Worst-Case Incremental Analysis for a Class of p-Facility Location Problems

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We consider a rather large class of p-facility location models including the p-median, p-center, and other related and more general models. For any such model of interest with p new facilities, let $v(p)$ denote the minimal objective function value and let n be the number of demand points. Given $1 \le p < q \le n$, we find easily computed positive constants $k(p, q)$, where $v(q)/v(p) \le k(p, q) \le 1$. These resulting inequalities relating $v(p)$ and $v(q)$ are worst case, since they are attained as equalities for a class of "hub-and-spoke" trees. Our results also provide insight into some demand point aggregation problems, where a graph of the function $v(q)$ can provide an upper bound on aggregation error. © 2002 Wiley Periodicals, Inc.

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1. INTRODUCTION

Let $V = \{v_1, \ldots, v_n\}$ be a set of n distinct points in some metric space M. For each pair of points $x, y \in M$, let $d(x, y)$ denote the distance between x and y. For any finite nonempty set Z of M, let

$$d(x, Z) = d(Z, x) = \min\{d(x, y) : y \in Z\}.$$

Suppose that with each $v_i, i = 1, \ldots, n$, is associated a positive weight w_i.

Let S be a subset of M satisfying $V \subset S$. The *p-median problem* is to find $X \subset S$, $|X| = p$, such that the function

$$f(X) = \sum_{i=1}^{n} w_i d(v_i, X),$$

is minimized. The set V is viewed as the set of customers (demand points), and S is the set of all possible sites for locating servers (supply points). X is then the set of p selected servers (medians). Note that the supply set S is not necessarily finite, for example, the Euclidean planar 1-median problem, which is recognized as the Weber problem. Nevertheless, for many location problems, S can be reduced *a priori* to a finite set, for example, the rectilinear planar p-median problem and the p-median problem on networks (see [8, 9]). Let $v(p)$ denote the optimal objective value of the p-median problem.

Since the points in V are distinct, $v(p)$ is a decreasing function of p with $v(n) = 0$. Although this function is defined only for integer values of p, we extend its domain to the interval $[1, n]$ by considering the linear interpolation induced by the integer values of p. In particular, $v(p)$ is a decreasing, continuous, piecewise linear function over $[1, n]$, satisfying $v(n) = 0$.

It is interesting and useful to explore marginal effects of adding more servers. The demand point aggregation work of Francis et al. [7] provides one reason of interest for the study of the function $v(p)$. For a number of well-known network location models, if q aggregate demand points replace the original n demand points, then, in a well-defined sense, it is known that the value $v(q)$ provides an upper bound on the maximum aggregation error. The structure of the function $v(q)$ indicates how the aggregation error decreases as q increases. Since the problem of computing $v(q)$ is generally NP-hard, the graphs of $v(q)$ have been laboriously obtained (to date) using heuristic approaches and curve fitting; see, for example, Francis et al. [4–6]. These graphs tend to obey the law

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of diminishing returns (convexity): As q increases, $v(q)$ decreases, but at a decreasing rate. It can be shown that when V is a set of points on the real line the function v is indeed convex. However, in general, in spite of the above computational observations, $v(p)$ is not convex. The next example (adapted from Broin and Lowe [3]) illustrates that this function may not be convex even when V is the node set of a tree network, and the distance function is the one induced by the edge lengths.

Example 1. *Consider a tree $T = (V,E)$ with a node set $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and an edge set $E = \{(v_1,v_2),(v_1,v_3),(v_1,v_4),(v_2,v_5),(v_3,v_6),(v_4,v_7)\}$. Suppose that all six edges have unit lengths and $w_i = 1, i = 1,\dots,7$. It is easy to see that $v(1) = 9, v(2) = 7$, and $v(3) = 4$. Hence, $v(2) > (v(1)+v(3))/2$, and the function $v(p)$ is not convex.*

Nevertheless, in this paper, we exhibit partial convexity properties of the function $v(p)$ for networks and general metric spaces for a generalized median problem, called the (b,p)-facility ordered median problem. (Its formal definition is given in the next section.) Several properties of this class of ordered median problems were discussed in Nickel and Puerto [14] and Francis et al. [7]. This class of facility location problems contains a number of the commonly known facility location models as special cases. In addition to the above p-median problem, the p-center, the centdian, and the k-centrum problems are also included in this class. The main result of our paper relates the optimal objective value for two distinct values of p by an inequality. This general result implies, for example, the following partial convexity result for the classical p-median problem on networks: Consider the line in the (p,v)-plane, connecting the solution to the 1-median problem, $(p,v) = (1, v(1))$, with the solution to the n-median problem, $(p,v) = (n,0)$. Then, for any value of p, the solution to the p-median problem, $(p, v(p))$, is on or below the above line. More generally, for any pair $p,q = 1,\dots,n, p < q$, the solution value to the q-median problem, $(q, v(q))$, is either on or below the line connecting the solutions to the p-median and n-median problems. Further, we exhibit "hub-and-spoke" tree networks where this linear interpolation exactly estimates $v(p)$.

2. THE (b, p)-FACILITY ORDERED MEDIAN PROBLEM

We now formally define the class of location problems that we study. This class contains many of the commonly known facility location models as special cases.

For each vector $u = (u_1,\dots,u_n) \in R^n$, let $\hat{u}$ be the vector obtained from u by sorting the components of u in nonincreasing order, that is, $\hat{u} = (u_{[1]},\dots,u_{[n]})$, where $u_{[1]} \geq \cdots \geq u_{[n]}$.

Let $b = (b_1,\dots,b_n)$ be a nonzero, nonnegative vector. For each vector $u = (u_1,\dots,u_n) \in R^n$, define $g_b(u_1,\dots,u_n) = \sum_{i=1}^n b_i u_{[i]}$. The *$(b,p)$-facility ordered median problem* is to find $X \subset S, |X| = p$, such that the function

$$f_b(X) = g_b(w_1 d(v_1,X),\dots,w_n d(v_n,X))$$

is minimized. The *demand-set (b,p)-facility ordered median problem* is to find $X \subset V, |X| = p$, such that the function

$$f_b(X) = g_b(w_1 d(v_1,X),\dots,w_n d(v_n,X))$$

is minimized.

The above ordered median problems were discussed in Nickel and Puerto [14] and Francis et al. [7].

It is easy to see that the classical p-median and p-center problems are obtained as special cases by setting $b = (1,\dots,1)$ and $b = (1,0,\dots,0)$, respectively. Another special case, which unifies the p-median and the p-center models, is the k-centrum problem, defined as follows:

For each vector $(u_1,\dots,u_n) \in R^n$, define $H_k(u_1,\dots,u_n)$ to be the sum of the k largest components of the vector $(u_1,\dots,u_n)$. The *p-facility k-centrum problem* is to find $X \subset S, |X| = p$, such that the function

$$f_k(X) = H_k(w_1 d(v_1,X),\dots,w_n d(v_n,X))$$

is minimized. Single-facility k-centrum problems were first discussed in Slater [15] and Andreatta and Mason [1, 2]. The reader is referred to Tamir [16] for structural and algorithmic results concerning the multifacility problem. The k-centrum model unifies the center and the median problems since the case $k = 1$ defines the p-center problem, while the case $k = n$ corresponds to the p-median problem. The case for arbitrary k corresponds to the special case of the ordered median problem, where the first k components of the vector b are equal to 1 and the last $n - k$ components are equal to 0.

Another well-known location problem, which is also a special case of the above ordered median problem, is the p-cent-dian problem. Given $0 \leq \alpha \leq 1$, *the p-cent-dian problem* is to find $X \subset S, |X| = p$, such that the function

$$\alpha f_n(X) + (1-\alpha) f_1(X)$$

is minimized. The cent-dian model was introduced by Halpern [10–12] and Handler [13] to obtain a good way to trade off the minimum (efficiency) and minimax (equity) approaches of the p-median and the p-center problems. To obtain the cent-dian model, set $b = (1,\alpha,\dots,\alpha)$.

3. INCREMENTAL ANALYSIS OF (b, p)-FACILITY ORDERED MEDIAN PROBLEM

Let $v_b(p)$ denote the optimal objective value of the (b,p)-facility ordered median problem. Clearly, for $p = 1,\dots,n$, $v_b(p)$ is a monotone function of p with $v_b(n) = 0$. We demonstrated in the Introduction that $v_b(p)$ may not be convex even for metric spaces induced by tree networks. Our goal is to derive some partial convexity results by relating the objective value $v_b(p)$ for two distinct values of p. We will need the following lemmas:

The first lemma was proven in Francis et al. [7].

Lemma 1. *Let $u = (u_1, \ldots, u_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in R^n. Suppose that $u \le y$. Then, $\hat{u} = (u_{[1]}, \ldots, u_{[n]}) \le \hat{y} = (y_{[1]}, \ldots, y_{[n]})$.*

Lemma 2. *Let $a_1 \ge \cdots \ge a_m$ be a sequence of m real numbers, and let $b_1 \ge \cdots \ge b_m$ be a sequence of m nonnegative real numbers. For $j = 1, \ldots, m-1$,*

$$(b_1 + \cdots + b_m)(b_1 a_{j+1} + \cdots + b_{m-j} a_m) \le (b_1 + \cdots + b_{m-j})(b_1 a_1 + \cdots + b_m a_m).$$

Proof. The validity of the result for $j = m-1$ follows directly from the nonnegativity of $b = (b_1, \ldots, b_m)$ and $a_1 \ge \cdots \ge a_m$. Suppose that $j \le m-2$. Since

$$(b_1 a_1 + \cdots + b_m a_m) \ge ((b_1 + \cdots + b_{j+1}) a_{j+1} + b_{j+2} a_{j+2} + \cdots + b_m a_m),$$

it will suffice to prove that

$$(b_1 + \cdots + b_m)(b_1 a_{j+1} + \cdots + b_{m-j} a_m) \le (b_1 + \cdots + b_{m-j}) ((b_1 + \cdots + b_{j+1}) a_{j+1} + b_{j+2} a_{j+2} + \cdots + b_m a_m).$$

Suppose, without loss of generality, that $b_1 > 0$. For each $k = 1, \ldots, m-j$, define $\beta_k = b_k/(b_1 + \cdots + b_{m-j})$. Also, for each $k = 1, \ldots, m$, define $\gamma_k = b_k/(b_1 + \cdots + b_m)$. Note that $\beta_k \ge 0, \gamma_k \ge 0$ and $\sum_{k=1}^{m-j} \beta_k = \sum_{k=1}^{m} \gamma_k = 1$. Using this notation, we will equivalently prove that

$$\left(1 - \sum_{t=j+2}^{m} \beta_{t-j}\right) a_{j+1} + \sum_{t=j+2}^{m} \beta_{t-j} a_t \le \left(1 - \sum_{t=j+2}^{m} \gamma_t\right) a_{j+1} + \sum_{t=j+2}^{m} \gamma_t a_t.$$

The last inequality is then equivalent to

$$\sum_{t=j+2}^{m} (\beta_{t-j} - \gamma_t) a_t \le a_{j+1} \sum_{t=j+2}^{m} (\beta_{t-j} - \gamma_t).$$

If $\beta_{t-j} = \gamma_t$, for all $t = j+2, \ldots, m$, then equality holds. Thus, suppose that this is not the case. Consider some index $t = j+2, \ldots, m$. Then, from the monotonicity and the nonnegativity properties of $b = (b_1, \ldots, b_m)$, we obtain

$$\gamma_t \le b_t/(b_1 + \cdots + b_{m-j}) \le b_{t-j}/(b_1 + \cdots + b_{m-j}) = \beta_{t-j}.$$

Finally, to complete the proof, we need to show that

$$\left(\sum_{t=j+2}^{m} (\beta_{t-j} - \gamma_t) a_t\right) \Big/ \left(\sum_{t=j+2}^{m} (\beta_{t-j} - \gamma_t)\right) \le a_{j+1}.$$

The left-hand side of the last inequality is a weighted average of $\{a_{j+2}, \ldots, a_m\}$, and, therefore, it is bounded above by a_{j+2}. Since, $a_{j+1} \ge a_{j+2}$, the validity of the inequality is now established. ∎

Theorem 1. *Let $b = (b_1, \ldots, b_n)$ be a nonzero, nonnegative vector, satisfying $b_1 \ge \cdots \ge b_n$. For $p = 2, \ldots, n$, suppose that the solution to the (b,p)-facility ordered median problem is attained by setting the p servers at $X^* = \{x_1, \ldots, x_p\}$, where exactly t servers, $0 \le t \le p$, are in $S - V$. Then, for any q, $1 \le p < q < n$,*

$$v_b(q) \left(\sum_{i=1}^{n-p+t} b_i\right) \le v_b(p) \left(\sum_{i=1}^{n-q+t} b_i\right).$$

Proof. Without loss of generality, assume that $\{x_1, \ldots, x_t\} \subset S - V, (x_{t+1}, \ldots, x_p) = (v_{n-(p-t)+1}, \ldots, v_n)$ and

$$w_1 d(X^*, v_1) \ge w_2 d(X^*, v_2) \ge \cdots \ge w_n d(X^*, v_n).$$

[Note that $w_{n-(p-t)+1} d(X^*, v_{n-(p-t)+1}) = \cdots = w_n d(X^*, v_n) = 0$.] Consider now a feasible solution to the q-median problem where servers are set at each node of the subset $X' = \{v_1, \ldots, v_{q-p}, x_1, \ldots, x_p\}$. (We augment $q-p$ servers to X^* and set them at $\{v_1, \ldots, v_{q-p}\}$.) Using Lemma 2 with $m = n - (p-t), j = q-p, (a_1, \ldots, a_m) = (w_1 d(X^*, v_1), \ldots, w_m d(X^*, v_m))$, and b_i for $i = 1, \ldots, m$, we have

$$\begin{aligned}
&(b_1 + \cdots + b_m)(w_{q-p+1} d(X^*, v_{q-p+1}) b_1 + \\
&\quad \cdots + w_m d(X^*, v_m) b_{m-j}) \\
&\quad \le (b_1 + \cdots + b_{m-j})(w_1 d(X^*, v_1) b_1 + \\
&\quad \cdots + w_m d(X^*, v_m) b_m) \\
&\quad = v_b(p)(b_1 + \cdots + b_{m-j}).
\end{aligned}$$

By definition, the last inequality is now written as

$$(b_1 + \cdots + b_m) g_b(0, \ldots, 0, w_{q-p+1} d(X^*, v_{q-p+1}), \ldots, w_m d(X^*, v_m), 0, \ldots, 0) \le v_b(p)(b_1 + \cdots + b_{m-j}).$$

Since $X^* \subset X'$, we obtain

$$w_j d(X', v_j) \le w_j d(X^*, v_j), j = 1, \ldots, n.$$

Using Lemma 1,

$$\begin{aligned}
f_b(X') &= g_b(0, \ldots, 0, w_{q-p+1} d(X', v_{q-p+1}), \ldots, w_m d(X', v_m), 0, \ldots, 0) \\
&\le g_b(0, \ldots, 0, w_{q-p+1} d(X^*, v_{q-p+1}), \ldots, w_m d(X^*, v_m), 0, \ldots, 0).
\end{aligned}$$

Therefore,

$$(b_1 + \cdots + b_m) f_b(X') \le v_b(p)(b_1 + \cdots + b_{m-j}).$$

Finally, note that from optimality $v_b(q) \le f_b(X')$. Therefore,

$$(b_1 + \cdots + b_m) v_b(q) \le v_b(p)(b_1 + \cdots + b_{m-j}).$$

Substituting $m = n - (p - t), j = q - p$, we have

$$v_b(q)\left(\sum_{i=1}^{n-p+t} b_i\right) \le v_b(p)\left(\sum_{i=1}^{n-q+t} b_i\right). \qquad \blacksquare$$

Remark 1. *Let $\delta = 1$ if the solution to the $(b, 1)$-median problem is attained at a point in V and $\delta = 0$ otherwise. The last theorem implies that for $p = 1, \ldots, n$*

$$v_b(p)\left(\sum_{i=1}^{n-\delta} b_i\right) \le v_b(1)\left(\sum_{i=1}^{n-p+1-\delta} b_i\right).$$

We point out that, for the k-centrum problem, $\sum_{i=1}^{j} b_i = \min(k, j)$, for $j = 1, \ldots, n$. For the cent-dian problem, $\sum_{i=1}^{j} b_i = 1 + (j-1)\alpha$, for $j = 1, \ldots, n$.

Recall that in the demand-set (b, p)-facility ordered median problem $S = V$, and, therefore, all servers are in V. Applying the above theorem to this discrete case with $q = p + 1$, we obtain

$$v_b(n-1)\Big/\left(\sum_{i=1}^{1} b_i\right) \le v_b(n-2)\Big/\left(\sum_{i=1}^{2} b_i\right) \le \cdots \le v_b(1)\Big/\left(\sum_{i=1}^{n-1} b_i\right).$$

In particular, for the p-median problem where $b = (1, \ldots, 1)$, for $p = 2, 3, \ldots, n-1$, we have

$$v_b(p)/(n-p) \le v_b(p-1)/(n-p+1).$$

The following "hub-and-spoke" tree example demonstrates that the above inequalities are the tightest possible for the demand-set ordered median problem.

Example 2. *Consider a star tree network ("hub-and-spoke tree"), $T = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ and $E = \{(v_1, v_2), \ldots, (v_1, v_n)\}$. Each edge has a unit length, and $w_i = 1, i = 1, \ldots, n$. For any vector b satisfying the hypothesis of Theorem 1, it is easy to verify that $v_b(p) = \sum_{i=1}^{n-p} b_i$, for $p = 1, \ldots, n-1$, and $v_b(n) = 0$. (Another equivalent example is a complete graph with unit lengths and node weights.)*

More generally, we have the following property for the demand-set (b, p)-facility ordered median problem:

Property 1. *Let $b = (b_1, \ldots, b_n)$ be a nonzero, nonnegative vector, satisfying $b_1 \ge \cdots \ge b_n$. Consider the demand-set (b, p)-facility ordered median problem. For each $i = 1, \ldots, n$, define $c_i = w_i \min_{j \ne i} d(v_i, v_j)$, and let $c = \min_{i=1,\ldots,n} c_i$. Suppose that $n \ge 3$. The following are equivalent:*

(1)

$$v_b(n-1)\left(\sum_{i=1}^{n-1} b_i\right) = v_b(1)b_1.$$

(2) *For all $p = 1, \ldots, n-1$,*

$$v_b(p)\left(\sum_{i=1}^{n-1} b_i\right) = v_b(1)\left(\sum_{i=1}^{n-p} b_i\right).$$

(3) *There exists a node $v_k \in V$, such that $w_i d(v_k, v_i) = c$, for all $v_i \in V, i \ne k$.*

Proof. We start by showing that (3) implies (2). For all $j, j = 1, \ldots, n-1$, let $b_j^* = \sum_{i=1}^{n-j} b_i$. Indeed, if there is a node v_k with $w_i d(v_k, v_i) = c$ for $i \ne k$, then by establishing servers at any subset $X \subset V$ which contains v_k, $|X| = p$, the objective value of the corresponding solution will be cb_p^*. But since cb_p^* is a lower bound on $v_b(p)$, we get $v_b(p) = cb_p^*$. Clearly, for $p = 1$, the server is optimally located at v_k, and so $v_b(1) = cb_1^*$. Thus, $v_b(p)b_1^* = v_b(1)b_p^*$.

Clearly, (2) implies (1). We conclude by showing that (1) implies (3).

Note that $v_b(n-1)$ is attained by establishing servers at all nodes but one, say v_t. Moreover, $v_b(n-1) = b_1 w_t \min_{j \ne t} d(v_j, v_t)$. The optimality of $v_b(n-1)$ implies that v_t must satisfy $w_t \min_{j \ne t} d(v_t, v_j) = c$, where c is defined above. Thus, $v_b(n-1) = cb_1$. Using (1), we conclude that

$$v_b(1) = cb_1^*.$$

Suppose that $v_b(1)$ is attained by establishing a server at node v_k. From the definition $w_i d(v_i, v_k) \ge c$ for each $i = 1, \ldots, n, i \ne k$. Therefore, if $\max_{i \ne k} w_i d(v_i, v_k) > c$, we would get $v_b(1) > cb_1^*$, contradicting $v_b(1) = cb_1^*$. We conclude that $\max_{i \ne k} w_i d(v_i, v_k) = c$, and, therefore, $w_i d(v_i, v_k) = c$ for each $i = 1, \ldots, n, i \ne k$. This proves (3). $\blacksquare$

We note in passing that for the important special case of the demand-set p-median problem, which has motivated our study, the following stronger property holds. For the sake of brevity, we omit the proof.

Property 2. *Consider the demand-set p-median problem. For each $i = 1, \ldots, n$, define $c_i = w_i \min_{j \ne i} d(v_i, v_j)$, and let $c = \min_{i=1,\ldots,n} c_i$. Suppose that $n \ge 3$. The following are equivalent:*

(1) *There exists an integer $p', 1 < p' < n$, such that*

$$v_b(p') = v_b(1)(n - p')/(n-1).$$

(2) *For all $p = 1, \ldots, n$,*

$$v_b(p) = v_b(1)(n-p)/(n-1).$$

(3) *There exists a node $v_k \in V$, such that $w_i d(v_k, v_i) = c$, for all $v_i \in V, i \ne k$.*

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