^n, v^n; \phi^n) \geq \frac{1}{2} C_1 \|\nabla \phi^n\|_{L^2(\Omega)}^2 + C_2 \|\phi^n - \phi_\Omega^n\|_{L^2(\Omega)}^2 = \frac{1}{2} C_1 \|\nabla(\phi^n - \phi_\Omega^n)\|_{L^2(\Omega)}^2 + C_2 \|\phi^n - \phi_\Omega^n\|_{L^2(\Omega)}^2$, we conclude that $\|\phi^n - \phi_\Omega^n\|_{H^1(\Omega)}^2 \rightarrow 0$.

Substituting the limit ϕ_Ω^n in (4.11) yields that the first term is a constant (since $h(x) > 0 \ \forall x$), which contradicts the fact that $\delta_u^2(u^n, v^n; \phi^n) \rightarrow 0$. Therefore we have to conclude that $\phi_\Omega^n = 0$. Thus, $\|\phi^n\|_{H^1(\Omega)}^2 \rightarrow 0$ or $\|u^n - u^{n-1}\|_{H^1(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$.

□

5. Fixed Point Convergence. The principle of the fixed point convergence proof is to show that the residual of equation (2.7) is a contraction. Adopting the notations of [9], let the residual $\mathcal{R}(u)$ be defined as

$$\mathcal{R}(u) := \mathcal{A}(v, u)u - \mathcal{G}(u). \quad (5.1)$$

For the l iteration,

$$\mathcal{R}(u^l) = \mathcal{A}(v, u^l)u^l - \mathcal{G}(u^l). \quad (5.2)$$

Consider the identity

$$u^{l+1} = u^l - \mathcal{A}(v, u^l)^{-1} \cdot (-\mathcal{A}(v, u^l)d^l)$$

where

$$d^l := u^{l+1} - u^l.$$

Note that

$$-\mathcal{A}(v, u^l)d^l = -\mathcal{A}(v, u^l)(u^{l+1} - u^l) = -\mathcal{A}(v, u^l)u^{l+1} + \mathcal{A}(v, u^l)u^l = -\mathcal{G}(u^l) + \mathcal{A}(v, u^l)u^l = \mathcal{R}(u^l).$$

Thus

$$u^{l+1} = u^l - \mathcal{A}(v, u^l)^{-1} \mathcal{R}(u^l). \quad (5.3)$$

We will show that $\mathcal{A}(v, u^l)^{-1}$ is uniformly bounded and $\mathcal{R}(u^l)$ converges to zero.

LEMMA 2. *Let $\zeta = \{\phi : \phi(x) \geq 0 \ \forall x, \phi \neq c \in \mathbb{R}\}$. Denote by $\Lambda \in C^0(\Omega) \cap \zeta$ the space of non negative and non constant functions. If the observed image $g \in L^\infty \cap \zeta$, then $u \in \Lambda$.*

Proof. By Eq. 2.7

$$2\beta \nabla \cdot ((v^2 + o_\epsilon) \nabla u) = h(-x) * \frac{(h * u - g)}{\sqrt{(h * u - g)^2 + \eta}}.$$

Assume that the minimal point of u is x_0 . At that point $\nabla u|_{x=x_0} = 0$ and therefore

$$o_\epsilon \nabla^2 u|_{x=x_0} \leq \nabla \cdot ((v^2 + o_\epsilon) \nabla u)|_{x=x_0} = (v^2 + o_\epsilon) \nabla^2 u|_{x=x_0} \leq (1 + o_\epsilon) \nabla^2 u|_{x=x_0}$$

The minimality implies $\nabla^2 u(x_0) \geq 0$. But since $g(x), h(x) \geq 0 \ \forall x$, $u(x)$ cannot achieve a negative minimum for every g and h in particular when h is a delta function. Now, Assume by contrary that u is a constant, then $h * u$ is constant and since $g \in \zeta$, it follows that $h * u - g \neq 0$, in contradiction to the E-L equation (2.7). $\square$

Now we have the needed tools and can continue with the proof of convergence of the fixed point iterations.

LEMMA 3. *The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are positive definite*

$$(u, \mathcal{A}_{1,2} u) > 0, \quad \forall u \in \Lambda,$$

where $(\cdot, \cdot)$ denotes an inner product.

Proof. The operator $\mathcal{A}_1$ can be expressed as

$$\mathcal{A}_1 u = \mathcal{H}^* \mathcal{M} \mathcal{H} u \quad (5.4)$$

where $\mathcal{H}$ denotes the convolution operator with $h(x)$, $\mathcal{H}^*$ is its adjoint, and

$$\mathcal{M}u = M(x)u = \frac{u}{\sqrt{(h * \tilde{u} - g)^2 + \eta}}$$

where $\tilde{u}$ is u as evaluated in the previous iteration. Since $M(x)$ is a real positive function, it can be decomposed as $M(x) = D(x) \cdot D(x)$. Together with the fact that the multiplication operator $\mathcal{M}$ is self adjoint, Eq. (5.4) can be replaced by

$$\mathcal{A}_1 = \mathcal{H}^* \mathcal{D}^* \mathcal{D} \mathcal{H}.$$

Now,

$$\begin{aligned} (u, \mathcal{A}_1 u) &= \int u \mathcal{H}^* \mathcal{D}^* \mathcal{D} \mathcal{H} u \, dx = \int u (\mathcal{D} \mathcal{H})^* \mathcal{D} \mathcal{H} u \, dx = \\ &= (u, (\mathcal{D} \mathcal{H})^* \mathcal{D} \mathcal{H} u) = (\mathcal{D} \mathcal{H} u, \mathcal{D} \mathcal{H} u) = \int |\mathcal{D} \mathcal{H} u|^2 \, dx > 0, \end{aligned} \quad (5.5)$$

since $\mathcal{H} > 0$ and $u \geq 0$, $\mathcal{H}u$ vanishes only in the case of $u \equiv 0$ which contradicts the requirement $u \in \Lambda$ as was proved in Lemma 2.

As for the second operator

$$(u, \mathcal{A}_2 u) = -2\beta \int_{\Omega} u \nabla \cdot ((v^2 + o_{\epsilon}) \nabla u) \, dx.$$

Using simple vector analysis manipulation and the divergence theorem yields

$$(u, \mathcal{A}_2 u) = -2\beta \int_{\partial\Omega} (u(v^2 + o_{\epsilon}) \nabla u) \cdot dn + 2\beta \int_{\Omega} \nabla u \cdot ((v^2 + o_{\epsilon}) \nabla u) \, dx. \quad (5.6)$$

The first term vanishes due to the Neumann boundary condition, thus

$$(u, \mathcal{A}_2 u) = 2\beta \int_{\Omega} (v^2 + o_{\epsilon}) |\nabla u|^2 \, dx > 0.$$

□

COROLLARY 1. *Since the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are positive definite, $\mathcal{A}_1 \mathcal{A}_2^{-1}$ is also positive definite.*

LEMMA 4. *The operator $\mathcal{A}^{-1}(v, u^l)$ is bounded by a constant $\gamma > 0$ independent on v and l ,*

$$(\phi, \mathcal{A}^{-1} \phi) \leq \gamma, \quad \phi \in \Lambda, \quad \|\phi\|^2 = 1.$$

Proof. We will look for γ which satisfies

$$(\phi, (\mathcal{A}_1 + \mathcal{A}_2) \phi) \geq \gamma.$$

The constant γ can be bounded by the lowest eigenvalue of the $\mathcal{A}_2$ operator,

$$\gamma \geq \inf_{\substack{\phi \in \Lambda, \\ \|\phi\|^2 = 1}} \{(\phi, \mathcal{A}_1 \phi) + (\phi, \mathcal{A}_2 \phi)\} > \inf_{\substack{\phi \in \Lambda, \\ \|\phi\|^2 = 1}} \{(\phi, \mathcal{A}_2 \phi)\} = \lambda_0,$$

where

$$\lambda_0 = \inf_{\substack{\phi \in \Lambda, \\ \|\phi\|^2=1}} -2\beta \int_{\Omega} \phi \nabla \cdot ((v^2 + o_\epsilon) \nabla \phi) \, dx.$$

By equation (5.6) and the Neumann boundary condition,

$$\lambda_0 = \inf_{\substack{\phi \in \Lambda, \\ \|\phi\|^2=1}} 2\beta \int_{\Omega} \nabla \phi \cdot ((v^2 + o_\epsilon) \nabla \phi) \, dx.$$

Hence,

$$\lambda_0 \geq 2\beta o_\epsilon \inf_{\substack{\phi \in \Lambda, \\ \|\phi\|^2=1}} \int_{\Omega} |\nabla \phi|^2 \, dx.$$

But

$$\inf_{\substack{\phi \in \Lambda, \\ \|\phi\|^2=1}} \int_{\Omega} |\nabla \phi|^2 \, dx > 0$$

since ϕ is not a constant.

□

We are now ready to prove the convergence of the fixed point scheme.

THEOREM 2. *The fixed point iteration (2.8) converges.*

Proof. Consider the residual function (5.1) in the $l+1$ iteration

$$\mathcal{R}(u^{l+1}) = \mathcal{A}(v, u^{l+1})u^{l+1} - \mathcal{G}(u^{l+1}).$$

Specifically,

$$\mathcal{R}(u^{l+1}) = h(-x) * \frac{h * u^{l+1}}{\sqrt{(h * u^{l+1} - g)^2 + \eta}} - 2\beta \nabla \cdot (v^2 \nabla u^{l+1}) - \mathcal{G}(u^{l+1}). \quad (5.7)$$

Let us now add and subtract

$$h(-x) * \frac{h * u^{l+1}}{\sqrt{(h * u^l - g)^2 + \eta}}$$

to equation (5.7), and reorder the terms. We obtain

$$\begin{aligned} \mathcal{R}(u^{l+1}) &= h(-x) * \frac{h * u^{l+1}}{\sqrt{(h * u^{l+1} - g)^2 + \eta}} - 2\beta \nabla \cdot (v^2 \nabla u^{l+1}) - \mathcal{G}(u^{l+1}) \\ &\quad + h(-x) * \frac{h * u^{l+1}}{\sqrt{(h * u^{l+1} - g)^2 + \eta}} - h(-x) * \frac{h * u^{l+1}}{\sqrt{(h * u^l - g)^2 + \eta}}. \end{aligned}$$

Note that by equation (2.8), the sum of the first and second terms is exactly $\mathcal{G}(u^l)$.

Thus,

$$\mathcal{R}(u^{l+1}) = \mathcal{G}(u^l) - \mathcal{G}(u^{l+1}) + h(-x) * \left\{ \frac{h * u^{l+1}}{\sqrt{(h * u^{l+1} - g)^2 + \eta}} - \frac{h * u^{l+1}}{\sqrt{(h * u^l - g)^2 + \eta}} \right\}.$$

Substituting equation (2.11) yields

$$\begin{aligned}\mathcal{R}(u^{l+1}) &= h(-x) * \left\{ \left[\frac{1}{\sqrt{(h * u^l - g)^2 + \eta}} - \frac{1}{\sqrt{(h * u^{l+1} - g)^2 + \eta}} \right] [g - h * u^{l+1}] \right\} \\ &= h(-x) * \left\{ \left[\frac{\sqrt{(h * u^{l+1} - g)^2 + \eta} - \sqrt{(h * u^l - g)^2 + \eta}}{\sqrt{(h * u^l - g)^2 + \eta} \cdot \sqrt{(h * u^{l+1} - g)^2 + \eta}} \right] [g - h * u^{l+1}] \right\}.\end{aligned}$$

Now, let $\sigma := \text{sign}(d^l) \cdot \text{sign}(g - h * u^{l+1}) \in \{+1, -1\}$.

Recall that $|a| - |b| \leq (a - b) \cdot \text{sign}(a - b)$ and $a/|a| = \text{sign}(a)$. When η is small enough one gets

$$\begin{aligned}\mathcal{R}(u^{l+1}) &\leq h(-x) * \left\{ \frac{(h * u^{l+1} - h * u^l) \cdot \text{sign}(d^l)}{|h * u^l - g| \cdot |h * u^{l+1} - g|} (g - h * u^{l+1}) \right\} \\ &= h(-x) * \left\{ \frac{h * d^l}{|h * u^l - g|} \cdot \text{sign}(d^l) \cdot \text{sign}(g - h * u^{l+1}) \right\}.\end{aligned}$$

Now, let $\sigma := \text{sign}(d^l) \cdot \text{sign}(g - h * u^{l+1}) \in \{+1, -1\}$.

Together with addition and subtraction of $\mathcal{A}_2(v)d^l$,

$$\begin{aligned}\mathcal{R}(u^{l+1}) &\leq \sigma \left[h(-x) * \frac{h * d^l}{|h * u^l - g|} - 2\beta \nabla \cdot (v^2 \nabla d^l) + 2\beta \nabla \cdot (v^2 \nabla d^l) \right] \\ &= \sigma [\mathcal{A}(v, u^l)d^l + 2\beta \nabla \cdot (v^2 \nabla d^l)] \\ &= \sigma [-\mathcal{R}(u^l) + 2\beta \nabla \cdot (v^2 \nabla (u^{l+1} - u^l))] .\end{aligned}\tag{5.8}$$

Substituting equation (5.3) in (5.8) results in

$$\begin{aligned}\mathcal{R}(u^{l+1}) &\leq -\sigma [\mathcal{R}(u^l) + 2\beta \nabla \cdot (v^2 \nabla \mathcal{A}^{-1}(v, u^l)\mathcal{R}(u^l))] \\ &= -\sigma [(I - \mathcal{Q}(v, u^l))\mathcal{R}(u^l)]\end{aligned}$$

where

$$\mathcal{Q}(v, u)\phi = -2\beta \nabla \cdot (v^2 \nabla (\mathcal{A}(v, u)^{-1}\phi)) = \mathcal{A}_2(\mathcal{A}_1 + \mathcal{A}_2)^{-1}\phi.$$

We will show now that the spectral radius $\rho(\mathcal{Q}) < 1$:

$$\begin{aligned}\mathcal{A}_2(\mathcal{A}_1 + \mathcal{A}_2)^{-1}\phi &= \lambda_n \phi, \\ (\mathcal{A}_1 + \mathcal{A}_2)^{-1}\phi &= \lambda_n \mathcal{A}_2^{-1}\phi, \\ (\phi, \phi) &= \lambda_n (\phi, (\mathcal{A}_1 + \mathcal{A}_2)\mathcal{A}_2^{-1}\phi) .\end{aligned}$$

Thus,

$$\frac{1}{\lambda_n} = \frac{(\phi, (I + \mathcal{A}_1\mathcal{A}_2^{-1})\phi)}{\|\phi\|^2}.$$

By Corollary 1, $\mathcal{A}_1\mathcal{A}_2^{-1}$ is positive definite and thus $\lambda_n < 1$. Consequently,

$$\|\mathcal{R}(u^{l+1})\|_2 \leq \|I - \mathcal{Q}(v, u^l)\|_2 \cdot \|\mathcal{R}(u^l)\|_2 = k \|\mathcal{R}(u^l)\|_2$$

where

$$k \leq 1 - \frac{1}{1 + \rho(\mathcal{A}_1\mathcal{A}_2^{-1})} = \frac{\rho(\mathcal{A}_1\mathcal{A}_2^{-1})}{1 + \rho(\mathcal{A}_1\mathcal{A}_2^{-1})} < 1.$$

The convergence of equation (5.3) is satisfied due to the contraction of $\mathcal{R}(u^l)$ and the boundness of the operator $\mathcal{A}^{-1}(v, u^l)$, as was shown in Lemma 4. By that the proof is completed.

□

6. Summary. An iterative method for variational image deconvolution and impulsive noise removal has been recently suggested by the authors. The calculus of variations approach leads to a nonlinear integro-differential equation which was linearized by the fixed point iteration. In this study, we explored the iterative scheme and proved the convergence of the fixed point iteration. This was accomplished by showing that the sequence of equation residuals is obtained by iterating a contractive mapping.

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