Discrete Applied Mathematics 47 (1993) 275–283
North-Holland

A unifying location model on tree graphs based on submodularity properties

Arie Tamir

Department of Statistics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat-Aviv, Tel Aviv 69978, Israel

Received 17 January 1990
Revised 27 June 1991

Abstract

Let $\mathscr{F}$ be the collection of nonempty subtrees of a given tree T. Each subtree is viewed as a potential facility. Let f be a real objective function defined on $\mathscr{F}$. The facility location model we consider is to select a subtree minimizing f. This model unifies and generalizes several facility location problems discussed in the literature. We prove that the most common objective functions used in facility location theory possess the submodularity property. In particular, the ellipsoid approach provides a unified framework for polynomial solvability.

Introduction

The uncapacitated facility location model is one of the classical problems in location theory. Given a finite set of customers located at fixed sites, the problem is to determine the locations for a set of servers. It is assumed that each server is uncapacitated and it can serve all customers. The objective is to minimize the sum of the set-up costs of the servers and the transportation costs of the customers. Usually the transportation cost functions are monotone nondecreasing with the distances the customers travel to their respective server. Indeed, if this is the case, each customer will travel to its closest server since all servers are assumed to be uncapacitated.

The above model is known to be NP-hard when defined on a general graph, even when the transportation costs are linear with the distance travelled [2]. It is polynomially solvable on tree graphs and some generalizations like series-parallel graphs [2, 4, 9].

Correspondence to: Professor A. Tamir, Department of Statistics, Tel Aviv University, Ramat-Aviv, Tel Aviv 69978, Israel.

The efficient solvability of the tree case has been supported and explained by relating it to fundamental concepts like convexity, chordality and total balancedness [9].

In this paper, we consider a variation of the model on tree graphs, where the sites selected for the servers must be connected. Specifically, focusing on tree networks the requirement here is that the subgraph induced by the serving centers must be connected. We refer the readers to the papers [1, 5, 6, 8, 10–14] for further discussion and motivation of this connectivity issue. In addition to the set-up costs of the individual servers we will also account for a connectivity cost. We assume that the objective function can be represented as the sum of two terms. The first is the total cost associated with the servers. It is independent of the customers. This term combines the connectivity cost and the set-up costs. The second term reflects the utilities of the individual customers, and it is expressed as a function of their distances to the servers.

Simple examples illustrate that the chordality and balancedness properties satisfied by the regular uncapacitated facility location model, do not extend to the version of the model where the connectivity between the servers is required. Thus, our goal in this paper is to identify another fundamental property that explains the polynomial solvability of the special cases of the latter version that have appeared recently in the literature. This property is the submodularity of the objective function of the location model. (See Section 2 for the exact definition.) It unifies the special cases, provides important and interesting generalizations and allows the polynomial solvability of the model by the ellipsoid approach [3].

The special cases that have motivated our study appeared in [5, 7]. Suppose that each node of a given tree, v_i, is associated with a desired service radius, say r_i, and a penalty term, p_i, for not being served within this radius. (In [7] $p_i = \infty$ for each node v_i.) The objective is to select servers with the above connectivity property that will minimize the sum of the connectivity cost and the total penalty cost. The connectivity cost is assumed to be linear in the total length of the connecting subtree. (There is no direct set-up cost for the serving facilities in [5, 7].) The penalty cost corresponds to the transportation cost in the general model we have introduced above. If the distance from customer v_i to its nearest server (the closest point of the connecting subtree) is at most r_i then there is no transportation cost; otherwise the latter cost is p_i.

The main contribution of this paper is in proving that the most common cost functions, used in facility location models, do possess the submodularity property when the underlying graph is a tree. We now list our submodularity results.

Starting with the total set-up cost of the servers we immediately note that submodularity holds when:

(1) Set-up cost is equal to the sum of the set-up costs of the individual facilities.

(2) Set-up cost is equal to the maximum over the individual set-up costs.

For the cost of the connecting subtree we have submodularity when:

(3) The subtree cost is equal to the sum of its edge lengths.

(4) The subtree cost is equal to its longest edge.

(5) The subtree cost is equal to its (edge) cut value. (In particular, the cost is equal to the number of neighbours.)

(6) The subtree cost is equal to its diameter, the longest (simple) path in the subtree.

For the total transportation cost function, we show submodularity for the following cases:

(7) The transportation cost is the sum of the customers transportation cost functions, which depend only on the respective distances to the nearest servers. (The dependence is not required to be monotone.)

(8) The transportation cost function of each customer is assumed to be monotone nondecreasing in the service distance, and the total transportation objective is the maximum over the individual costs.

Note that (7) corresponds to the classical "median" objective in location theory, while (8) captures the objective of "center" models.

For cases (1), (3), (5) and (7) we establish a stronger property, i.e., the objectives there are in fact modular.

Since submodularity is preserved under addition, we conclude that our general model can involve any objective representable as the sum of the eight cases listed above.

We point out that the above submodularity results depend crucially on our supposition that the subgraph induced by the sites selected for the servers must be connected. When we remove this connectivity requirement the model reduces to the classical uncapacitated facility location minimization model [2]. It is well known (see [2] and the references cited there) that the objective of this minimization model is supermodular even for general graphs. However, it is easy to see that the transportation cost term of the objective ((7) or (8) above) is not submodular even for tree graphs.

1. The formal location model

Let $T = (V, E)$ be an undirected tree graph with node set V, and edge set E. Each edge $e \in E$ has a positive length, α_e. A subgraph of T, $S = (V', E')$ is a *subtree*, if it is connected, $V' \neq \emptyset$, $V' \subseteq V$, and $E' \subseteq E$. If $S^1 = (V^1, E^1)$ and $S^2 = (V^2, E^2)$ are subtrees of T, we say that S^1 is contained in S^2 ($S^1 \subseteq S^2$) if $V^1 \subseteq V^2$ and $E^1 \subseteq E^2$. We say that S^1 intersects S^2 if V^1 intersects V^2, and we define the intersecting subtree, $S^1 \cap S^2 = (V^1 \cap V^2, E^1 \cap E^2)$. Also, if S^1 intersects S^2 we define their union (subtree), $S^1 \cup S^2 = (V^1 \cup V^2, E^1 \cup E^2)$.

The edge lengths of T induce a distance function on T. For any pair of nodes, v, u in V we let $d(u, v)$ denote the sum of the lengths of the edges on $P(u, v)$, the unique simple path connecting u and v. If $S^1 = (V^1, E^1)$ and $S^2 = (V^2, E^2)$ are subtrees of T we define $d(S^1, S^2) = \min\{d(v, u) \mid v \in V^1, u \in V^2\}$.

Consider the following location model. Given is a finite collection of subtrees, $\{S^i\}$, $i = 1, \dots, p$. Each subtree S^i represents a "customer" or a demand region that will be served by a new facility, "server", which must be established on T. We assume that the server is also modelled by a subtree. It is uncapacitated and can serve all customers. If S is a server, then customer S^i, $i = 1, \dots, p$, is served at the closest point in S, i.e., the service distance of S^i is $d(S, S^i)$. There exists a (nonempty) family of subtrees, $\mathscr{F}$, from which the server can be selected. The selection is determined by the optimality criterion used. Motivated by the models presented in the introduction, we assume that the objective function, f, depends both on the "size" of S, e.g. length, diameter, total node weights, and the distances of S from the p customers, $S^1, \dots, S^p$. The dependence on the distances is formulated as follows. For $i = 1, \dots, p$, let $\bar{u}_i$ be the utility function associated with S^i. $\bar{u}_i$ is a real function of the distance to the server, S. Since we also wish to allow "obnoxious" customers, e.g., nuclear reactors or garbage depots, we do not, in general, require $\bar{u}_i$ to be monotone. We let $u_i(S) = \bar{u}_i(d(S, S^i))$, $i = 1, \dots, p$. We will also refer to $u_i(S)$ as the transportation cost function of S^i. The total transportation cost is given by $h(u_1(S), \dots, u_p(S))$, where $h(x_1, \dots, x_p)$ is a given real function defined for all p-tuples, $x_1, \dots, x_p$. In the so-called "median" models, $h(x_1, \dots, x_p) = x_1 + \cdots + x_p$, while for the "center" models $h(x_1, \dots, x_p) = \max\{x_1, \dots, x_p\}$. The dependence of the objective f on the "size" of the facility is denoted by $L(S)$. This term is independent of the customers. It reflects the set-up costs of the serving facilities and the cost of connecting them together (see the introduction). We now assume that for each S in $\mathscr{F}$

$$f(S) = L(S) + h(u_1(S), \dots, u_p(S)). \tag{1.1}$$

The location model is defined by

$$\min\{f(S) \mid S \in \mathscr{F}\}. \tag{1.2}$$

For example the minimum cost partial covering subtree problem in [5] corresponds to the following case. For $i = 1, \dots, p$, let v_i be a node in V. Then $S^i = (V^i, E^i)$, $V^i = \{v \in V \mid d(v, v^i) \le r_i\}$. Also

$$u_i(S) = \begin{cases} 0, & \text{if } d(S, S^i) = 0, \\ p_i, & \text{otherwise.} \end{cases}$$

$L(S)$, the size function in [5] is given by the total edge lengths in S.

In the next section, we present several results on the submodularity of the objective in (1.1).

2. Submodularity properties on trees

Consider a family $\mathscr{F}$ of (nonempty) subtrees of a given tree $T = (V, E)$. $\mathscr{F}$ is a *lattice family* if for every pair S_1 and S_2 in $\mathscr{F}$ both $S_1 \cup S_2$ and $S_1 \cap S_2$ are in $\mathscr{F}$. $\mathscr{F}$ is an *intersecting family* if for every pair S_1 and S_2 in $\mathscr{F}$ such that $S_1 \cap S_2$ is nonempty, both $S_1 \cup S_2$ and $S_1 \cap S_2$ are in $\mathscr{F}$. The family of all subtrees of T is an intersecting family but not a lattice. If $\mathscr{F}$ is an intersecting family, let $\mathscr{F}_v$, $v \in V$, denote the subfamily of $\mathscr{F}$ consisting of all the members of $\mathscr{F}$ containing the node v. Let $\bar{V} \subseteq V$ be the set of nodes v for which $\mathscr{F}_v$ is nonempty. Clearly, $\mathscr{F}_v$ is a lattice. Let f be a real function defined on an intersecting family $\mathscr{F}$. f is *isotone* (*antitone*) on $\mathscr{F}$ if for every pair S_1 and S_2 in $\mathscr{F}$ with $S_1 \subseteq S_2$,

$$f(S_1) \leq f(S_2) \quad (f(S_1) \geq f(S_2)).$$

f is *submodular* on $\mathscr{F}$ if for every pair S_1 and S_2 in $\mathscr{F}$ with $S_1 \cap S_2 \neq \emptyset$,

$$f(S_1 \cup S_2) + f(S_1 \cap S_2) \leq f(S_1) + f(S_2). \tag{2.1}$$

f is *supermodular* if $-f$ is submodular. f is *modular* on $\mathscr{F}$ if it is both submodular and supermodular.

As mentioned in the introduction our interest is in minimizing some objective, f, over an intersecting family of subtrees, $\mathscr{F}$. If f is isotone there exists a simple straightforward scheme to locate a minimizer. Let v be a node of T such that $\mathscr{F}_v$ is nonempty, i.e., $v \in \bar{V}$. Consider the subminimization of f over $\mathscr{F}_v$. The minimizer is the least element of the lattice $\mathscr{F}_v$, i.e., the intersection of all members in $\mathscr{F}_v$. Let $S(v)$, $v \in \bar{V}$, denote this subtree. Therefore, a minimizer over $\mathscr{F}$ is in the set $\{S(v)\}$, $v \in \bar{V}$. Special cases of this model are the minimal length covering subtree model in [7], and the minimal node cardinality covering subtree model in [8]. The isotone case is not even rich enough to unify the extension of [7] as presented in [5].

Motivated by [5, 7] and other models that we later discuss we suggest the framework where f is submodular. The attractiveness of this model follows from its unification property as well as its wealth of theory known today [3]. In particular, using the ellipsoid approach in [3] we can now minimize any submodular function over an intersecting family in (strongly) polynomial time.

In the remainder of this section we will prove several modularity and submodularity properties on families of subtrees of a given tree. Since our main motivation comes from location problems most of those properties will model and unify objective functions which are often used in this field.

2.1. *Modular functions*

We start with modular functions on $\mathscr{F}$ which depend only on the server but not on the customers in $\mathscr{F}^1 = \{S^1, \dots, S^p\}$.

Suppose that each edge $e \in E$ is associated with a weight (not necessarily nonnegative) α_e. (α_e can be viewed as the length of e.) Also, assume that each node $v \in V$ has a weight β_v. Let $S = (V', E')$ be a subtree of $T = (V, E)$. Define

$$\alpha(S) = \sum_{e \in E'} \alpha_e, \qquad \beta(S) = \sum_{v \in V'} \beta_v. \tag{2.1.1}$$

The next result follows directly from the definition.

Proposition 2.1. *Let $\mathscr{F}$ be an intersecting family of subtrees. The functions $\alpha(S)$ and $\beta(S)$ defined in* (2.1.1) *are both modular on $\mathscr{F}$.*

Another example of a modular function is the (edge) cut function. For each subtree S in $\mathscr{F}$ let $X(S)$ denote the set of edges of T connecting a node in S with a node in its complement, $T - S$. The value of the cut, $C(S)$, is given by

$$C(S) = \sum_{e \in X(S)} \alpha_e.$$

It is well known that if $\alpha_e \geq 0$, for each $e \in E$, $C(S)$ is submodular even on general graphs. (In this case $C(S)$ is defined over the collection of the subsets of nodes of a graph.) The next theorem shows that the cut function is modular when restricted to an intersecting family of subtrees of T, even without the nonnegativity assumption on $\{\alpha_e\}$, $e \in E$.

Theorem 2.2. *Let S_1 and S_2 be two subtrees with $S_1 \cap S_2 \neq \emptyset$ and $S_1 \cup S_2 \neq T$. Then*

$$C(S_1 \cup S_2) + C(S_1 \cap S_2) = C(S_1) + C(S_2). \tag{2.1.2}$$

Proof. Define the following pairwise disjoint sets of edges:

$$A_1 = \{e \in E \mid e \text{ connects a node in } S_1 - S_2 \text{ with a node in } T - (S_1 \cup S_2)\},$$

$$A_2 = \{e \in E \mid e \text{ connects a node in } S_2 - S_1 \text{ with a node in } T - (S_1 \cup S_2)\},$$

$$A_3 = \{e \in E \mid e \text{ connects a node in } S_1 \cap S_2 \text{ with a node in } T - (S_1 \cup S_2)\},$$

$$A_4 = \{e \in E \mid e \text{ connects a node in } S_1 \cap S_2 \text{ with a node in } S_1 - S_2\},$$

$$A_5 = \{e \in E \mid e \text{ connects a node in } S_1 \cap S_2 \text{ with a node in } S_2 - S_1\}.$$

The tree property implies that there is no edge connecting $S_1 - S_2$ with $S_2 - S_1$. Therefore, we obtain the following representation of the four cut sets.

$$X(S_1) = A_1 \cup A_5 \cup A_3,$$

$$X(S_2) = A_2 \cup A_4 \cup A_3,$$

$$X(S_1 \cup S_2) = A_1 \cup A_2 \cup A_3,$$

$$X(S_1 \cap S_2) = A_3 \cup A_4 \cup A_5.$$

Since the sets A_i, $1 \leq i \leq 5$, are pairwise disjoint the validity of (2.1.2) follows directly from this representation. □

Next we turn to modular functions which depend on the distances of the server from the customers. We first prove the following lemma.

Lemma 2.3. *Let S, S_1 and S_2 be nonempty subtrees of T. Suppose that $S_1 \cap S_2$ is nonempty. Let y be a closest point to S in $S_1 \cup S_2$, and suppose that y is in S_1. Then*

$$d(S, S_1 \cup S_2) = d(S, S_1) \quad \text{and} \quad d(S, S_1 \cap S_2) = d(S, S_2).$$

Proof. We have

$$d(S, S_1 \cup S_2) \leq d(S, S_1) \leq d(S, y) = d(S, S_1 \cup S_2).$$

Therefore, $d(S, S_1 \cup S_2) = d(S, S_1)$. To prove the second equality suppose first that y is also in S_2. In this case we obtain

$$d(S, y) = d(S, S_1 \cup S_2) \leq d(S, S_2) \leq d(S, S_1 \cap S_2) \leq d(S, y).$$

Thus, $d(S, S_1 \cap S_2) = d(S, S_2)$. Finally, suppose that every closest point to S in $S_1 \cup S_2$ is in $S_1 - S_2$. In particular, S does not intersect S_2. Let x be a closest point to S in S_2. Since there is a unique (simple) path connecting each pair of points on the tree, every path connecting x to S contains a point y of S_1. Also, the unique path connecting x and y contains some point z which is in $S_1 \cap S_2$. Therefore

$$d(S, S_2) = d(S, x) \geq d(S, z) \geq d(S, S_1 \cap S_2) \geq d(S, S_2).$$

This completes the proof. □

Theorem 2.4. *Let $\mathscr{F}^1 = \{S^1, \ldots, S^p\}$ be a collection of subtrees of T. Let $\mathscr{F}$ be an intersecting family of subtrees of T. For $i = 1, \ldots, p$ let $\bar{u}_i$ be a real function. For each S in $\mathscr{F}$ define $u_i(S) = \bar{u}_i(d(S, S^i))$, $i = 1, \ldots, p$. Then, any linear combination of the functions u_i, $i = 1, \ldots, p$, defined on $\mathscr{F}$ is modular.*

Proof. It will suffice to prove that $u_i(S)$, $i = 1, \ldots, p$, is modular over $\mathscr{F}$. Indeed, let S_1 and S_2 be two subtrees in $\mathscr{F}$ with $S_1 \cap S_2 \neq \emptyset$. Consider the subtree S^i used to define $u_i(S)$. From Lemma 2.3 we may assume without loss of generality that $d(S^i, S_1 \cup S_2) = d(S^i, S_1)$ and $d(S^i, S_1 \cap S_2) = d(S^i, S_2)$. Therefore,

$$u_i(S_1 \cup S_2) \equiv \bar{u}_i(d(S^i, S_1 \cup S_2)) = \bar{u}_i(d(S^i, S_1)) \equiv u_i(S_1),$$

and

$$u_i(S_1 \cap S_2) \equiv \bar{u}_i(d(S^i, S_1 \cap S_2)) = \bar{u}_i(d(S^i, S_2)) \equiv u_i(S_2).$$

This completes the proof of the theorem. □

2.2. Submodular functions

Given the edge weights $\{\alpha_e\}$, $e \in E$, and the node weights $\{\beta_v\}$, $v \in V$, we define for any subtree $S = (V', E')$,

$$\bar{\alpha}(S) = \max_{e \in E'} \{\alpha_e\}, \qquad \bar{\beta}(S) = \max_{v \in V'} \{\beta_v\}. \tag{2.2.1}$$

The following analogue of Proposition 2.1 follows directly from the definition.

Proposition 2.5. *Let $\mathscr{F}$ be an intersecting family of subtrees. The functions $\bar{\alpha}(S)$ and $\bar{\beta}(S)$ defined in (2.2.1) are both submodular on $\mathscr{F}$.*

To state the next submodularity result, let $L(S)$ be the diameter of S, i.e., the maximum length of a (simple) path in S.

Theorem 2.6. *Suppose that all edge weights $\{\alpha_e\}$, $e \in E$, are nonnegative. Let $\mathscr{F}$ be an intersecting family of subtrees of T. For each $S \in \mathscr{F}$, let $L(S)$ denote the diameter of S. Then $L(S)$ is submodular over $\mathscr{F}$.*

Proof. Let S_1 and S_2 be subtrees in $\mathscr{F}$. Suppose that $S_1 \cap S_2$ is nonempty. Define $S = S_1 \cup S_2$. $L(S)$ is the length of a longest simple path in S. Let x and y be two nodes in S such that $L(S) = d(x, y)$.

Suppose first that both x and y belong to S_1. Then, using the monotonicity of $L(S)$ we obtain $L(S_1) \leq L(S_1 \cup S_2) = d(x, y) \leq L(S_1)$ and $L(S_1 \cap S_2) \leq L(S_2)$. Therefore, $L(S_1 \cup S_2) + L(S_1 \cap S_2) \leq L(S_1) + L(S_2)$. (The latter inequality holds also when both x and y are in S_2.)

Suppose without loss of generality that $x \in S_1$ and $y \in S_2$. Let u and v be two nodes (not necessarily distinct) in $S_1 \cap S_2$ such that $L(S_1 \cap S_2) = d(u, v)$. Let $\bar{T}$ be the union of the two paths $P(y, u)$ and $P(y, v)$. Define w to be the closest point in $\bar{T}$ to x. Without loss of generality suppose that w is on $P(y, u)$. Then $d(x, z) = d(x, w) + d(w, z)$ for $z = y, u, v$, and $d(y, u) = d(y, w) + d(w, u)$. Therefore

$$
\begin{aligned}
L(S_1) + L(S_2) &\geq d(x, v) + d(y, u) \\
&= d(x, w) + d(w, v) + d(y, w) + d(w, u) \\
&= d(x, y) + d(w, v) + d(w, u) \geq d(x, y) + d(v, u) \\
&= L(S) + L(S_1 \cap S_2). \qquad \square
\end{aligned}
$$

The next submodularity result depends on the distances of the server from the customers. It is motivated by "center" models, often used in location theory. The objective there is to minimize the maximum transportation cost over all customers. (In this respect note that Theorem 2.4 includes the objective of the "median" models.)

Theorem 2.7. *Let $\mathscr{F}^1 = \{S^1, \ldots, S^p\}$ be a collection of subtrees of T. Let $\mathscr{F}$ be an intersecting family of subtrees of T. For $i = 1, \ldots, p$, let $\bar{u}_i$ be a monotone nondecreasing real function. For each S in $\mathscr{F}$ define $u_i(S) = \bar{u}_i(d(S, S^i))$, $i = 1, \ldots, p$. Then the function $g(S)$, defined by*

$$g(S) = \max\{u_1(S), \ldots, u_p(S)\}$$

is submodular on $\mathscr{F}$.

Proof. The monotonicity property of u_i, $i = 1, \ldots, p$, implies that $g(S)$ is antitone, i.e., if $S_1 \subseteq S_2$ then $g(S_1) \geq g(S_2)$.

To prove the submodularity of g consider S_1 and S_2 in $\mathscr{F}$ with $S_1 \cap S_2 \neq \emptyset$. Let i, $i = 1, \ldots, p$, be such that $g(S_1 \cap S_2) = \bar{u}_i(d(S^i, S_1 \cap S_2))$. Using Lemma 2.3 we assume without loss of generality that $d(S^i, S_1 \cap S_2) = d(S^i, S_2)$. Therefore

$$g(S_1 \cap S_2) = \bar{u}_i(d(S^i, S_1 \cap S_2)) = \bar{u}_i(d(S^i, S_2)) \leq g(S_2).$$

From the antitonicity of g we have $g(S_1 \cup S_2) \leq g(S_1)$. Thus,

$$g(S_1 \cup S_2) + g(S_1 \cap S_2) \leq g(S_1) + g(S_2). \qquad \square$$

3. Concluding remarks

The general model presented above provides a unified framework for the polynomial solvability of many location problems. Using the results in [3] we conclude that the above model can be solved in strongly polynomial time. The theory developed in [3] allows us to introduce certain additional constraints without affecting the polynomial solvability. Here are two examples of such constraints. Suppose first that there is a subfamily $\bar{\mathscr{F}}$ of the given intersecting family $\mathscr{F}$. $\bar{\mathscr{F}}$ consists of subtrees that are not feasible, e.g., zoning considerations. If $\bar{\mathscr{F}}$ is a *clutter*, i.e., it does not contain a pair of subtrees, say S_1 and S_2, with $S_1 \subseteq S_2$, then the optimal subtree in $\mathscr{F} - \bar{\mathscr{F}}$ can be obtained in polynomial time. A second constraint is a parity requirement. The optimal subtree in $\mathscr{F}$ with an even number of nodes can be selected efficiently.

References

[1] V. Aaronson Huston and C.S. Revelle, Maximal covering tree problems, Transportation Sci. 23 (1989) 288–299.

[2] G. Cornuejols, G.L. Nemhauser and L.A. Wolsey, The uncapacitated facility location problem, in: P.B. Mirchandani and R.L. Francis, eds., Discrete Location Theory (Wiley, New York, 1990) 119–171.

[3] M. Grötschel, L. Lovasz and A. Schrijver, Geometric Algorithms and Combinatorial Optimization (Springer, Berlin, 1988).

[4] R. Hassin and A. Tamir, Efficient algorithms for optimization and selection on series-parallel graphs, SIAM J. Algebraic Discrete Methods 7 (1986) 379–389.

[5] T.U. Kim, T.J. Lowe and J.E. Ward, The minimum cost partial covering subtree of a tree, Tech. Rept., Krannert Graduate School of Management, Purdue University, West Lafayette, IN (1989).

[6] T.U. Kim, T.J. Lowe, J.E. Ward and R.L. Francis, A minimum length covering subgraph of a network, Ann. Oper. Res. 18 (1989) 245–260.

[7] T.U. Kim, T.J. Lowe, J.E. Ward and R.L. Francis, A minimum length covering subtree of a tree, Naval Res. Logist. 37 (1990) 309–326.

[8] R.K. Kincaid, T.J. Lowe and T.L. Morin, The location of central structures in trees, Comput. Oper. Res. 15 (1988) 103–113.

[9] A. Kolen and A. Tamir, Covering problems, in: P.B. Mirchandani and R.L. Francis, eds., Discrete Location Theory (Wiley, New York, 1990) 263–304.

[10] E. Minieka, The optimal location of a path or tree in a tree network, Networks 15 (1985) 309–321.

[11] C.A. Morgan and P.J. Slater, A linear algorithm for a core of a tree, J. Algorithms 1 (1980) 247–258.

[12] M.B. Richey, Optimal location of a path or tree on a network with cycles, Networks 20 (1990) 391–407.

[13] P.J. Slater, On locating a facility to service areas within a network, Oper. Res. 29 (1981) 523–531.

[14] P.J. Slater, Locating central paths in a network, Transportation Sci. 16 (1982) 1–18.